1. Compute the following limits. Please distinguish between “\( \lim f(x) = \infty \)”, “\( \lim f(x) = -\infty \)” and “limit does not exist even allowing for infinite values”.

(a) \( \lim_{x \to -1} x^2 + x - 1 \)

Solution: Since any polynomial is continuous, \( \lim_{x \to -1} x^2 + x - 1 = (-1)^2 + (-1) - 1 = 1 - 1 - 1 = -1 \).

(b) \( \lim_{x \to -3} \frac{x^2 + 2x - 3}{x + 3} \)

Solution: We can’t just substitute \( x = -3 \), as it will give denominator zero. The numerator also becomes zero. However, factoring the numerator works:

\[
\lim_{x \to -3} \frac{x^2 + 2x - 3}{x + 3} = \lim_{x \to -3} \frac{(x - 1)(x + 3)}{x + 3} = \lim_{x \to -3} (x - 1) = -4
\]

Note: this problem can also be solved by using L’Hospital’s rule.

(c) \( \lim_{t \to 0} \frac{\sqrt{2} - t - \sqrt{2}}{t} \)

Solution: Again, substituting \( t = 0 \) gives meaningless expression \( 0/0 \); however, multiplying the numerator by the conjugate expression \( \sqrt{2} - t + \sqrt{2} \) works:

\[
\lim_{t \to 0} \frac{\sqrt{2} - t - \sqrt{2}}{t} = \lim_{t \to 0} \frac{(\sqrt{2} - t - \sqrt{2})(\sqrt{2} - t + \sqrt{2})}{t(\sqrt{2} - t + \sqrt{2})} = \lim_{t \to 0} \frac{(2 - t) - 2}{t(\sqrt{2} - t + \sqrt{2})} = \lim_{t \to 0} \frac{2 - t}{t(\sqrt{2} - t + \sqrt{2})} = \lim_{t \to 0} \frac{-1}{\frac{2\sqrt{2}}{2\sqrt{2}}} = -1
\]

Note: this problem can also be solved by using L’Hospital’s rule.

(d) \( \lim_{x \to 0} x \sin \pi \left( x^2 + \frac{1}{x^2} \right) \)

Solution: Since \(-1 < \sin \pi \left( x^2 + \frac{1}{x^2} \right) < 1\), we see that

\[
-|x| \leq x \sin \pi \left( x^2 + \frac{1}{x^2} \right) \leq |x|
\]

Since \( \lim_{x \to 0} |x| = \lim_{x \to 0} -|x| = 0 \), by Squeeze Theorem, \( \lim_{x \to 0} x \sin \pi \left( x^2 + \frac{1}{x^2} \right) = 0 \).
(e) \( \lim_{x \to \infty} \frac{x^3 + 2x + 1}{x^3 - 2x + 1} \)

Solution:

\[
\lim_{x \to \infty} \frac{x^3 + 2x + 1}{x^3 - 2x + 1} = \lim_{x \to \infty} \frac{1 + \frac{2}{x^2} + \frac{1}{x^3}}{1 - \frac{2}{x^2} + \frac{1}{x^3}} = 1 = 1
\]

(f) \( \lim_{x \to \pi/2} \frac{\cos x}{2x - \pi} \)

Solution: Direct substituiton \( x = \pi/2 \) gives \( \frac{0}{0} \) which is meaningless. Thus, we can use L’Hospital’s rule, which gives

\[
\lim_{x \to \pi/2} \frac{\cos x}{2x - \pi} = \lim_{x \to \pi/2} \frac{-\sin x}{2} = \frac{-1}{2}
\]
2. Compute the derivatives of the following functions

(a) \( f(x) = x^3 - 12x^2 + x + 2\pi \)

**Solution:** \( f'(x) = 3x^2 - 24x + 1 \)

(b) \( f(x) = (2x + 1)\sin(x) \)

**Solution:** \( f'(x) = (2x + 1)\sin(x) + (2x + 1)(\sin(x))' = 2\sin(x) + (2x + 1)\cos(x) \)

(c) \( g(s) = \sqrt{1 + e^{2s}} \)

**Solution:** By chain rule, using \( u = 1 + e^{2s} \):
\[
\frac{dg}{ds} = \frac{dg}{du} \frac{du}{ds} = \frac{d(\sqrt{u})}{du} \frac{d(1 + e^{2s})}{ds} = \frac{1}{2\sqrt{u}} 2e^{2s} = \frac{e^{2s}}{\sqrt{1 + e^{2s}}} 
\]

(d) \( h(t) = \frac{1+e^t}{1-e^t} \)

**Solution:** By quotient rule,
\[
h'(t) = \frac{(1+e^t)'(1-e^t) - (1+e^t)(1-e^t)'}{(1-e^t)^2} = \frac{e^t(1-e^t) - (1+e^t)(-e^t)}{(1-e^t)^2} \\
= \frac{e^t - e^t + e^t + (e^t)^2}{(1-e^t)^2} = \frac{2e^t}{(1-e^t)^2} 
\]

(e) \( f(x) = (2x + 2)^{10} \)

**Solution:** By chain rule,
\[
f'(x) = 10(2x + 2)^9(2x + 2)' = 20(2x + 2)^9 
\]

(f) \( g(x) = x^{(\sin x)} \)

**Solution:** We will use logarithmic derivative:
\[
(ln g(x))' = (\ln x^{(\sin x)})' = ((\sin x) \ln x)' = (\sin x)' \ln x + (\sin x)(\ln x)' = (\cos x) \ln x + (\sin x) \frac{1}{x} 
\]

Thus, using \( (\ln g)' = \frac{g'}{g} \), we get
\[
g'(x) = g(x)(\ln g(x))' = x^{(\sin x)}[(\cos x) \ln x + \frac{\sin x}{x}] 
\]
3. Let $f(x) = xe^{(-x^2)}$.

(a) Find asymptotes of $f(x)$ (hint: $f(x) = \frac{x}{e^{(x^2)}}$)

Solution: This function is continuous everywhere, so there are no vertical asymptotes. To find horizontal asymptotes, we need to compute $\lim_{x \to \pm \infty} f(x)$. Writing $f(x) = \frac{x}{e^{(x^2)}}$, we see that as $x \to \infty$, both numerator and denominator have limit $\infty$. Thus, we can not use quotient rule (it would give $\frac{\infty}{\infty}$, which is meaningless); however, we can use L'Hospital’s rule:

$$\lim_{x \to \infty} \frac{x}{e^{(x^2)}} = \lim_{x \to \infty} \frac{1}{2xe^{(x^2)}} = 0$$

since $\lim_{x \to \infty} 2xe^{(x^2)} = \infty$. Similar computation gives

$$\lim_{x \to -\infty} f(x) = 0$$

Thus, the horizontal asymptote is $y = 0$.

(b) Compute the derivative of $f(x)$

Solution: $f'(x) = (x)'e^{(-x^2)} + x \left(e^{(-x^2)}\right)' = e^{(-x^2)} + x \left(-2xe^{(-x^2)}\right) = (1-2x^2)e^{(-x^2)}$

(c) On which intervals is $f(x)$ increasing? decreasing?

Solution: $f(x)$ is increasing when $f'(x) > 0$, i.e. $(1-2x^2)e^{(-x^2)} > 0$. Since $e^{(-x^2)} > 0$, it is equivalent to $1-2x^2 > 0$, i.e. $1 < 2x^2$, or $x^2 < 1/2$. Solutions of this last inequality are $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. So $f(x)$ is increasing on $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

Same argument shows that $f(x)$ decreasing on $(-\infty, -\frac{1}{\sqrt{2}})$ and on $(\frac{1}{\sqrt{2}}, \infty)$

(d) Sketch a graph of $f(x)$ using the results of the previous parts and the fact that $f(0) = 0$. 

[Graph of $f(x) = xe^{(-x^2)}$ showing intervals of increase and decrease, and vertical and horizontal asymptotes.]
4. Let \( f(x) = \frac{1}{\sqrt{1+x}} \). Write the linear approximation for \( f(x) \) near \( x = 0 \) and use it to estimate \( f(0.1) \).

Solution: General formula is \( f(x) \approx f(a) + f'(a)(x - a) \). In this case, \( a = 0 \), \( f(a) = \frac{1}{\sqrt{1+0}} = 1 \). To find \( f'(0) \), compute \( f'(x) \) and then substitute \( x = 0 \):

\[
\begin{align*}
f(x) &= (1 + x)^{-1/2} \\
f'(x) &= -\frac{1}{2}(1 + x)^{-3/2}
\end{align*}
\]

Thus, \( f'(0) = -\frac{1}{2} \). Therefore,

\[
f(x) \approx 1 - \frac{1}{2}(x - 0) = 1 - \frac{x}{2}
\]

Substituting \( x = 0.1 \), we get

\[
f(0.1) \approx 1 - \frac{0.1}{2} = 1 - 0.05 = 0.95
\]
5. Let \( f(x) = -2x^3 + 6x^2 - 3 \).

(a) Compute \( f', f'' \).

\[ f'(x) = -6x^2 + 12x \]
\[ f''(x) = -12x + 12 \]

(b) On which intervals is \( f(x) \) increasing/decreasing?

\[ f(x) \] is increasing when \( f'(x) > 0: \]
\[ -6x^2 + 12x > 0 \]
\[ -6x(x - 2) > 0 \]

Since the graph of \(-6x^2 + 12x\) is a parabola with the branches going down, this expression is positive between the roots, i.e. for \( 0 < x < 2 \). Thus, \( f'(x) > 0 \) on the interval \((0, 2)\), and \( f(x) \) is increasing on \((0, 2)\).

Similar argument shows that \( f'(x) < 0 \) on \((-\infty, 0)\) and on \((2, \infty)\); thus, on these intervals \( f(x) \) is decreasing.

(c) On which intervals is \( f(x) \) concave up/down?

\[ f(x) \] is concave up when \( f''(x) > 0 \), i.e. \(-12x + 12 > 0\), or \( 1 - x > 0 \), \( x < 1 \). Therefore, \( f(x) \) is concave up on \((-\infty, 1)\) and concave down on \((1, \infty)\).

(d) Find all critical points of \( f(x) \). Which of them are local maximums? local minimums? neither? Justify your answer.

\[ f''(x) = 0 \], i.e.
\[ -6x^2 + 12x = 0 \]
\[ x^2 - 2x = 0 \]
\[ x(x - 2) = 0 \]

So the critical points are \( x = 0, x = 2 \).

Since \( f(x) \) is decreasing for \( x < 0 \) and increasing for \( 0 < x < 2 \), by first derivative test, \( x = 0 \) is a local minimum. Similarly, since \( f(x) \) is increasing for \( 0 < x < 2 \) and decreasing for \( x > 2 \), \( x = 2 \) is a local maximum.

6. It is known that the polynomial \( f(x) = x^3 - x - 1 \) has a unique real root. Between which two whole numbers does this root lie? Justify your answer.

\[ f(-2) = -7 \]
\[ f(-1) = -1 \]
\[ f(0) = -1 \]
\[ f(1) = -1 \]
\[ f(2) = 5 \]

Thus, we see that \( f(x) \) changes sign on the interval \([1, 2]\). Since any polynomial is continuous, by Intermediate Value Theorem \( f(x) \) must have a root somewhere on this interval. Thus, the root is between 1 and 2.

7. It is known that for a rectangular beam of fixed length, its strength is proportional to \( w \cdot h^2 \), where \( w \) is the width and \( h \) is the height of the beam’s cross-section.

Find the dimensions of the strongest beam that can be cut from a 12” diameter log (thus, the cross-section must be a rectangle with diagonal 12”).

![Diagram of a rectangle with diagonal 12 inches](image)

**Solution:** The dimensions of the beam are width \( w \) and height \( h \). They must satisfy the conditions \( h \geq 0, w \geq 0 \). In addition, since the diagonal of the cross-section must be 12 inches, Pythagorean theorem gives \( h^2 + w^2 = 12^2 = 144 \). Thus, we need to find the maximum of the function \( wh^2 \), where \( h, w \) are real numbers subject to the above conditions.

Let us rewrite everything in terms of \( w \). Then \( h = \sqrt{144 - w^2} \); restrictions \( h \geq 0, w \geq 0 \) give \( 0 \leq w \leq 12 \), and the strength is given by

\[ s(w) = w(\sqrt{144 - w^2})^2 = w(144 - w^2) = -w^3 + 144w \]

So we need to find the maximum of this function on the interval \([0, 12]\).

\[ f'(w) = -3w^2 + 144, \] so critical points are when

\[ -3w^2 + 144 = 0 \]
\[ 144 = 3w^2 \]
\[ w^2 = 48 \]
\[ w = \pm \sqrt{48} = \pm \sqrt{16 \cdot 3} = \pm 4\sqrt{3} \]

Thus, on \([0, 12]\) there is a unique critical point, \( w = 4\sqrt{3} \).

To find the maximum, we compare the values of the function at the critical point and the endpoints:

\[ f(0) = 0(144 - 0^2) = 0 \]
\[ f(12) = 12(144 - 12^2) = 0 \]
\[ f(4\sqrt{3}) = 4\sqrt{3}(144 - (4\sqrt{3})^2) = 4\sqrt{3}(144 - 48) = 4\sqrt{3} \cdot 96 \]
Clearly, the largest value is $f(4\sqrt{3})$; thus, this is the maximum. So the best width is $4\sqrt{3}$, and the corresponding height is $h = \sqrt{144 - w^2} = \sqrt{96} = 4\sqrt{6}$.

8. The curve defined by the equation

$$y^2(y^2 - 4) = x^2(x^2 - 5)$$

is known as the “devil’s curve”. Use implicit differentiation to find the equation of the tangent line to the curve at the point $(0; -2)$.

*Solution:* Rewriting the equation in the form

$$y^4 - 4y^2 = x^4 - 5x^2$$

and taking derivative of both sides, we get $y'(4y^3 - 8y) = 4x^3 - 10x$, so

$$y' = \frac{4x^3 - 10x}{4y^3 - 8y}$$

Substituting $x = 0, y = -2$, we get $y' = 0$, so the tangent line is horizontal and the equation of the tangent line is $y = -2$. 

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