MAT 319: SUGGESTED PROJECTS

OCT 26, 2006

Rules

You can work individually or in groups (up to 4 students in a group). Each group must select a project from the list below; if you prefer, you can also choose a different topic after discussing it with me. The choice of groups and projects should be done by **Wed, Nov. 1**.

Each group will make a short oral presentation during the last two weeks of the semester highlighting one or two of the main arguments. (Each group should divide the topic into separate parts, so that each member of the group contributes their own bit of the presentation: there should be 5 minutes each.) Each person will also **individually** write a 6-8 page paper on the topic chosen. (This can be handwritten (especially the formulas etc) but must be legible.)

I am most interested in your constructing valid arguments and explaining the mathematics clearly, though you can give a small amount of background. You should outline the whole project but concentrate your calculations on your special part. The essays should describe specific examples, not just be general and vague.

The written papers are **due on Dec. 4**. The oral presentations will be done in the last two weeks of the semester; details will be decided later.

Plagiarism

The line between plagiarism and legal borrowing is as follows: you can borrow the ideas from any sources (books, internet, discussions with other members of your group) but final writing should be your own and reflect your understanding of the material — not cut-and-paste. Any sources you use (other than discussions with other members of your group) must be clearly acknowledged.

In general, you are strongly advised to use books rather than online sources for your projects: internet is full of texts by people who have very strong opinions and incomplete understanding. Books are usually much more reliable.

Projects

1. **Projects on Sequences.**

Project 1: Square roots. Example 3.3.5 gives a sequence that calculates $\sqrt{a}$. This sequence is derived from Newtons method of finding roots of an equation. (You can find this in any first year Calculus text.) Explain this. Use the algorithm to calculate $\sqrt{5}$ correct to 3 decimal places. How many terms do you need? Explain the error estimate. There is another way finding square roots by a long division process, a method that "every school boy used to learn" — perhaps 100 years ago. Find out about this and explain it. Is it based on the same or a different algorithm?

Project 2: The Fibonacci sequence. This is the well known sequence 1, 1, 2, 3, 5, 8, ...defined inductively by the equation $F_{n+1} = F_n + F_{n-1}$. It turns out that the ratios of successive terms $r_n = F_{n+1}/F_n$ form a sequence that is eventually monotone and converges to the number known as the Golden Ratio Φ (the positive root of the equation $x^2 = x + 1$.) Use the method of Ex 3.3.5 to explain this. Explore the effect of starting with pairs of different numbers, eg with 1, 5 or with -2, 1. Do these sequences also converge? If so, what are their limits?

Project 3: Contractive sequences. Instead of the rule $F_{n+1} = F_n + F_{n-1}$, consider the sequence defined by $s_{n+1} = as_n + (1-a)s_{n1}$, where $0 < a < 1$. Thus s_{n+1} is a weighted average of s_n and s_{n1}. This sequence is contractive and converges. The case $a = 1/2$ is explained on p. 82 in Ex. 3.5.6(a). You could discuss a different case (say $a = 1/4$ or $a = 1/3$.) Contractive sequence are also used in Ex 3.5.10 to find roots of certain polynomials. Use this method to find a root of the equation $x^4 + 2x - 2 = 0$ lying between 0 and 1 correct to 5 decimal places. (This is a good project for three people; one could explain contractive sequences and the other two could do the different examples.)

Project 4: A divergent sequence. Investigate the sequence (x_n) where $x_n = \sin n$. Show how to construct a subsequence of x_n that converges to $+1$ and another that converges to -1. What properties of the sine function and the number π do you use in your argument? Can you show that this sequence has a subsequence that sconverges to any given number between -1 and 1? It may be helpful to look at Example 3.4.6(c) where it is shown that this sequence is divergent.

2. **Projects on series.**

Project 5: Binary and ternary expansions. Find the binary and ternary representations of the numbers 3/8, 1/3, 2/7, and 32. Explain (using the Completeness Axiom) why any infinite sequence $S = .10011100111...$of zeros and ones represents a unique real number x_S. Show how to get a geometric series from the ternary representation of 3/8 and sum it to 3/8. Is it true that S is eventually periodic iff x_S is rational? Explain. This project is based on the end of Ch 2.5. See also Ex. 3.7.2 (a) (geometric series). It is somewhat related to Project 8.

Project 6: Variations on the harmonic series. If $\{m_1, m_2, \dots\}$ is the collection of all natural numbers that end in 6 then $\sum_{k=1}^{\infty} \frac{1}{m_k}$ diverges. (Hint: adapt the proof that the harmonic series $\sum \frac{1}{n}$ is divergent.)

The next result is more unexpected. Let $\{n_1, n_2, \dots\}$ be the collection of all natural numbers that do NOT use the digit 6 in their decimal expansion (e.g., 345145 is in this collection while 3456247 is not.) Show that $\sum_{k=1}^{\infty} \frac{1}{n_k}$ converges to a number less than 80. What can you say about $\sum_{k=1}^{\infty} \frac{1}{p_k}$ if $\{p_1, p_2, \dots\}$ is the collection of natural numbers that do not involve 4 in their decimal expansion?

Note: This is an adaptation of ex 16 on p 263. You will have to read about the convergence of sequences of positive numbers from Section 3.7 and Ch 9, specially Ex. 3.7.2 (a) (geometric series), 3.7.6(b) (harmonic series) and the Comparison test 3.7.7

Project 7: The harmonic series and rearrangements. Show how to rearrange the terms of the alternating harmonic series $\sum_{N=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \dots$ so that the series (a) converges to 2, and (b) diverges. (In a rearrangement you permute the order of the terms but not their signs, eg you might consider $1+1/3-1/2+1/5+1/7-1/4\dots$). Can you find a rearrangemet of the terms to make this sequence converge to 5?

Note: This is an example of a conditionally convergent sequence, ie. the corresponding sequence of positive terms does not converge. You will find relevant definitions on p 89, and p 255. cf also Ex 3.3.3(b), Ex 3.7.6(b).

Project 8: Euler's product. Give an accurate definition of infinite product $\prod_{n=1}^{\infty} a_n$ as a limit. Describe the relation between convergence of the sum $\sum a_n$ and product $\prod(1+a_n)$. Accurately prove Euler's formula:

$$\prod_{i=1}^{\infty} \frac{1}{1-p_i^{-s}} = \sum_{n=1}^{\infty} n^{-s}, \qquad s > 1$$

where $p_1 = 2$, $p_2 = 3$, $p_3 = 5\dots$ is the sequence of all prime numbers.

3. **Projects on Sets.**

Project 9: The Cantor set $\mathbb{F}$. This is a paradoxical subset of $[0,1]$ discovered by Cantor. It is obtained from the unit interval by first removing the open interval $I_1 = (1/3, 2/3)$ (which leaves two intervals each of length 1/3), then removing the middle thirds of these two intervals (leaving four intervals each of length 1/9), then removing the middle thirds of these four intervals which gives you 8 intervals of length 1/27 and so on .... See p. 316318. Give a detailed description of this set $\mathbb{F}$ as an infinite intersection. Show it is closed (cf Def 11.1.2) and that all its points are cluster points. Explain its relation to ternary expansions, ie expansions to base 3. Show that its complement in $[0,1]$ is the union of infinitely many disjoint open intervals of total length 1. So you might think youd taken away all the points in $[0,1]$. But in fact $\mathbb{F}$ has uncountably many points. Explain clearly and in detail why this is so.

Project 10: Countable and Uncountable sets. Give a variety of examples of Countable and Uncountable sets. For example, show that the set of all sequences (x_n) where $x_n = 0, 1,$ or 2 is uncountably infinite. Show that its subset consisting of sequences with only a finite number of nonzero entries is countably infinite. What about the set of sequences that are eventually periodic i.e. they are periodic if you ignore a finite set of terms at the beginning?

4. **More theoretical projects.** Uniform continuity and convergence are the most important concepts covered in MAT 320 and not in MAT 319. Projects 11 and 12 use the Weierstrass M-test (on p 268), which involves uniform convergence.

Project 11: Uniform continuity. Give examples of continuous functions that are not uniformly continuous. Explain why any continuous function defined on the interval $[a, b]$ is uniformly continuous. This concept is important when defining the Riemann integral: explain.

Project 12: Uniform convergence. Explain why the uniform limit of continuous functions is continuous (p 235). Give an example of a sequence of continuous functions that converges pointwise to a discontinous limit. Explain the Weierstrass M-test.

The next two projects concern examples whose discovery astounded the mathematics community of the day. There are many possible references for these examples, and you can use them if you prefer. The main thing is to describe the examples carefully and then explain why they have the properties claimed.

Project 13: A continuous nowhere differentiable function. Give a careful explanation of the example on p 354. Why is your function not differentiable at the point $1/\sqrt{2}$? (You will need to understand the definition of the derivative from 6.1.1.)

Project 14: A space filling curve. Explain the example on p 355 or you might be able to find another one online. Explain why your curve goes through an arbitrary point in the square, say $(1/\sqrt{2}, 1/\sqrt{3})$. Draw some pictures, using a computer graphics system if possible. This is related to fractal curves, snowflakes etc.

Project 15: Non-standard Analysis. Before rigorous definition of limit was given in 19th century, people somtimes described limits by saying that the difference gets infinitley small. It is not rigorous: there are no "infinitely small" real numbers. However, in the 20th century it was shown that in fact this language can be made rigoroius as well: it is possible to define some set of numbers which, i addition to usual real numbers, also contains "infinitely small" and "infinitely large" numbers. Give an overview of this. Give a definition of continuous function in this language.