Some common tautologies

1. Modus ponens \((P \land (P \Rightarrow Q)) \Rightarrow Q\)
2. \(((P \Rightarrow Q) \land (\neg Q)) \Rightarrow \neg P\)
3. \(((P \Rightarrow Q) \land (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)\)
4. De Morgan’s laws: \((\neg (P \lor Q)) \iff (\neg P \land \neg Q)\)
5. \((\neg (P \land Q)) \iff (\neg P \lor \neg Q)\)
6. Contrapositive: \((P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P)\)
7. \((P \lor Q) \land \neg Q \Rightarrow P\)
8. \((\neg (P \Rightarrow Q)) \iff (P \land \neg Q)\)

Some methods of proof

**Direct proof:** To prove \(P \Rightarrow Q\), assume \(P\); derive \(Q\) (you are allowed to use that \(P\) is true when doing this). Then you can conclude that \(P \Rightarrow Q\) is true (without any assumptions!).

**Indirect proof, also known as proof by contradiction:** To prove that \(P\) is true, assume \(\neg P\); derive from this a contradiction (you are allowed to use that \(P\) is false when doing this). Then you can conclude that \(P\) is true (without any assumptions!).

Important special case: to prove \(P \Rightarrow Q\), you can assume \(\neg (P \Rightarrow Q)\) (which, by (8) above, is the same as \(P \land \neg Q\)) and get a contradiction.

**Proof by cases:** If \(P_1 \lor P_2 \lor \ldots\) is true, and you have proved that \(P_1 \Rightarrow Q\), \(P_2 \Rightarrow Q\), \ldots, then you can conclude that \(Q\) is true.

**Induction**

**Proofs by induction:** if \(P(n)\) is some statement depending on an integer number \(n\), and we have checked that

- (Induction base) For \(n = 1\), \(P(n)\) is true
- (Induction step) For any \(n\), if \(P(n)\) is true then \(P(n + 1)\) is also true.

Then \(P(n)\) is true for all integer \(n \geq 1\).

**Strong induction** if \(P(n)\) is some statement depending on an integer number \(n\), and we have checked that

- (Induction base) For \(n = 1\), \(P(n)\) is true
- (Induction step) For any \(n\), if \(P(k)\) is true for \(k = 1, \ldots, n\), then then \(P(n + 1)\) is also true.

Then \(P(n)\) is true for all integer \(n \geq 1\).

**Proofs with quantifiers**

1. De Morgans laws: \(\neg (\exists x \ P(x)) \iff \forall x \ \neg P(x)\)
2. \(\neg (\forall x \ P(x)) \iff \exists x \ \neg P(x)\)
3. Given that \(\forall x \in M \ P(x)\) (where \(P(x)\) is some statement) and that \(c \in M\), we can conclude \(P(c)\).
4. If you know that \(\exists x \ P(x)\) is true, you can say “let us choose an \(a\) such that \(P(a)\) is true”.
   Warning: to denote this chosen value, you should use a variable which was not used so far in your arguments!
5. To prove \(\exists x \ P(x)\), it suffices to give one example of \(x\) for which \(P(x)\) is true.
(6) To prove $\forall x P(x)$, you have to give a proof of $P(x)$ which would work for all possible values of $x$. Proving it in one (or two, or five... ... special cases is not enough. Thus, a typical proof of $\forall x \in M P(x)$ begins: “Let $x$ be an arbitrary element of $M$...”

To disprove $\forall x P(x)$, it suffices to give one example of $x$ for which $P(x)$ is false.