Practice Midterm 2 Solutions
MAT 125

Spring 2006

Name:______________________________ ID number: ____________________

Recitation number (e.g., R01):__________
(for evening lecture, use "ELC 4")

Lecture 1	MWF 9:35–10:30	An, Daniel
R01	M 11:45am–12:40pm	Solorzano, Pedro
R02	Th 3:50pm– 4:45pm	Ostrovsky, Stanislav
R03	W 11:45am–12:40pm	Solorzano, Pedro
R04	Tu 11:20am–12:15pm	Basu, Somnath
R05	Tu 11:20am–12:15pm	Han, Zhigang
R31	M 10:40am–11:35am	Patu, Ionel
Lecture 2	TuTh 2:20pm – 3:40pm	Kirillov, Alexander
R06	M 11:45am–12:40pm	Zeng, Huayi
R07	F 11:45am–12:40pm	Nowicki, Jan
R08	W 9:35am-10:30am	Ma, Xin
R09	Tu 3:50pm– 4:45pm	Ostrovsky, Stanislav
R10	F 8:30am–9:25am	Ma, Xin
Lecture 3	MW 3:50pm–5:10pm	Chen, Je-Wei
R11	M 9:35am–10:30am	Poole, Thomas
R12	F 10:40am–11:35am	Panok, Lena
R13	W 2:20pm–3:15pm	Poole, Thomas
R14	Tu 11:20am-12:15pm	Lyberg, Ivar
R15	Th 11:20am–12:15pm	Lyberg, Ivar
R32	M 2:20pm– 3:15pm	Guo, Weixin
Evening Lec 4	TuTh 6:50pm–8:10pm	Bulawa, Andrew

Please answer each question in the space provided. Please write full **solutions**, not just answers. Unless otherwise marked, **answers without justification will get little or no partial credit**. Cross out anything the grader should ignore and circle or box the final answer. Do **NOT** round answers.

No books, notes, or calculators!

Do not open the exam until instructed by proctor!

1. Compute the following limits. Please distinguish between "limit is equal to ∞", "limit is equal to $-\infty$" and "the limit doesn't exist even allowing for infinite values":

   (a) $\lim_{x\to\infty} \frac{x^3+2x+1}{x^3-15x}$
   Solution:
   $$\lim_{x\to\infty} \frac{x^3+2x+1}{x^3-15x} = \lim_{x\to\infty} \frac{1+\frac{2}{x^2}+\frac{1}{x^3}}{1-\frac{15}{x^2}} = \frac{1}{1} = 1$$

   (b) $\lim_{x\to 2-} \frac{x^2-2x-3}{x^2-5x+6}$
   Solution:
   As $x \to 2-$,
   $$x^2-2x-3 \to 2^2-4-3 = -3$$
   $$x^2-5x+6 \to 4-10+6 = 0$$
   Thus, we have limit of the form $\frac{\text{negative number}}{0}$. This gives either ∞ or $-\infty$, depending on whether the denominator is positive or negative. To find this, we factor $x^2-5x+6 = (x-2)(x-3)$, so as $x \to 2-$, $x-3 \to -1$ and $x-2 \to 0-$; thus, $x^2-5x+6 \to 0+$. So we have limit of the form $\frac{\text{negative number}}{0+} = -\infty$

   (c) $\lim_{x\to 3+} \frac{x^2-2x-3}{x^2-5x+6}$
   Solution: Straightforward computation gives $\frac{0}{0}$ which is useless. We try to factor:
   $$\frac{x^2-2x-3}{x^2-5x+6} = \frac{(x+1)(x-3)}{(x-2)(x-3)} = \frac{x+1}{x-2}$$
   Thus,
   $$\lim_{x\to 3+} \frac{x^2-2x-3}{x^2-5x+6} = \lim_{x\to 3+} \frac{x+1}{x-2} = \frac{4}{2} = 2$$

   (d) $\lim_{x\to\infty} \frac{1}{e^{(x^2)}+1}$
   Solution: As $x \to \infty$, $x^2 \to \infty$, so $e^{x^2} \to \infty$. Thus, $\frac{1}{e^{x^2}+1} \to 0$.

2. Calculate derivatives of the following functions:

   (a) $3(x+\sqrt{x})$
   Solution:
   $$(3(x+\sqrt{x}))' = 3(x+x^{1/2})' = 3(1+\tfrac{1}{2}x^{-1/2})$$

   (b) $xe^x - 17x$
   Solution:
   $$(xe^x-17x)' = (xe^x)' - 17 = x'e^x + x(e^x)' - 17$$
   $$= e^x + x(e^x) - 17 = e^x(1+x) - 17$$

(c) $\frac{2x}{x+1}$

Solution: By quotient rule,

$$\left(\frac{2x}{x+1}\right)' = \frac{(2x)'(x+1)-(x+1)'2x}{(x+1)^2} = \frac{2(x+1)-2x}{(x+1)^2} = \frac{2}{(x+1)^2}$$

(d) $\frac{1+\sqrt{x}}{1-\sqrt{x}}$

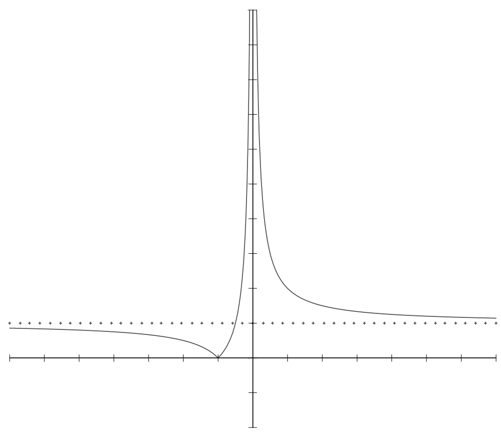
Solution: By quotient rule,

$$\begin{aligned}\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)' &= \frac{(1+\sqrt{x})'(1-\sqrt{x})-(1+\sqrt{x})(1-\sqrt{x})'}{(1-\sqrt{x})^2} \\ &= \frac{\frac{1}{2\sqrt{x}}(1-\sqrt{x})-(1+\sqrt{x})\frac{1}{2\sqrt{x}}}{(1-\sqrt{x})^2} = \frac{(1-\sqrt{x})-(1+\sqrt{x})}{2\sqrt{x}(1-\sqrt{x})^2} \\ &= \frac{-2\sqrt{x}}{2\sqrt{x}(1-\sqrt{x})^2} = -\frac{1}{(1-\sqrt{x})^2}\end{aligned}$$

3. (*15 points*) Let $f(x) = \left|1 + \frac{1}{x}\right|$.

(a) Sketch the graph of f and identify the asymptotes.

Solution: The graph is obtained by plotting the graph of $1 + \frac{1}{x}$ and then flipping the part of the graph which is below the x axis:



Since $\lim_{x\to\infty} \left|1 + \frac{1}{x}\right| = \left|\lim_{x\to\infty}(1 + \frac{1}{x})\right| = |1| = 1$, and same for $x \to -\infty$, the horizontal asymptote is $y = 1$.

Vertical asymptote is $x = 0$: everywhere else this function is continuous (see part b), and as $x \to 0$, $1 + \frac{1}{x} \to \pm\infty$, so $\left|1 + \frac{1}{x}\right| \to \infty$.

(b) Find all values of x for which f is not continuous.

Solution: f is continuous everywhere where it is defined, i.e. everywhere except $x = 0$. So it is not continuos when $\boxed{x = 0}$.

(c) Find all values of x for which f is not differentiable (you do not have to calculate the derivative).

Solution: First of all, f is not differentiable for $x = 0$, since f is not defined at this point. Next, looking at the graph we see that it has a corner at $x = -1$ (when $1 + \frac{1}{x} = 0$); thus, at this point f is not differentiable. It can be verified by direct computation of derivative as limit:

$$
\begin{aligned}
f'(-1) &= \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} \\
&= \lim_{h\to 0}\left(\left|1+\tfrac{1}{-1+h}\right| - \left|1+\tfrac{1}{-1}\right|\right)h^{-1} \\
&= \lim_{h\to 0}\left(\left|\tfrac{1+(-1)+h}{-1+h}\right| - 0\right)h^{-1} = \lim_{h\to 0}\left|\tfrac{h}{-1+h}\right|h^{-1}
\end{aligned}
$$

For $h>0$ and small, $\frac{h}{-1+h}<0$, so $\left|\frac{h}{-1+h}\right| = \frac{-h}{-1+h}$ and

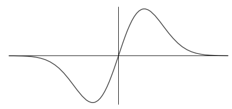
$$\lim_{h\to 0+}\left|\tfrac{h}{-1+h}\right|h^{-1} = \lim_{h\to 0+}\tfrac{-h}{-1+h}h^{-1} = \tfrac{-1}{-1+0} = 1$$

Similarly, for $h<0$

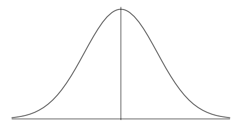
$$\lim_{h\to 0+}\left|\tfrac{h}{-1+h}\right|h^{-1} = \lim_{h\to 0+}\tfrac{h}{-1+h}h^{-1} = \tfrac{1}{-1+0} = -1$$

Since these limits do not match, $f'(-1)$ does not exist.
Thus, the answer is $\boxed{x=0,-1}$.

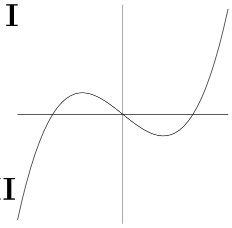
4. Match the graphs of functions **I–IV** below with the graphs of their derivatives **A–D**. (Justification is not required.)



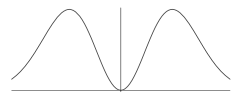
I



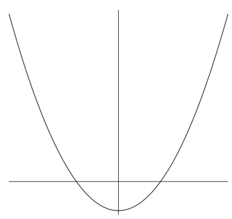
II



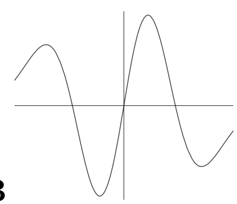
III



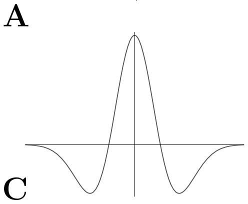
IV



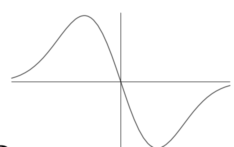
A



B



C



D

Solution: Looking at graph I, we see that $f_I'(0) > 0$; the only function among A–D which satisfies this is C, so

I–C

Similarly, by looking at points where f increases/decreases/has derivative zero, we get the remaining ones:

II–D

III–A

IV–B

5. Let $f(x) = x^3 - 3x^2 - 9x + 7$.

(a) Calculate f'

Solution: $\boxed{f'(x) = 3x^2 - 6x - 9}$

(b) Calculate f''

Solution: $\boxed{f''(x) = 6x - 6}$

(c) On which intervals does f increase? decrease?

Solution: f increases when $f'(x) > 0$, which gives

$$\begin{gathered} 3x^2 - 6x - 9 > 0 \\ x^2 - 2x - 3 > 0 \\ (x-3)(x+1) > 0 \\ \boxed{x < -1 \text{ or } x > 3} \end{gathered}$$

Similarly, f decreases when $f'(x) < 0$, which gives $\boxed{x \in (-1, 3)}$.

(d) On which intervals is f concave up?

Solution: f is concave up when $f''(x) > 0$, i.e. when $6x - 6 > 0, 6(x-1) > 0, \boxed{x > 1}$.

6. Find all tangent lines to the graph of $f(x) = 1/x$ which have slope $m = -1/4$; write equations of each of these tangent lines.

Solution: The slope of tangent line to the graph of f at $x = a$ is $f'(a)$. Thus, to find for which a we have slope $-1/4$, we have to solve

$$\begin{aligned} f'(a) &= -1/4 \\ -\frac{1}{a^2} &= -1/4 \\ a^2 &= 4 \\ a &= \pm 2 \end{aligned}$$

Now let us find the equation of the tangent line for each of these a, using general formula: $y = f(a) + f'(a)(x - a)$.

For $a = 2$:

$$y = \frac{1}{2} + \left(-\frac{1}{4}\right) \cdot (x - 2) = \frac{1}{2} + \frac{1}{2} - \frac{x}{4} = \boxed{1 - \frac{x}{4}}$$

For $a = -2$:

$$y = \frac{1}{-2} + \left(-\frac{1}{4}\right) \cdot (x + 2) = -\frac{1}{2} - \frac{1}{2} - \frac{x}{4} = \boxed{-1 - \frac{x}{4}}$$

7. (a) Write the linear approximation for the function $g(x) = \frac{1}{e^x+1}$ near $x = 0$.

   Solution: General formula is

   $$g(a + h) \approx g(a) + g'(a)h$$

   (or, equivalently, denoting $x = a + h$, $g(x) = g(a) + g'(a)(x - a)$).
   In this case $a = 0$, $g(a) = \frac{1}{e^0+1} = \frac{1}{2}$, and

   $$g'(x) = \frac{-(e^x + 1)'}{(e^x + 1)^2} = \frac{-e^x}{(e^x + 1)^2}$$

   so $g'(0) = \frac{-1}{2^2} = -\frac{1}{4}$. This gives

   $$\boxed{g(h) \approx \tfrac{1}{2} - \tfrac{1}{4}h}$$

   (b) Use the linear approximation you found in the previous part to estimate $\frac{1}{e^{0.01}+1}$.

   Solution:

   $$g(0.01) \approx \tfrac{1}{2} - \tfrac{0.01}{4} = 0.5 - 0.0025 = 0.4975$$