MAT 200 Handout 1

March 3, 2016

Some common tautologies

This is not the full list, only the most common ones.

(1) Modus ponens $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
(2) $((P \Rightarrow Q) \wedge (\neg Q)) \Rightarrow \neg P$
(3) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
(4) De Morgan's laws: $(\neg(P \vee Q)) \iff (\neg P) \wedge (\neg Q)$
(5) $(\neg(P \wedge Q)) \iff (\neg P) \vee (\neg Q)$
(6) Contrapositive: $(P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P)$
(7) $(P \vee Q) \wedge \neg Q \Rightarrow P$
(8) $\neg(P \Rightarrow Q) \iff (P \wedge \neg Q)$

Some methods of proof

Direct proof: To prove $P \Rightarrow Q$, assume P; derive Q (you are allowed to use that P is true when doing this). Then you can conclude that $P \Rightarrow Q$ is true (without any assumptions!).

Indirect proof, also known as proof by contradiction: To prove that P is true, assume $\neg P$; derive from this a contradiction (you are allowed to use that P is false when doing this). Then you can conclude that P is true (without any assumptions!).

Important special case: to prove $P \Rightarrow Q$, you can assume $\neg(P \Rightarrow Q)$ (which, by (8) above, is the same as $P \wedge \neg Q$) and get a contradiction

Proof by cases: If $P_1 \vee P_2 \vee \dots$ is true, and you have proved that $P_1 \Rightarrow Q$, $P_2 \Rightarrow Q$, $\dots$, then you can conclude that Q is true.

Induction

Proofs by induction: if $P(n)$ is some statement depending on an integer number n, and we have checked that

- (Induction base) For $n = 1$, $P(n)$ is true
- (Induction step) For any n, if $P(n)$ is true then $P(n+1)$ is also true.

Then $P(n)$ is true for all integer $n \geq 1$.

Strong induction if $P(n)$ is some statement depending on an integer number n, and we have checked that

- (Induction base) For $n = 1$, $P(n)$ is true
- (Induction step) For any n, if $P(k)$ is true for $k = 1, \dots, n$, then then $P(n+1)$ is also true.

Then $P(n)$ is true for all integer $n \geq 1$.

Proofs with quantifiers

(1) De Morgans laws: $\neg\big(\exists x\ P(x)\big) \iff \forall x\ \neg P(x)$
(2) $\neg\big(\forall x\ P(x)\big) \iff \exists x\ \neg P(x)$
(3) Given that $\forall x \in M\ P(x)$ (where $P(x)$ is some statement) and that $c \in M$, we can conclude $P(c)$.
(4) If you know that $\exists x\ P(x)$ is true, you can say "let us choose an a such that $P(a)$ is true". Warning: to denote this chosen value, you should use a variable which was not used so far in your arguments!
(5) To prove $\exists x\ P(x)$, it suffices to give one example of x for which $P(x)$ is true.

(6) To prove $\forall x\ P(x)$, you have to give a proof of $P(x)$ which would work for all possible values of x. Proving it in one (or two, or five.....) special cases is not enough. Thus, a typical proof of $\forall x \in M\ P(x)$ begins: "Let x be an arbitrary element of M..."

To **disprove** $\forall x\ P(x)$, it suffices to give one example of x for which $P(x)$ is false.