Homework 2, due on Thursday, September 15

1. Prove that the following statements about real numbers $a, b, c$ are equivalent to each other:

   (1) $a \in (b - c, b + c),$
   (2) $b - c < a < b + c,$
   (3) $|a - b| < c,$
   (4) $b \in (a - c, a + c),$
   (5) $c \in (\max(b - a, a - b), +\infty).$

2. Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R},$ and let $A + B$ and $A - B$ be, respectively, the set of all sums $a + b$ and all differences $a - b,$ where $a \in A$ and $b \in B.$ Prove that $\sup(A + B) = \sup A + \sup B$ and $\sup(A - B) = \sup A - \inf B.$

3. Modify and prove the statements of the preceding problem for the products and quotients. Should any additional restrictions on the sets be imposed?

4. Problem 4.4 from the textbook.
Solutions

1

(1) \iff (2): this is definition of \((b-c, b+c)\)

(2) \iff (3): if \(|a-b| \leq c\), then

\[-c < a-b < c\]

adding \(b\) to both sides, we get

\[b-c < a < b+c\]

Conversely, if (2) holds, then subtract \(b\) to get

\[-c < a-b < c\]

so

\[|a-b| < c\]

(2) \iff (4): \hspace{1cm} \text{by same argument as above,}

(3) \iff a-c < b < a+c

\[\iff b \in (a-c, a+c)\]

(5) \iff (3):

(5) \iff \(c > \max(a-b, b-a)\)

\[\iff (c > a-b) \text{ and } (c > b-a)\]

\[\iff -c < a-b < c\]
2 a. For $+$:

Since for any $a \in A$, we have $a \leq \sup A$ for any $b \in B$, we have $b \leq \sup B$ this shows that for any $a \in A, b \in B$

$$a + b \leq \sup A + \sup B$$

So $\sup A + \sup B$ is an upper bound for $A + B$; thus,

$$\sup (A + B) \leq \sup (A) + \sup (B).$$

On the other hand, for any $\varepsilon > 0$, one can find $a \in A \text{ s.t. } a > (\sup A) - \frac{\varepsilon}{2}$ and $b \in B \text{ s.t. } b > (\sup B) - \frac{\varepsilon}{2}$

(otherwise, $\sup A - \frac{\varepsilon}{2}$ would be an upper bound for $A$). Thus,

$$\forall \varepsilon > 0 \; \exists a \in A, \; b \in B \; \text{s.t.} \; a + b > (\sup A + \sup B) - \varepsilon$$

Therefore, number $(\sup A + \sup B) - \varepsilon$ can not be an upper bound of $A + B$.

So, $\sup A + \sup B$ is the lowest upper bound of $A + B$. 
For $A \setminus B$: follows from the previous and the fact that $\sup(-B) = -\inf(B)$

Indeed: 

$\exists X$ is an upper bound for $-B$ 

$\forall x < X$ 

$-X$ is a lower bound for $B$

Thus, least upper bound for $-B$

$\sup(-B)$ (greatest lower bound for $B$)
One possible modification:

If all elements of $B$ are positive, then

$$\sup(A \cdot B) = \sup(A) \cdot \sup(B).$$

If, in addition, $\inf B > 0$, then

$$\sup(A / B) = \frac{\sup(A)}{\inf(B)}$$

Proof: If $a \in A$, then $0 < a \leq \sup(A)$

If $b \in B$, then $b \leq \sup(B)$

Thus, multiplying by $a$, we get

$$a \cdot b \leq a \cdot \sup(B) \leq \sup(A) \cdot \sup(B)$$

which shows that $\sup(AB) \leq \sup(A) \cdot \sup(B)$

To prove equality, it suffices to prove that for any $\varepsilon > 0$, one can find $a \in A$, $b \in B$ s.t.

$$ab > \sup(A) \cdot \sup(B) - \varepsilon$$

To do that, choose $a \in A$, $b \in B$ so that

$$|ab - a \cdot \sup(B)| < \varepsilon / 2$$ (x)

$$|\sup(B)| < \sup(A)$$

$$|a \cdot \sup(B) - \sup(A) \cdot \sup(B)| < \varepsilon / 2$$ (xx)

(see Figure.)
\[
\begin{align*}
\text{To get } (\#), \text{ take } b > \sup(B) - \frac{\varepsilon}{2a} \\
\text{To get } (\#\#), \text{ take } a > \sup(A) - \frac{\varepsilon}{2 \cdot \sup(B)}
\end{align*}
\]
Together \((\#), (\#\#)\) imply that
\[
|ab - \sup(A) \cdot \sup(B)| < \varepsilon
\]
so
\[
ab > \sup(A) \cdot \sup(B) - \varepsilon
\]
so \(\sup(A) \cdot \sup(B) - \varepsilon\) can not be an \(\varepsilon\) upper bound for \(A \cdot B\).

**Note:** this was a really hard problem.

For quotients, the rule follows from
the rule for products and
\[
\sup(1/B) = \frac{1}{\inf(B)} \quad \text{if } \inf(B) > 0.
\]
4.  
   a. \( \inf = 0 \)
   
   b. \( \inf = 0 \)
   
   c. \( \inf = 2 \)
   
   d. \( \inf = e \)
   
   e. \( \inf = 0 \)
   
   f. \( \inf = 0 \)
   
   g. \( \inf = 0 \)
   
   h. \( \inf = 2 \)
   
   i. \( \inf = 0 \)
   
   j. \( \inf = 1 - \frac{1}{3} = \frac{2}{3} \)
   
   k. \( \inf = 0 \)
   
   l. Not bounded below
   
   m. \( \inf = -2 \)
   
   n. \( \inf = -\sqrt{2} \)
   
   o. Not bounded below
   
   p. \( \inf = 1 \)
   
   q. \( \inf = 0 \)
   
   r. \( \inf = 1 \)
   
   s. \( \inf = 0 \)