Practice Final Exam Solutions
MAT 125

May 8, 2006

1. Compute the following limits. Please distinguish between "$\lim f(x) = \infty$", "$\lim f(x) = -\infty$" and "limit does not exist even allowing for infinite values".

   (a) $\lim_{x\to -1} x^2 + x - 1$

   Solution: Since any polynomial is continuous, $\lim_{x\to-1} x^2+x-1 = (-1)^2+(-1)-1 = 1-1-1 = -1$.

   (b) $\displaystyle\lim_{x\to -3} \frac{x^2+2x-3}{x+3}$

   Solution: We can't just substitute $x=-3$, as it will give denominator zero. The numerator also becomes zero. However, factoring the numerator works:

   $$\lim_{x\to-3}\frac{x^2+2x-3}{x+3} = \lim_{x\to-3}\frac{(x-1)(x+3)}{x+3} = \lim_{x\to-3}(x-1) = -4$$

   Note: this problem can also be solved by using L'Hospital's rule.

   (c) $\displaystyle\lim_{t\to 0}\frac{\sqrt{2-t}-\sqrt{2}}{t}$

   Solution: Again, substituting $t=0$ gives meaningless expression $0/0$; however, multiplying the numerator by the conjugate expression $\sqrt{2-t}+\sqrt{2}$ works:

   $$\begin{aligned}\lim_{t\to0}\frac{\sqrt{2-t}-\sqrt{2}}{t} &= \lim_{t\to0}\frac{(\sqrt{2-t}-\sqrt{2})(\sqrt{2-t}+\sqrt{2})}{t(\sqrt{2-t}+\sqrt{2})}\\ &= \lim_{t\to0}\frac{(2-t)-2}{t(\sqrt{2-t}+\sqrt{2})} = \lim_{t\to0}\frac{-t}{t(\sqrt{2-t}+\sqrt{2})}\\ &= \lim_{t\to0}\frac{-1}{(\sqrt{2-t}+\sqrt{2})} = \frac{-1}{2\sqrt{2}}\end{aligned}$$

   Note: this problem can also be solved by using L'Hospital's rule.

   (d) $\displaystyle\lim_{x\to0} x\sin\pi\left(x^2+\frac{1}{x^2}\right)$

   Solution: Since $-1 < \sin\pi(x^2+\frac{1}{x^2}) < 1$, we see that

   $$-|x| \le x\sin\pi\left(x^2+\frac{1}{x^2}\right) \le |x|$$

   Since $\lim_{x\to0}|x| = \lim_{x\to0} -|x| = 0$, by Squeeze Theorem, $\lim_{x\to0} x\sin\pi\left(x^2+\frac{1}{x^2}\right) = 0$

(e) $\lim_{x \to \infty} \frac{x^3 + 2x + 1}{x^3 - 2x + 1}$

Solution:

$$\lim_{x \to \infty} \frac{x^3 + 2x + 1}{x^3 - 2x + 1} = \lim_{x \to \infty} \frac{1 + \frac{2}{x^2} + \frac{1}{x^3}}{1 - \frac{2}{x^2} + \frac{1}{x^3}} = \frac{1}{1} = 1$$

(f) $\lim_{x \to \pi/2} \frac{\cos x}{2x - \pi}$

Solution: Direct substituiton $x = \pi/2$ gives $\frac{0}{0}$ which is meaningless. Thus, we can use L'Hospital's rule, which gives

$$\lim_{x \to \pi/2} \frac{\cos x}{2x - \pi} = \lim_{x \to \pi/2} \frac{-\sin x}{2} = \frac{-1}{2}$$

2. Compute the derivatives of the following functions

(a) $f(x) = x^3 - 12x^2 + x + 2\pi$
Solution: $f'(x) = 3x^2 - 24x + 1$

(b) $f(x) = (2x+1)\sin(x)$
Solution: $f'(x) = (2x+1)'\sin(x) + (2x+1)(\sin(x))' = 2\sin(x) + (2x+1)\cos(x)$

(c) $g(s) = \sqrt{1+e^{2s}}$
Solution: By chain rule, using $u = 1 + e^{2s}$:

$$\frac{dg}{ds} = \frac{dg}{du}\frac{du}{ds} = \frac{d(\sqrt{u})}{du}\frac{d(1+e^{2s})}{ds} = \frac{1}{2\sqrt{u}}2e^{2s} = \frac{e^{2s}}{\sqrt{1+e^{2s}}}$$

(d) $h(t) = \frac{1+e^t}{1-e^t}$
Solution: By quotient rule,

$$\begin{aligned} h'(t) &= \frac{(1+e^t)'(1-e^t) - (1+e^t)(1-e^t)'}{(1-e^t)^2} = \frac{e^t(1-e^t) - (1+e^t)(-e^t)}{(1-e^t)^2} \\ &= \frac{e^t - (e^t)^2 + e^t + (e^t)^2}{(1-e^t)^2} = \frac{2e^t}{(1-e^t)^2} \end{aligned}$$

(e) $f(x) = (2x+2)^{10}$
Solution: By chain rule,

$$f'(x) = 10(2x+2)^2(2x+2)' = 20(2x+2)^9$$

(f) $g(x) = x^{(\sin x)}$
Solution: We will use logarithmic derivative:

$$\begin{aligned} (\ln g(x))' &= (\ln x^{(\sin x)})' = ((\sin x)\ln x)' = (\sin x)'\ln x + (\sin x)(\ln x)' \\ &= (\cos x)\ln x + (\sin x)\frac{1}{x} \end{aligned}$$

Thus, using $(\ln g)' = \frac{g'}{g}$, we get

$$g'(x) = g(x)(\ln g(x))' = x^{(\sin x)}[(\cos x)\ln x + \frac{\sin x}{x}]$$

3. Let $f(x) = xe^{(-x^2)}$.

(a) Find asymptotes of $f(x)$ (hint: $f(x) = \frac{x}{e^{(x^2)}}$)
Solution: This function is continuous everywhere, so there are no vertical asymptotes. To find horizontal asymptotes, we need to compute $\lim_{x\to\pm\infty} f(x)$. Writing $f(x) = \frac{x}{e^{(x^2)}}$, we see that as $x \to \infty$, both numerator and denominator have limit ∞. Thus, we can not use quotient rule (it would give $\frac{\infty}{\infty}$, which is meaningless); however, we can use L'Hospital's rule:

$$\lim_{x\to\infty} \frac{x}{e^{(x^2)}} = \lim_{x\to\infty} \frac{1}{2xe^{(x^2)}} = 0$$

since $\lim_{x\to\infty} 2xe^{(x^2)} = \infty$. Similar computation gives

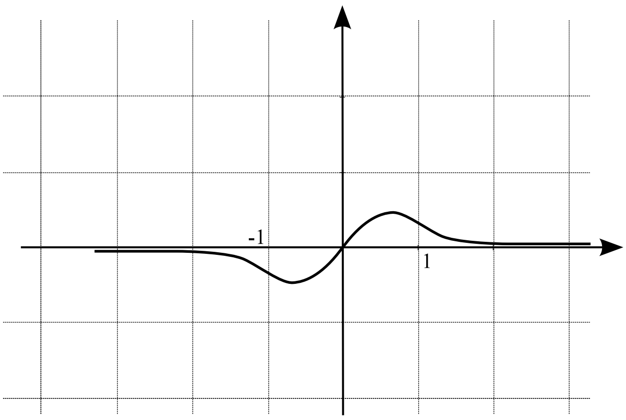
$$\lim_{x\to-\infty} f(x) = 0$$

Thus, the horizontal asymptote is $y = 0$.

(b) Compute the derivative of $f(x)$
Solution: $f'(x) = (x)'e^{(-x^2)} + x\left(e^{(-x^2)}\right)' = e^{(-x^2)} + x\left(-2xe^{(-x^2)}\right) = (1-2x^2)e^{(-x^2)}$

(c) On which intervals is $f(x)$ increasing? decreasing?
Solution: $f(x)$ is increasing when $f'(x) > 0$, i.e. $(1 - 2x^2)e^{(-x^2)} > 0$. Since $e^{(-x^2)} > 0$, it is equivalent to $1 - 2x^2 > 0$, i.e. $1 < 2x^2$, or $x^2 < 1/2$. Solutions of this last inequality are $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. So $f(x)$ is increasing on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.
Same argument shows that $f(x)$ decreasing on $(-\infty, -\frac{1}{\sqrt{2}})$ and on $(\frac{1}{\sqrt{2}}, \infty)$

(d) Sketch a graph of $f(x)$ using the results of the previous parts and the fact that $f(0) = 0$.



4. Let $f(x) = \frac{1}{\sqrt{1+x}}$. Write the linear approximation for $f(x)$ near $x = 0$ and use it to estimate $f(0.1)$.

Solution: General formula is $f(x) \approx f(a) + f'(a)(x - a)$. In this case, $a = 0$, $f(a) = \frac{1}{\sqrt{1+0}} = 1$. To find $f'(0)$, compute $f'(x)$ and then substitute $x = 0$:

$$f(x) = (1 + x)^{-1/2}$$
$$f'(x) = -\frac{1}{2}(1 + x)^{-3/2}$$

Thus, $f'(0) = -\frac{1}{2}$. Therefore,

$$f(x) \approx 1 - \frac{1}{2}(x - 0) = 1 - \frac{x}{2}$$

Substituting $x = 0.1$, we get

$$f(0.1) \approx 1 - \frac{0.1}{2} = 1 - 0.05 = 0.95$$

5. Let $f(x) = -2x^3 + 6x^2 - 3$.

   (a) Compute f', f''.
   Solution:

$$f'(x) = -6x^2 + 12x$$
$$f''(x) = -12x + 12$$

   (b) On which intervals is $f(x)$ increasing/decreasing?
   Solution: $f(x)$ is increasing when $f'(x) > 0$:

$$-6x^2 + 12x > 0$$
$$-6x(x-2) > 0$$

   Since the graph of $-6x^2 + 12x$ is a parabola witht he branches going down, this expression is positive between the roots, i.e. for $0 < x < 2$. Thus, $f'(x) > 0$ on the interval $(0, 2)$, and $f(x)$ is increasing on $(0, 2)$.
   Similar argument shows that $f'(x) < 0$ on $(-\infty, 0)$ and on $(2, \infty)$; thus, on these intervals $f(x)$ is decreasing.

   (c) On which intervals is $f(x)$ concave up/down?
   Solution: $f(x)$ is concave up when $f''(x) > 0$, i.e. $-12x + 12 > 0$, or $1 - x > 0$, $x < 1$. Threfore, $f(x)$ is concave up on $(-\infty, 1)$ and concave down on $(1, \infty)$.

   (d) Find all critical points of $f(x)$. Which of them are local maximums? local minimums? neither? Justify your answer.
   Solution: Critical points are where $f''(x) = 0$, i.e.

$$-6x^2 + 12x = 0$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$

   So the critical points are $x = 0$, $x = 2$.
   Since $f(x)$ is decreasing for $x < 0$ and increasing for $0 < x < 2$, by first derivative test, $x = 0$ is a local minimum. Similarly, since $f(x)$ is increasing for $0 < x < 2$ and decreasing for $x > 2$, $x = 2$ is a local maximum.

6. It is known that the polynomial $f(x) = x^3 - x - 1$ has a unique real root. Between which two whole numbers does this root lie? Justify your answer.

   Solution: Computing the values of $f(x)$ for several whole values of x, we get

   $f(-2) = -7$

   $f(-1) = -1$

   $f(0) = -1$

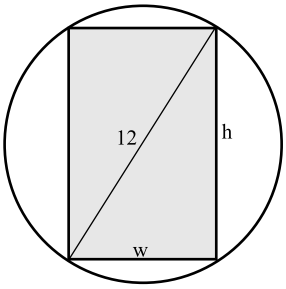
$f(1) = -1$

$f(2) = 5$

Thus, we see that $f(x)$ changes sign on the intrerval $[1, 2]$. SInce any polynomial is continuous, by Intermediate Value Theorem $f(x)$ must have a root somewhere on this interval. Thus, the root is between 1 and 2.

7. It is known that for a rectangular beam of fixed length, its strength is proportional to $w \cdot h^2$, where w is the width and h is the height of the beam's cross-section.

   Find the dimensions of the strongest beam that can be cut from a 12" diameter log (thus, the cross-section must be a rectangle with diagonal 12").



Solution: The dimensions of the beam are width w and height h. They must satisfy the conditions $h \geq 0$, $w \geq 0$. In addition, since the diagonal of the cross-section must be 12 inches, Pythagorean theorem gives $h^2 + w^2 = 12^2 = 144$. Thus, we need to find the maximum of the function wh^2, where h, w are real numbers subject to the above conditions.

Let us rewrite everything in terms of w. Then $h = \sqrt{144 - w^2}$; restrictions $h \geq 0$, $w \geq 0$ give $0 \leq w \leq 12$, and the strength is given by

$$s(w) = w(\sqrt{144 - w^2})^2 = w(144 - w^2) = -w^3 + 144w$$

So we need to find the maximum of this function on the interval $[0, 12]$.

$f'(w) = -3w^2 + 144$, so critical points are when

$$\begin{aligned} &-3w^2 + 144 = 0 \\ &144 = 3w^2 \\ &w^2 = 48 \\ &w = \pm\sqrt{48} = \pm\sqrt{16 \cdot 3} = \pm 4\sqrt{3} \end{aligned}$$

Thus, on $[0, 12]$ there is a unique critical point, $w = 4\sqrt{3}$.

To find the maximum, we compare the values of the function at the critical point and the endpoints:

$$\begin{aligned} &f(0) = 0(144 - 0^2) = 0 \\ &f(12) = 12(144 - 12^2) = 0 \\ &f(4\sqrt{3}) = 4\sqrt{3}(144 - (4\sqrt{3})^2) = 4\sqrt{3}(144 - 48) = 4\sqrt{3} \cdot 96 \end{aligned}$$

Clearly, the largest value is $f(4\sqrt{3})$; thus, this is the maximum. So the best width is $4\sqrt{3}$, and the corresponding height is $h = \sqrt{144 - w^2} = \sqrt{96} = 4\sqrt{6}$.

8. The curve defined by the equation

$$y^2(y^2 - 4) = x^2(x^2 - 5)$$

is known as the "devil's curve". Use implicit differentiation to find the equation of the tangent line to the curve at the point $(0; -2)$.

Solution: Rewriting the equation in the form

$$y^4 - 4y^2 = x^4 - 5x^2$$

and taking derivative of both sides, we get $y'(4y^3 - 8y) = 4x^3 - 10x$, so

$$y' = \frac{4x^3 - 10x}{4y^3 - 8y}$$

Subsituting $x = 0, y = -2$, we get $y' = 0$, so the tangent line is horizaontal and the equation of the tangent line is $y = -2$.