MAT 314: HOMEWORK 9

DUE TH, MAY 9, 2019

Throughout this problem set, F is a field of characteristic zero.

1. Let $p(x) \in F[x]$ be a polynomial of degree n. Let L be the splitting field of p, and let $x_1, \dots, x_n \in L$ be the roots of p. Define

$$\Delta = \prod_{i<j} (x_i - x_j) \in L$$

(a) Prove that for every $g \in Gal(L/F)$, we have $g(\Delta) = sgn(g)\Delta$, where $sgn(g)$ is the sign of the corresponding permutation (recall that $Gal(L/F) \subset S_n$).
(b) Prove that $D = \Delta^2 \in F$ (D is called the discriminant of p).
(c) Prove that $\Delta \in F$ iff $Gal(L/F) \subset A_n$ (where A_n is the subgroup of even permutations).

2. Describe the Galois groups of the follwoing polynomials over $\mathbb{Q}$:
(a) $x^3 - 3x + 1$
(b) $x^3 - 3x + 3$
(You can use without proof the result I quoted in class: for a cubic polynomial $x^3 + px + q$, the discriminant is $D = -4p^3 - 27q^2$.)

3. Let G be a group, with subgroups $G_1, G_2 \subset G$ such that G_1 is a normal subgroup of G_2. Let $\varphi\colon G \to H$ be a group homomorphism.
(a) Prove that $\varphi(G_1)$ is a normal subgroup of $\varphi(G_2)$.
(b) Prove that if G_2/G_1 is commutative, then so is $\varphi(G_2)/\varphi(G_1)$.

4. A group G is called *solvable* if there exists a finite collection of subgroups

$$\{1\} \subset G_1 \subset G_2 \subset \cdots \subset G_k = G$$

such that each G_i is normal subgroup of G_{i+1} and the quotient G_{i+1}/G_i is commutative.
(a) Prove that S_3, S_4 are solvable
(b) Use the previous problem to prove that if G is solvable, then any quotient of G is also solvable.