MAT 314: HOMEWORK 8

DUE TH, APRIL 25, 2019

Throughout this problem set, F is a field of characteristic zero.

1. Show that any extension of F of degree 2 is normal.

2. Let $F \subset E$ be an algebraic extension. We say that elements $\alpha, \alpha' \in E$ are conjugate (over F) if they are roots of the same irreducible polynomial $f \in F[x]$. We define $\deg_F(\alpha) = \deg f$.

(a) Show that an element $\alpha \in E$ can have no more than $n = \deg_F(\alpha)$ conjugates in E (including itself); if E is algebraically closed, then α has exactly n conjugates.

(b) Assume that α has exactly $n = \deg_F(\alpha)$ conjugates $\alpha_1, \dots, \alpha_n$ in E. Define the elements

$$N = \prod_i \alpha_i T = \sum_i \alpha_i$$

Show that then $N, T \in F$.

3. Let $F \subset E \subset K$ be a chain of extensions such that K is a finite normal extension of F. Show that then K is also a normal extension of E. Is it true that E is always a normal extension of F?

4. Recall that the cyclotomic polynomials $\Phi_n(x) \in \mathbb{Z}[x]$ are defined by

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

where the product is taken over all divisors d of n (including 1 and n); as we have shown in the previous homework, $\deg(\Phi_n) = \varphi(n)$, where $\varphi(n)$ is the Euler's function.

(a) Prove that in any field E (possibly of positive characteristic), if $\zeta \in E$ is a primitive root of 1 of degree n (i.e. $\zeta^n = 1$ but $\zeta^k \neq 1$ for all $k < n$), then ζ is a root of Φ_n. Deduce from this that the number of primitive roots of 1 of degree n in E is at most $\varphi(n)$

(b) Prove that if $G \subset E^*$ is a finite subgroup of the group E^* of all nonzero elements of E with respect to multiplication, then G is cyclic (hint: if $n = |G|$, then for any $x \in G$, $x^n = 1$).