MAT 319: HOMEWORK 5

DUE TH, OCT 13

Note that our exposition in class was different from the textbook. We have used the theorem about nested intervals (below); on the other hand, we didn't use the notion of $\limsup$ and $\liminf$.

Theorem. *Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$, ... be a sequence of nested intervals:*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

Then there exists a common point: $\exists c \in \mathbb{R} : c \in [a_k, b_k]$ for all k.

1. Let $I_1, I_2 \dots,$ be a sequence of nested intervals as in the theorem. Assume in addition that $l_k = \text{length}(I_k) = b_k - a_k \to 0$ as $k \to \infty$. Prove that then, the common point is unique.

2. Let s_n be a sequence such that for all n, we have

$$|s_{n+1} - s_n| < 2^{-n}$$

Prove that then s_n is a Cauchy sequence and thus must converge.

3. Let sequence a_n be defined by $a_1 = 1$, $a_{n+1} = (a_n + 1)/3$. Prove:
(a) For all n, $a_n > 1/2$
(b) a_n is a decreasing sequence
(c) Show that $\lim a_n$ exists and find this limit.

4. Prove the lemma discussed in class: a number A is a limit of some subsequence of sequence s_n if and only if every interval $(A - \varepsilon, A + \varepsilon)$ contains infinitely many terms of the sequence.

5. A set S is called closed if for any sequence s_n of elements of S, if the limit $\lim s_n$ exists and is finite, then this limit is itself in S.
(a) Prove that interval $[0, 1]$ is closed.
(b) Prove that interval $(0, 1)$ is not closed.
(c) Is the set of rational numbers closed? all irrational numbers?
(d) Prove that if S, T are closed sets, then their intersection $S \cap T$ is also closed.

6. A number $L \in \mathbb{R}$ is called a subsequential limit of sequence s_n if it is a limit of some subsequence of s_n.

Construct a sequence which has exactly three subsequential limits.