Practice Midterm 1 Solutions

MAT 125, Spring 2006

Name:	**ID #:**	**Section:**

Please answer each question in the space provided. Show your work whenever possible. Unless otherwise marked, **answers without justification will get little or no partial credit**. Cross out anything the grader should ignore and circle or box the final answer.

(1) Calculate the following limits

(a) $\lim_{x \to 2} 3x^2 + x - 2$

Solution: Since $f(x) = 3x^2 + x - 2$ is continuous, $\lim_{x \to 2} f(x) = f(2) = 12$.

(b) $\lim_{y \to -3} |y + 3|$

Solution: For $y > -3$, $y + 3 > 0$ so $|y + 3| = y + 3$. Thus, $\lim_{y \to (-3)+} |y+3| = \lim_{y \to (-3)+} y+3 = (-3)+3 = 0$. Similarly, $\lim_{y \to (-3)-} |y + 3| = \lim_{y \to (-3)-} -(y + 3) = -((-3) + 3) = 0$. Since one-sided limits coincide, $\lim_{y \to -3} |y + 3| = 0$.

(c) $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$

Solution: In this case, plugging in $x = 2$ is impossible because in this case both the numerator and denominator are zero. Instead, we can factor the numerator, using the formula for roots of quadratic equation, to get

$$\begin{aligned} \lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} &= \lim_{x \to 2} \frac{(x + 3)(x - 2)}{x - 2} \\ &= \lim_{x \to 2} (x + 3) = 5 \end{aligned}$$

(d) $\lim_{q \to 2} \frac{2q^2+5}{\sqrt{q+2}}$

Solution: This function is continuous, thus $\lim_{q \to 2} f(q) = f(2) = (2 \cdot 4 + 5)/\sqrt{4} = 13/2 = 6.5$

(e) $\lim_{t \to 3} \frac{\sqrt{t}-\sqrt{3}}{t-3}$

Solution: Again, both numerator and denominator have limit zero, so we can not use the quotient rule; instead, we can multiply both numerator and denominator by $\sqrt{t} + \sqrt{3}$:

$$\begin{aligned} \lim_{t \to 3} \frac{\sqrt{t} - \sqrt{3}}{t - 3} &= \lim_{t \to 3} \frac{(\sqrt{t} - \sqrt{3})(\sqrt{t} + \sqrt{3})}{(t - 3)(\sqrt{t} + \sqrt{3})} \\ &= \lim_{t \to 3} \frac{t - 3}{(t - 3)(\sqrt{t} + \sqrt{3})} = \lim_{t \to 3} \frac{1}{\sqrt{t} + \sqrt{3}} = \frac{1}{2\sqrt{3}} \end{aligned}$$

(f) $\lim_{s \to 0} s^2 \cos\left(s + \frac{1}{s}\right)$

Solution: Denote $f(s) = s^2 \cos\left(s + \frac{1}{s}\right)$.
Since $-1 \le \cos\left(s + \frac{1}{s}\right) \le 1$, we have $-s^2 \le f(s) \le s^2$. Since $\lim_{s \to 0} s^2 = \lim_{s \to 0}(-s^2) = 0$, by squeeze theorem we have $\lim_{s \to 0} f(s) = 0$.

(2) Calculate

$$\lim_{x\to(\pi/2)-}\frac{1+\tan x}{1-\tan x}$$

Solution: First, let us see what happens with $t=\tan x$ as $x\to(\pi/2)-$. By definition, $\tan x=\sin x/\cos x$. As $x\to(\pi/2)-$, we know that $\sin x\to 1$ and $\cos x\to 0$. Thus, we can't use quotient rule to compute the limit of $\tan x$ (in fact, this limit does not exist).

However, we can rewrite the expression as follows:

$$\begin{aligned}
&\lim_{x\to(\pi/2)-}\frac{1+\tan x}{1-\tan x}=\lim_{x\to(\pi/2)-}\frac{1+\frac{\sin x}{\cos x}}{1-\frac{\sin x}{\cos x}}\\
&=\lim_{x\to(\pi/2)-}\frac{\cos x+\sin x}{\cos x-\sin x}
\end{aligned}$$

(by multiplying both numerator and denominator by $\cos x$). Now we can use the quotient rule: since $\sin,\cos$ are continuous, as $x\to(\pi/2)-$, we have

$$\begin{aligned}
\sin x&\to\sin(\pi/2)=1\\
\cos x&\to\cos(\pi/2)=0
\end{aligned}$$

so

$$\lim_{x\to(\pi/2)-}\frac{\cos x+\sin x}{\cos x-\sin x}=\frac{0+1}{0-1}=-1$$

Note: this is a difficult problem — I probably wouldn't include such a problem in the actual exam.

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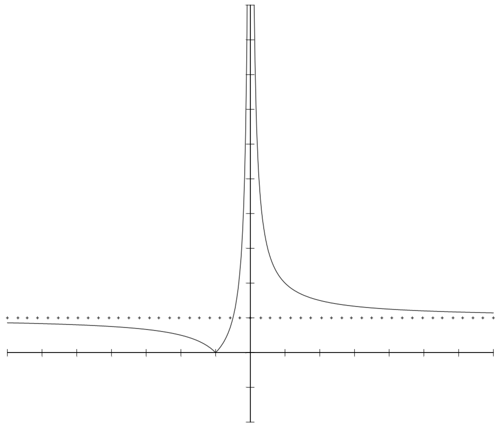
(3) Let $f(x)=\left|1+\frac{1}{x}\right|$.
  (a) Sketch the graph of f and identify the asymptotes.
  (b) Find all values of x for which f is not continuous.

Solution: The graph is shown below; it is obtained from the graph of $y=\frac{1}{x}$ by shifting it one unit up (this gives graph of $y=1+\frac{1}{x}$) and then reflecting the part of the graph below the x-axis.



The asymptotes are: horizontal: $y = 1$ and vertical: $x = 0$.

Since the functions $1/x$ and $|x|$ are continuous, $f(x)$ is also continuous. Thus, the only discontinuity points are when the function is not defined, that is, at $x = 0$.

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(4) Find

$$\lim_{x \to 1} e^{(x^2 - x - 1)}.$$

Between which two integers (whole numbers) does the answer lie?

Solution: Since this function is continuous, $\lim_{x\to 1} e^{(x^2-x-1)} = e^{1^2-1-1} = e^{-1} = 1/e$. Since $e \approx 2.7\ldots$, $0 < 1/e < 1$.

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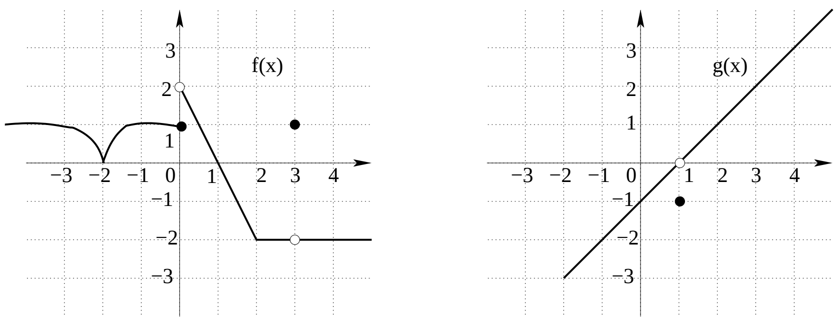
(5) Use the graphs of $f(x)$ and $g(x)$ below to compute each of the following quantities. If the quantity is not defined, say so.



$f(0)$ $\quad\quad \lim_{x\to 0+} f(x) \quad\quad \lim_{x\to 0-} f(x) \quad\quad \lim_{x\to 0} f(x)$

$\lim_{x\to 1} g(x) \quad\quad \lim_{x\to 1} f(x)-g(x) \quad\quad \lim_{x\to 3}(2f(x)-f(3))$

Solution: $f(0)=1$; $\lim_{x\to 0+} f(x)=2$; $\lim_{x\to 0-} f(x)=1$;
$\lim_{x\to 0} f(x)$ does not exist, since the one-sided limits are different;
$\lim_{x\to 1} g(x)=0$;
$\lim_{x\to 1} f(x)-g(x)=\lim_{x\to 1} f(x)-\lim_{x\to 1} g(x)=0-0=0$;
$\lim_{x\to 3}(2f(x)-f(3))=\left(2\lim_{x\to 3} f(x)\right)-f(3)=2(-2)-1=-5.$

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(6) Consider the function

$$f(t)=\begin{cases}\frac{t}{t-1} & t\geq 0\\ t+1 & t<0\end{cases}$$

(a) At which points is this function continuous?
(b) Find the left and right limits, if they exist, at $t=0$.

Solution:
For $t<0$, this function is given by $f(t)=t+1$, so it is continuos.
For $t>0$, this function is given by $f(t)=\frac{t}{t-1}$, so it is continuos wherever defined. Thus, it is continuous at all points where denominator is non-zero, i.e. $t\neq 1$
It remians to consider the point $t=0$. At this point, function is defined by different formulas on two sides of this point. To check whether it is continuous, we compute the one-sided limits.

$$\lim_{t\to 0+} f(t)=\lim_{t\to 0+}\frac{t}{t-1}=\frac{0}{0-1}=0$$
$$\lim_{t\to 0-} f(t)=\lim_{t\to 0-}(t+1)=0+1=1$$

Since these limtis are not equal, limit $\lim_{t\to 0} f(t)$ does not exist. So $f(t)$ is not continuous at 0.
Thus, $f(t)$ is continuous everywhere except $t=0$, $t=1$.

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(7) Explain, without using a graphing calculator, why the equation $x^5-3x+1=0$ must have a solution with $0<x<1$.

Solution: $f(0)=1$ and $f(1)=1-3+1=-1$. Since $f(x)$ is continuous, it follows from the Intermediate Value Theorem that $f(x)=0$ for some $0<x<1$.
