MAT 127: Calculus C, Spring 2015
Course Summary II

Extremely Important: sequences vs. series (do not mix them or their convergence/divergence tests up!!!); what it means for a sequence or series to converge or diverge;

$$\boxed{\boxed{\text{“2 limits”} = \text{“no limit”} = \text{“diverge”}}}$$

systems of 2 autonomous first-order differential equations and phase-plane portraits; Euler's formula

Very Important: convergence/divergence tests for sequences and series; equilibrium/stationary points for systems of 2 autonomous first-order differential equations; general structure of solutions of ODEs (the number of C's)

Important: limit rules for sequences and series; computing limits of convergent sequences and sums of convergent series; sketching graphs of a solution to a system of 2 autonomous first-order differential equations as functions of time from phase trajectory and vice versa; finding solutions linear homogeneous second-order equations with constant coefficients and to initial-value problems involving such equations

F: Finding Solutions of Some Second-Order Differential Equations

The general solution of a *second-order linear homogeneous* equation with constant coefficients

$$y'' + by' + cy = 0, \quad b, c = const, \quad y = y(x), \tag{F1}$$

is determined by the two roots, r_1 and r_2, of the *associated polynomial*

$$r^2 + br + c = 0. \tag{F2}$$

The general solution can be of two or three different forms, depending on whether one is looking for complex or real solutions:

$$\boxed{y'' + by' + cy = 0, \quad y = y(x) \quad \Longrightarrow \quad y(x) = C_1 e^{r_1 x} + C_2 x e^{r_1 x} \qquad \text{if} \quad r_1 = r_2 \quad \Longleftrightarrow \quad b^2 = 4c}$$

$$\boxed{y'' + by' + cy = 0, \quad y = y(x) \quad \Longrightarrow \quad y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} \qquad \text{if} \quad r_1 \neq r_2 \quad \Longleftrightarrow \quad b^2 \neq 4c}$$

If the coefficients b and c are real, the roots r_1 and r_2 of (F2) are either real or complex conjugates of each other. In the latter case, Euler's formula,

$$\boxed{\boxed{e^{i\theta} = \cos\theta + i \sin\theta}}$$

can be used to extract the general real solution from the general complex solution:

$$\boxed{y'' + by' + cy = 0 \quad \Longrightarrow \quad y(x) = C_1 e^{px} \cos qx + C_2 e^{px} \sin qx, \quad p = -\tfrac{1}{2}b, \ q = \tfrac{1}{2}\sqrt{4c - b^2}, \quad \text{if } b^2 < 4c}$$

The numbers p and q are related to the roots r_1 and r_2 by $r_1, r_2 = p \pm iq$.

Here are some consequences of Euler's formula:

$$\mathrm{e}^{-\mathrm{i}\theta} = \cos\theta - \mathfrak{i}\sin\theta, \qquad \cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2}, \qquad \sin\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathfrak{i}}.$$

The *double-angle formulas* follow from Euler's formula:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \qquad \text{and} \qquad \sin 2\theta = 2\cos\theta\cdot\sin\theta,$$

as do the more general formulas:

$$\cos(\alpha\pm\beta) = \cos\alpha\cdot\cos\beta \mp \sin\alpha\cdot\sin\beta \qquad \text{and} \qquad \sin(\alpha\pm\beta) = \sin\alpha\cdot\cos\beta \pm \cos\alpha\cdot\sin\beta.$$

Please derive all these from Euler's formula.

G: Systems of 2 Autonomous First-Order Differential Equations

G.1 A system of 2 autonomous first-order differential equations is a system of the form

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y) \end{cases} \qquad (x,y) = \big(x(t), y(t)\big), \tag{G1}$$

where f and g are some functions of x and y. For example,

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y) = \frac{1}{20}x - \frac{1}{500}xy \\ \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y) = -\frac{1}{10}y + \frac{1}{1000}xy \end{cases} \qquad (x,y) = \big(x(t), y(t)\big). \tag{G2}$$

The systems (G1) and (G2) are called autonomous because they do not involve t explicitly. A solution of such a system is a pair of functions $(x,y) = (x(t), y(t))$ which satisfy both equations at the same time; neither $x = x(t)$ nor $y = y(t)$ separately is a solution (of the system). In order to check that a given pair of functions solves a system, simply compute LHS and RHS of the first equation for the two given functions and check that they are equal, and then compute LHS and RHS of the second equation for the two given functions and check that they are equal. This is usually not difficult; actually finding such pairs of functions is difficult.

G.2 The first step in analyzing the system (G1) is to find the constant solutions or the equilibrium points of (G1). These are the points (x_i, y_i) in the xy-plane such that each constant function $(x(t), y(t)) = (x_i, y_i)$ is a solution of (G1). The physical interpretation of this is that if the system starts at an equilibrium point, it stays there forever. In mathematical terms, this means that if the initial value (x_0, y_0) of a solution $(x,y) = (x(t), y(t))$ to (G1) is an equilibrium point, then $(x(t), y(t)) = (x_0, y_0)$ for all t. Since the derivative of a constant function is zero, the constant function $(x(t), y(t)) = (x_i, y_i)$ is a solution of (G1) if and only if $f(x_i, y_i) = (0,0)$ and $g(x_i, y_i) = (0,0)$. Thus,

$$\boxed{(x_i, y_i) \text{ is equilibrium pt for } \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,y) \end{cases} \quad (x,y) = (x(t), y(t)) \quad \Longleftrightarrow \quad \begin{cases} f(x_i, y_i) = 0 \\ g(x_i, y_i) = 0 \end{cases}}$$

Thus, in order to find the equilibrium points for (G1) or constant solutions of (G1), we only need to solve the system

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \tag{G3}$$

This system does not involve any derivatives! For example, we find the equilibrium points for (G2) by solving:

$$\begin{cases} f(x,y) = \frac{1}{20}x - \frac{1}{500}xy = 0 \\ g(x,y) = -\frac{1}{10}y + \frac{1}{1000}xy = 0 \end{cases} \iff \begin{cases} \frac{1}{500}(25-y) = 0 \\ -\frac{1}{1000}y(100-x) = 0 \end{cases} \iff \begin{cases} x=0 \text{ or } y=25 \\ y=0 \text{ or } x=100 \end{cases} \tag{G4}$$

Thus, the equilibrium points of the system (G2) are $(0,0)$ and $(100,25)$; they are indicated by the two large dots on the first sketch in Figure 1.

WARNING: While it is usually not hard to find the equilibrium points of (G1), some care is often needed. For example, after the last step in (G4), we need to determine all pairs (x,y) such that one of the two conditions on the top line is satisfied, so that $\frac{\mathrm{d}x}{\mathrm{d}t}=0$, **and** one of the two conditions on the bottom line is satisfied, so that $\frac{\mathrm{d}y}{\mathrm{d}t}=0$. This is different from finding (x,y) such that any two of the four conditions in (G4) are satisfied; so $(x,y)=(0,25)$ is *not* an equilibrium point. Thus, it is essential to keep the conditions for $\frac{\mathrm{d}x}{\mathrm{d}t}=0$ and the condition for $\frac{\mathrm{d}y}{\mathrm{d}t}=0$ separately, e.g. on separate lines.

Note: In general, you can expect the last system in (G4) to be of the form

$$\begin{cases} f_1(x,y) = 0 \text{ or } f_2(x,y) = 0 \text{ or } f_m(x,y) = 0 \\ g_1(x,y) = 0 \text{ or } g_2(x,y) = 0 \text{ or } g_n(x,y) = 0 \end{cases}$$

where $f_1, \ldots, f_m$ are some functions that factor f and $g_1, \ldots, g_n$ are some functions that factor g. In order to find the equilibrium solutions of (G1), or equivalently all pairs of numbers (x,y) solving (G3), we then need to find ALL solutions of *each* of the mn systems of equations

$$\begin{cases} f_i(x,y) = 0 \\ g_j(x,y) = 0 \end{cases} \qquad i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n.$$

In the case of (G4) above, we get $2\cdot 2=4$ systems:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \qquad \begin{cases} x = 0 \\ 100 - x = 0 \end{cases} \qquad \begin{cases} 25 - y = 0 \\ y = 0 \end{cases} \qquad \begin{cases} 25 - y = 0 \\ 100 - x = 0 \end{cases}$$

Each of these is a system of linear equations, which happens to be easy to solve; in general, some of these systems may not be so easy to solve (you should still be able to do so if they are linear!). In this case, the second and third systems of equations have no solutions, while the first and the forth give us $(x,y) = (0,0)$ and $(x,y) = (100,25)$, respectively.

G.3 We are interested in knowing what happens with the point $(x(t), y(t))$, where $(x,y)=(x(t),y(t))$ is a solution of (G1), as t increases. One special property of systems of autonomous equations is that if $(x,y)=(x(t),y(t))$ is a solution of such a system, e.g. of (G1), then so is

$$(\tilde{x},\tilde{y}) = \big(\tilde{x}(t),\tilde{y}(t)\big) = \big(x(t-a), y(t-a)\big)$$

for any fixed constant a. As the time parameter t increases, the points $(x(t),y(t))$ and $(\tilde{x}(t),\tilde{y}(t))$ trace the same path in the xy-plane, but $(\tilde{x}(t),\tilde{y}(t))$ is delayed by time a. Thus, the behavior of a solution $(x(t),y(t))$ of (G1) is well-represented by the directed curve in the xy-plane traced by $(x(t),y(t))$ as t increases. Such a curve is called a phase trajectory for the system (G1). It shows every point in the xy-plane the path $(x(t),y(t))$ passes through as t increases, though it does not specify at what value of t the solution $(x,y)=(x(t),y(t))$ arrives at each given point (except possibly for $t=0$).

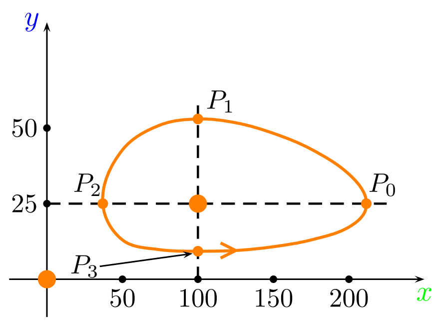
Different phase trajectories for the same system generally do not intersect, but may converge at some point. It is usually much easier to find explicit xy-equations describing phase trajectories for a system of differential equation than actual solutions. These curves in the xy-plane (the trajectories, not solutions, which are functions, *not* curves) satisfy the differential equation obtained by dividing the second equation in (G1) by the first and viewing y as a function of x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}, \qquad y = y(x).$$

Solutions to this equation in a specific case are analyzed in Section 7.6 using the direction field for the xy-differential equation. In the case of (G2), we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\frac{1}{10}y + \frac{1}{1000}xy}{\frac{1}{20}x - \frac{1}{500}xy} = -\frac{y(100-x)}{2x(25-y)}, \qquad y = y(x).$$

This equation is separable and thus solvable. From this we find that the phase trajectories in the first quadrant of the xy-plane (not including the axes) are closed curves circling around the only equilibrium point in the first quadrant.



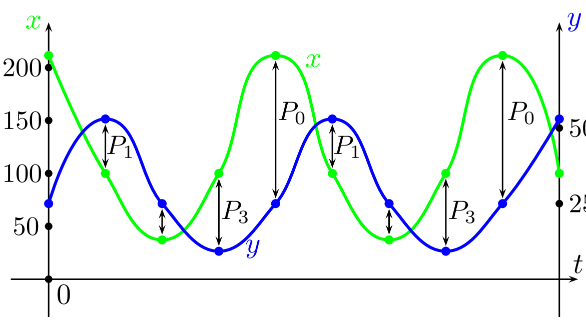


Figure 1: The left diagram shows a phase trajectory and the equilibrium points (the two large dots) for the system (G2); a solution $(x(t), y(t))$ of (G2) goes around this curve counter-clockwise as t increases. The right diagram shows the corresponding graphs of $x = x(t)$ and $y = y(t)$ as functions of time.

G.4 Another way to represent a solution $(x, y) = (x(t), y(t))$ to the system (G1) is by sketching the graphs of $x = x(t)$ and $y = y(t)$ as functions of time. It is most appropriate to do so with the same horizontal t-axis, but different vertical x- and y-axes. The second diagram in Figure 1 shows rough graphs of the functions $x = x(t)$ and $y = y(t)$ for the solution $(x(t), y(t))$ of (G2) whose phase trajectory is shown in the first diagram. The t-axis is shared by the two graphs, but the vertical axes are different. In particular, the x-axis has nothing to do with the y-graph; so $y(0) \approx 25$, not 75. Similarly, the y-axis has nothing to do with the x-graph; so at the last time point shown on the graph $x \approx 100$, and not 35. The intersection points of the two graphs are pretty much irrelevant as x and y may represent very different quantities, in addition to the x-axis and y-axes having different scales. What the sketch does tell us is the x-value(s) at the time(s) when y-value was something and vice versa, usually without specifying the corresponding value of time t. For example, the first time x was about 45 (the first x-min), y was about 25; because both graphs are periodic, $y \approx 25$ when x reaches about 45 for the second time. Both of these facts are indicated by the tiny vertical unlabeled line segments in the second diagram in Figure 1. This tells us that roughly $(45, 25)$ should be a point on the corresponding phase trajectory (labeled P_2 on the first diagram in Figure 1). Furthermore, this is the left-most point on the trajectory (because this is the minimal value of x on the graph) and the trajectory passes through this point at least twice. Since a solution of (G1) is determined by its value at $t=0$ (or any other fixed value of t), the latter implies that the corresponding trajectory keeps on going around a simple closed curve in the xy-plane.

G.5 One of the central (and hard!) themes of Section 7.6 is to roughly sketch the graphs of $x=x(t)$ and $y=y(t)$ as functions of t from the phase trajectory traced by $(x,y)=(x(t),y(t))$ in the xy-plane and vice versa. In order to do so, first determine the extremal points of the phase trajectory or of the graphs (whichever you are given) and the limiting behavior if any. For example, the phase trajectory in the first diagram in Figure 1 has four extremal points that are traversed in the order $P_0, P_1, P_2, P_3, P_0, P_1, \ldots$; it has no limiting behavior (it keeps on circling around instead of approaching some point). If our trajectory instead spiraled down to the point (100,25), it would have had lots of extremal of points (one after each quarter-turn) and would have also approached (100,25) as $t \longrightarrow \infty$. The trajectory in the second sketch in Figure 2 limits to (1150,200) as $t \longrightarrow \infty$.

After determining the extremal points of a given phase trajectory, mark the x- and y-coordinates of each of them above the same t-point and do so in the order the extremal points are traversed. So if we start at P_0 in the first sketch in Figure 1, mark $x=210$ and $y=25$ above $t=0$ (however, remember that the x and y-scales may not be the same). After that, mark the x and y-coordinates of P_1, 100 and say 52, over some $t=t_1>0$. Then mark the x and y-coordinates of P_2, say 45 and 25, over some $t=t_2>t_1$; continue on to P_3 and then $P_4=P_0$. In order to indicate the periodic behavior of the trajectory, this should be done for at least one full period (so the x and y-coordinates of at least P_0 must be marked over two different t-values); it is preferable to continue for slightly longer. If the trajectory has a limiting point, the graphs should be done for long enough to indicate that they approach some asymptotes, such as in the first sketch in Figure 2. Make sure to distinguish between the x-points and the y-points (for example, use a pencil and a blue pen). Once you have marked the coordinates of the extremal points, connect the x-points by a smooth curve which is monotonic between any two of them and do the same for the y-points. To get a more precise sketch, you could also use coordinates of the non-extremal points on the phase trajectory, but using just the extremal points and the limiting behavior will suffice in most cases. Note that the t-axis should have **no** indication of scale; the only labels on it should be t on the very right and 0, provided the starting point of the trajectory is given. The scales of the x and y-axes and the labels on them in the “graphs sketch” should be analogous to the scales and the labels on these axes in the xy-plane.

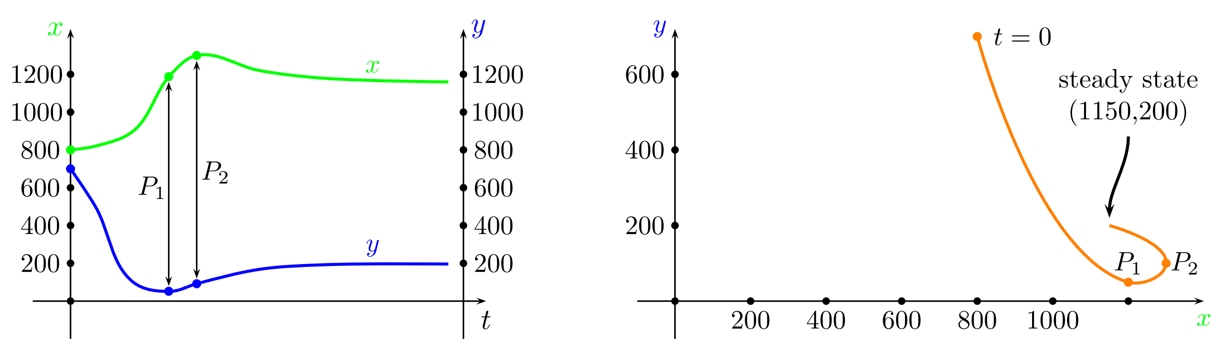


Figure 2: The left diagram shows graphs of some functions $x=x(t)$ and $y=y(t)$. The right diagram shows the phase trajectory traced by $(x,y)=(x(t),y(t))$ in the xy-plane.

If you start with graphs of $x=x(t)$ and $y=y(t)$, begin by marking the peaks and sags on each of the two graphs as well as the point on the other graph on the same vertical line as each of the peaks and sags. The x-values and y-values at these points give extremal points on the corresponding phase trajectory; you should also mark the starting point $(x_0, y_0) = (x(0), y(0))$. Once you have marked the extremal points in the xy-plane, connect them by a smooth curve in the order of increasing t so that the curve does not change its general direction (e.g. up and to the right) between any two of the points. If the graphs are periodic, the phase trajectory will be a closed curve. If the graphs have asymptotes, the phase trajectory will have a limiting point which is approached from the last extremal marked point (it is possible that there are infinitely many extremal points approaching the limiting point). For example, in the case of the first sketch in Figure 2, the starting values of x and y give the point $(800, 700)$ in the xy-plane. Then mark the sag on the y-curve along with the point on the x-curve directly above it and the peak on the x-curve along with the point on the y-curve directly below it. The x-value and y-value of the first pair give the point $P_1 \approx (1200, 50)$ in the xy-plane; the second pair gives the point $P_2 \approx (1300, 100)$ in the xy-plane. Connect the three points by a smooth curve in the xy-plane that does not change the general direction and after passing P_2 heads toward $(1150, 200)$; this is because the x-graph approaches $x = 1150$ and the y-graph approaches $y = 200$ as $t \longrightarrow \infty$. In the case of the first diagram in Figure 2, the x- and y-scales are the same, but usually this is not the case.

In general, various features of a phase trajectory in the xy-plane correspond to some features of the graphs of $x = x(t)$ and $y = y(t)$. Here is a partial "dictionary":

phase trajectory in xy-plane	graphs of $x = x(t)$ and $y = y(t)$	Examples
starting point P_0 (at $t = 0$)	points above $t = 0$	Figs 1,2 above
local left/right-most point	local min/max in x-graph	in MIIf09 solutions
local bottom/top point	local min/max in y-graph	Figs 3,4 on p543 in Stewart
cycle (closed loop)	periodic	Figs 1 above, Figs 3,4 on p543
limiting point	horizontal asymptote	Fig 2 above, in MIIf09 solutions
spiraling down to a point	decaying oscallations around horizontal line	Fig for 7.2 11

Note: The process of going between a phase trajectory and corresponding graphs, in either direction, does not require knowing the corresponding system of differential equations; for example, it was not specified on some of the homework exercises. Knowing the system may help you check that you have completed this process correctly. If $P_i = (x_i, y_i)$ is a horizontally extremal point of a phase trajectory for the system (G1), then $f(x_i, y_i) = 0$; if it is a vertically extremal point of a phase trajectory for the system (G1), then $g(x_i, y_i) = 0$. In the case of the system (G2) and the first sketch in Figure 1, the former means that the y-coordinates of P_0 and P_2 are both 25; the latter means that the x-coordinates of P_1 and P_3 are both 100. Similarly, if the graph of $x = x(t)$ as a function of t has a peak or a sag at x_i and the y-value on the same vertical line is y_i, then $f(x_i, y_i) = 0$; if the y-graph has a peak or a sag at y_i and the x-value on the same vertical line is x_i, then $g(x_i, y_i) = 0$. In the case of the system (G2) and the second sketch in Figure 1, the latter means that $y = 25$ on a vertical line passing through an x-peak or x-sag; the former means that $x = 100$ on a vertical line passing through a y-peak or y-sag.

G.6 In Section 7.6, systems of 2 autonomous first-order differential equations are used to model interactions of two species. In such cases, $x(t)$ denotes the population of one of the species at time t, while $x(t)$ denotes the population of the other species. Such a system normally has an equilibrium point $(0, 0)$ corresponding to no population of either species. With $y = 0$, the $\frac{\mathrm{d}x}{\mathrm{d}t}$ equation in (G1) describes the growth rate of the first species in the absence of the second; with $x = 0$, the $\frac{\mathrm{d}y}{\mathrm{d}t}$ equation in (G1) describes the growth rate of the second species in the absence of the first. Each of these reduced equations is likely to be an exponential growth/decay equation or a logistic growth equation. In the exponential growth case, the population of the species increases exponentially in the absence of the other species; in the exponential decay case, the population decays out to 0 in the absence of the other species. In the logistic growth case, the population approaches the carrying capacity; this gives an equilibrium point $(K,0)$ or $(0,K)$, where K is the carrying capacity for the first population or the second population. There may well be other equilibrium points, with both populations nonzero; these correspond to the two populations precisely matched up to "support" each other (including possibly by one feeding on the other). The terms in the $\frac{\mathrm{d}x}{\mathrm{d}t}$ equation that involve y indicate whether the second species has a positive or negative effect on the first; the terms in the $\frac{\mathrm{d}y}{\mathrm{d}t}$ equation that involve x indicate whether the first species has a positive or negative effect on the second. From considering these terms, it should be possible to determine whether the system models a predator-prey relation $(+/-)$, that of cooperation for mutual benefit $(+/+)$, or of competition for common resources $(-/-)$.

In the case of (G2),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{20}x \quad \text{if } y=0, \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{10}y \quad \text{if } x=0.$$

So, in the absence of the second species, the first obeys an exponential growth equation and thus increases exponentially with time; in the absence of the first species, the second obeys an exponential decay equation and thus eventually dies out. Since xy has a negative coefficient in the $\frac{\mathrm{d}x}{\mathrm{d}t}$ equation in (G2) and positive in the $\frac{\mathrm{d}y}{\mathrm{d}t}$ equation, the presence of the second species has negative effect on the first and the presence of the first species has positive effect on the second. This suggests that the first species is prey and the second is predator.

H: Sequences

H.1 A sequence is an infinite string of numbers. It can be specified in several ways:

- list the numbers; for example, $-1, 1/2, -1/6, 1/24, -1/120, \ldots$;
- give a formula for the n-th number in the sequence; for example $a_n = (-1)^n/n!$ for $n \geq 1$;
- through recursive definition; for example, $a_1 = -1$, $a_{n+1} = -a_n/(n+1)$ for $n \geq 1$, or $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$. The first of these sequences is the same as the two sequences above; the second one is the famous Fibonacci sequence.

A sequence does not have to start with a_1; it could start with a_0 or with any other a_{n_0}, as long as a_n is specified for all $n \geq n_0$. Since it is just a string of numbers, the first number could be called a_1, or a_0, or a_{-10}; the second number in the sequence would then have to be called a_2, a_1, or a_{-9}, respectively.

H.2 Given a sequence $a_1, a_2, \ldots$, we'd like to know whether it gets closer and closer to some number a or there is no such number. In the former case, the sequence is said to converge to a and this is written as $\lim\limits_{n\longrightarrow\infty} a_n = \infty$; in the latter case, the sequence is said to diverge. In many cases, it is fairly straightforward to determine whether a sequence converges (and if so to what limit) or diverges. For example, if

$$a_n = (-1)^n \frac{\sqrt{n^4+n^2}}{n^2+\sqrt{n^2+1}}\,,$$

simply divide top and bottom by n^2 (you have to divide by the same thing!):

$$\begin{aligned} a_n &= (-1)^n \frac{\sqrt{n^4+n^2}/n^2}{n^2/n^2+\sqrt{n^2+1}/n^2} = (-1)^n \frac{\sqrt{n^4/n^4+n^2/n^4}}{1+\sqrt{n^2/n^4+1/n^4}} \\ &= (-1)^n \frac{\sqrt{1+1/n^2}}{1+\sqrt{1/n^2+1/n^4}}\,; \end{aligned} \tag{H1}$$

so the fraction approaches $\sqrt{1}/(1+\sqrt{0})=1$, but the sign alternates. So the terms a_n with n odd converge to -1, while the terms a_n with n even converge to 1; thus, the entire sequence *diverges* (there is no *single* number to which all of the terms approach):

$$\boxed{\text{“2 limits”} = \text{“no limit”} = \text{“diverge”}}$$

The above trick of dividing top and bottom of a fraction by a power of n is the **most common approach** to dealing with sequences given by fractions; in MAT 125, a similar trick involved dividing by a power of x. In order to reduce your chances of making minor computational errors, which may even alter the qualitative answer (and thus be heavily penalized), it is best not to skip steps in a computation like (H1); in particular, be careful when taking a power of n under a square root.

H.3 Some sequences are of the form $a_n = f(b_n)$ for some fairly simple function f and some fairly simple sequence b_n. For example, if $a_n = \mathrm{e}^{1/n}$, then the sequence $b_n = 1/n$ converges to 0 and since e^x is continuous at 0,

$$\lim_{n\longrightarrow\infty} \mathrm{e}^{1/n} = \mathrm{e}^{\lim_{n\longrightarrow\infty} 1/n} = \mathrm{e}^0 = 1.$$

On the other hand, the sequence $a_n = \cos(\pi n)$ is divergent, since it alternates between 1 and -1.

H.4 If $a_n = f(n)$ for some function $f = f(x)$ defined on the positive real line, then

$$\lim_{n\longrightarrow\infty} a_n = \lim_{x\longrightarrow\infty} f(x)$$

if the second limit exists; the first limit may exist even if the second does not. This may allow using l'Hospital for limits of functions. For example, if $a_n = (\ln n)/n$, then

$$\lim_{n\longrightarrow\infty} a_n = \lim_{x\longrightarrow\infty} \frac{\ln x}{x} = \lim_{x\longrightarrow\infty} \frac{(\ln x)'}{x'} = \lim_{x\longrightarrow\infty} \frac{1/x}{1} = 0.$$

If $a_n = n \cdot \sin(1/n)$, then

$$\lim_{n\longrightarrow\infty} a_n = \lim_{x\longrightarrow\infty} x\sin(1/x) = \lim_{x\longrightarrow 0} \frac{\sin x}{x} = \lim_{x\longrightarrow 0} \frac{(\sin x)'}{x'} = \lim_{x\longrightarrow 0} \frac{\cos x}{1} = 1.$$

This approach is *not* suitable for many sequences, including those involving $n!$ and $(-1)^n$.

H.5 If a sequence $\{a_n\}$ is defined recursively as $a_{n+1} = f(a_n)$, for some function f and with some initial condition, and it converges to a, then $a = f(a)$; this is obtained by taking the limit of both sides of $a_{n+1} = f(a_n)$. So *if* the sequence a_n is known to have a limit, one simply needs to solve the equation $a = f(a)$; it may have several solutions, but it should be possible to rule out all but one of them as possible limits (perhaps only one solution of $a = f(a)$ is non-negative and $a_n > 0$ for all n).

This trick applies in

8.1 Example 12: $a_1 = 2,\ a_{n+1} = \dfrac{a_n + 6}{2}$

8.1 #48: $a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2a_n},$ $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$

8.1 #54: $a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n},$ $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$

Note: given the right-most presentations of sequences on the second and third lines above, you should be able to convert them to the recursive definitions in the middle of the two lines.

H.6 Before applying the trick in **H.5**, one has to know that the sequence $\{a_n\}$ has a limit at all. The convergence/divergence test for sequences which is suitable for all three examples in **H.5** is the Monotonic Sequence Theorem:

> if $a_n \le a_{n+1}$ and $a_n \le M$ for all n ($\ge$ some N), then $\{a_n\}$ converges and $\lim\limits_{n\longrightarrow\infty} a_n \le M$
> if $a_n \ge a_{n+1}$ and $a_n \ge m$ for all n ($\ge$ some N), then $\{a_n\}$ converges and $\lim\limits_{n\longrightarrow\infty} a_n \ge m$

In the first case, the sequence is increasing with n and is climbing below some "roof" M; as it keeps climbing, but cannot escape past the roof, it must approach some level below the roof (or the roof itself). In the second case, the sequence is decreasing with n and is descending toward some "floor" m; as it keeps descending, but cannot escape past the floor, it must approach some level above the floor (or the floor itself).

H.7 The second main convergence test for sequences is the *Squeeze Theorem for Sequences:*

> if $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences such that $b_n \le a_n \le c_n$ for all n ($\ge$ some N), $\{b_n\}$ and $\{c_n\}$ converge, and $\lim\limits_{n\longrightarrow\infty} b_n = \lim\limits_{n\longrightarrow\infty} c_n$, then $\{a_n\}$ also converges and
>
> $$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} b_n = \lim_{n\longrightarrow\infty} c_n$$

You might apply this to a specified sequence $\{a_n\}$ and appropriately chosen simpler sequences $\{b_n\}$ and $\{c_n\}$ which converge to the *same* limit and squeeze $\{a_n\}$ in between. For example, if

$$a_n = \frac{n}{n+1} + \frac{\cos n}{n},$$

you might take $b_n = n/(n+1) - 1/n$ and $c_n = n/(n+1) + 1/n$ so that

$$b_n \le a_n \le c_n, \qquad \lim_{n\longrightarrow\infty} b_n = \lim_{n\longrightarrow\infty} c_n = \lim_{n\longrightarrow\infty} \frac{n/n}{n/n + 1/n} = 1;$$

this implies that $\{a_n\}$ also converges and its limit is also 1. If $a_n = 7^n/n!$, then

$$a_{7+n} = \frac{7^{7+n}}{(7+n)!} = \frac{7^7}{7!} \cdot \frac{7}{8} \cdot \frac{7}{9} \cdot \ldots \cdot \frac{7}{7+n} \le a_7 \cdot \left(\frac{7}{8}\right)^n;$$

so the sequence a_{7+n} is squeezed between the constant sequence $b_n = 0$ and the geometric sequence $c_n = a_7(7/8)^n$ which converges to 0 by **H.8** below. This implies that the sequences $\{a_n\}$ also converges to 0. The practical use of the *Squeeze Theorem for Sequences* is rather limited though. For example, in the first case above, you know that $|\cos n| \le 1$ and thus $(\cos n)/n \longrightarrow 0$; so

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} \frac{n}{n+1} = 1.$$

The second case is best dealt with by using the *Ratio Test for Sequences*; see **H.9** below.

H.8 A sequence of the form $c, cr, cr^2, cr^3, \ldots$ is called geometric. It is not difficult to determine whether it converges:

> the geometric *sequence* $c, cr, cr^2, cr^3, \ldots$ with $c \neq 0$
>
> - converges if $-1 < r \le 1$ (to 0 if $-1 < r < 1$; to 1 if $r = 1$);
> - diverges if $r \le -1$ or $r > 1$.

(H2)

Note that the convergence statement for geometric *series*, (I3) below, is slightly different.

H.9 Another convergence test that works well for some sequences is the *Ratio Test for Sequences*:

> - if $\lim_{n\longrightarrow\infty} \dfrac{|a_{n+1}|}{|a_n|} < 1$, then $\lim_{n\longrightarrow\infty} a_n = 0$;
> - $\lim_{n\longrightarrow\infty} \dfrac{|a_{n+1}|}{|a_n|} > 1$ or $a_{n+1}/a_n \longrightarrow \infty$, then the sequence a_n diverges;
> - $\lim_{n\longrightarrow\infty} \dfrac{|a_{n+1}|}{|a_n|} = 1$, then this test says nothing.

For example, for the sequence $a_n = (-1)^n 7^n/n!$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{7^{n+1}/(n+1)!}{7^n/n!} = \frac{7^{n+1}}{7^n} \cdot \frac{n!}{(n+1)!} = \frac{7^n \cdot 7}{7^n} \cdot \frac{n!}{n! \cdot (n+1)} = \frac{7}{n+1} \longrightarrow \frac{7}{\infty + 1} = 0.$$

Since $0 < 1$, this sequence converges to 0. On the other hand, for the sequence $a_n = 2^n/n$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}/(n+1)}{2^n/n} = \frac{2^{n+1}}{2^n} \cdot \frac{n}{n+1} = \frac{2^n \cdot 2}{2^n} \cdot \frac{1}{n/n+1/n} = 2\frac{1}{1+1/n} \longrightarrow 2\frac{1}{1+1/\infty} = 2\frac{1}{1+0} = 2.$$

Since $2 > 1$, this sequence diverges (actually "converges" to ∞).

Since *RT for Sequences* can detect convergence of sequences $a_1, a_2, \ldots$ with limit 0 only (and even of only some of these), it works with few sequences. However, whenever it is applicable, *RT for Sequences* determines the limit of convergent sequences immediately. *RT for Sequences* has a good chance of working for sequences that involve factorials and powers n (e.g. $n!$, 3^n, n^n), but has *no* chance of working for sequences that involve only powers of n (e.g. n^3).

H.10 Finally, there are *Limit Rules for Convergent Sequences*, which are more or less as expected:

if $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is any number,

$$\lim_{n\longrightarrow\infty}(a_n \pm b_n) = \lim_{n\longrightarrow\infty} a_n \pm \lim_{n\longrightarrow\infty} b_n\,, \qquad \lim_{n\longrightarrow\infty} ca_n = c\cdot\lim_{n\longrightarrow\infty} a_n$$

$$\lim_{n\longrightarrow\infty}(a_n b_n) = \big(\lim_{n\longrightarrow\infty} a_n\big)\cdot\big(\lim_{n\longrightarrow\infty} b_n\big) \qquad \lim_{n\longrightarrow\infty}\frac{a_n}{b_n} = \frac{\lim_{n\longrightarrow\infty} a_n}{\lim_{n\longrightarrow\infty} b_n} \quad \text{if } \lim_{n\longrightarrow\infty} b_n \neq 0$$

The second equation on the first line is a special case of the first equation on the second line: just take $b_n = c$ for all n. Note that the sequences $\{a_n \pm b_n\}$, $\{a_n b_n\}$, and $\{a_n/b_n\}$ can converge even if the sequences $\{a_n\}$ and $\{b_n\}$ do not; in such cases, the limit rules are useless. Typically the limit rules are used to compute limits of sequences; in some cases they could also be used to test for convergence. For example, *if* the sequence $\{a_n\}$ converges, then the sequence $\{a_n \pm b_n\}$ converges if and only if the sequence $\{b_n\}$ does.

I: Series

I.1 A series (or infinite series) is the sum of all terms in a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots\,, \tag{I1}$$

where $a_1, a_2, \ldots$ is some sequence. The lower limit in the summation need not be 1; if a_0 is the first term of the corresponding sequence, then the lower limit in the sum is 0. Associated to the infinite sum (I1) is the sequence of partial sums,

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \qquad s_n = \sum_{k=1}^{k=n} a_k.$$

The (infinite) series (I1) is said to converge to s if the sequence $\{s_n\}$ of the *partial sums* (**not** the original sequence $\{a_n\}$!!!) converges to s; if the partial sums s_n do not have a limit, the series (I1) is said to diverge. Thus, if the series (I1) converges to some number s, then the partial sums s_n approach s and so

$$a_n = s_n - s_{n-1} = \big(s_n - s\big) - \big(s_{n-1} - s\big)$$

approaches 0. This gives the **most important statement** regarding convergence of series:

if the sequence $\{a_n\}$ does not converge or it converges, but $\lim_{n\longrightarrow\infty} a_n \neq 0$,
then the series $\sum_{n=1}^{\infty} a_n$ does not converge (I2)

For example, the series $\sum_{n=1}^{\infty}(-1)^n$ and $\sum_{n=1}^{\infty}\cos(n\pi/2)$ do not converge, because neither of the sequences $\{(-1)^n\}$ and $\{\cos(n\pi/2)\}$ converges to 0 (in fact, neither of the two sequences converges at all). The partial sums s_n in the first case alternate between -1 and 0 and so indeed do not approach any number; in the second case, the partial sums cycle through $0, -1, -1, 0$ and so do not approach any number either.

WARNING: The most important statement about convergence of power series **can never be used** to conclude that a series converges; this is the reason that there are *lots of* other convergence tests for series. For example, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, according to the *p-Series Test* in (I9) below, even though $1/n \longrightarrow 0$.

I.2 Computing the sum of an infinite series is usually difficult, but possible in some cases. A geometric series is the sum of a geometric sequence and so has the form $\sum_{n=0}^{\infty} cr^n$. The sequence of partial sums in this case is

$$s_0 = c, \qquad s_1 = c + cr, \qquad s_2 = c + cr + cr^2, \qquad \ldots$$
$$s_n = c + cr + \ldots + cr^n = c(1 + r + \ldots + r^n) = \begin{cases} \frac{1-r^{n+1}}{1-r}c, & \text{if } r \neq 1; \\ (n+1)c, & \text{if } r = 1. \end{cases}$$

If $c \neq 0$ and $|r| \geq 1$, by the last line the sequence s_n does not converge; if $|r| < 0$, then $r^{n+1} \longrightarrow 0$ and so $s_n \longrightarrow 1/(1-r)$. Since the convergence of the series $\sum_{n=0}^{\infty} cr^n$ is the same the convergence of the sequence s_n (but **not** of a_n), we find that

$$\boxed{\begin{array}{l} \displaystyle\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r} \quad \text{if } \ |r| < 1 \text{ (note the lower limit on the sum)} \\ \displaystyle\sum_{n=0}^{\infty} cr^n \quad \text{does not converge if } |r| \geq 1 \text{ and } c \neq 0 \end{array}} \tag{I3}$$

In the second case, the sequence $a_n = cr^n$ being summed does not converge to 0 by (H2); thus, the conclusion in this case also follows from the most important statement about convergence of series in (I2) above.

As an application, we can write the number $2.1\overline{37} = 2.1373737\ldots$ as a simple fraction:

$$\begin{aligned} 2.1\overline{37} &= 2.1 + .037 + .037 \cdot \frac{1}{100} + .037 \cdot \frac{1}{100^2} + \ldots = \frac{21}{10} + \frac{37/1000}{1 - \frac{1}{100}} = \frac{21}{10} + \frac{37/10}{99} \\ &= \frac{21 \cdot 99 + 37}{990} = \frac{2116}{990} = \frac{1058}{495} \end{aligned}$$

This is another example when skipping steps might increase the chance of a computational error.

I.3 Infinite series can also be summed up in the cases of telescoping cancellation. Such series have the form

$$\sum_{n=1}^{\infty} \big(b_n - b_{n+m}\big) = (b_1 - b_{1+m}) + (b_2 - b_{2+m}) + \ldots + (b_{1+m} - b_{1+2m}) + (b_{2+m} - b_{2+2m}) + \ldots$$

for some fixed integer $m \geq 0$ or can be put into this form after some algebraic manipulations (the lower limit can be anything). Note that lots of terms above cancel in pairs. If $n \geq m$, the n-th partial sum is then

$$\begin{aligned} s_n = a_1 + a_2 + \ldots + a_n &= (b_1 - b_{1+m}) + (b_2 - b_{2+m}) + \ldots + (b_n - b_{n+m}) \\ &= \sum_{k=1}^{k=m} b_k - \sum_{k=n+1}^{k=n+m} b_k, \end{aligned} \tag{I4}$$

since the second term in the k-th pair cancels with the first term in the $(k+m)$-th, provided $k \le n-m$; this leaves the first terms in the first m pairs and the second terms in the last m pairs. As $n \longrightarrow \infty$, the first sum on the second line in (I4) does not change; so the sequence $\{s_n\}$ (and thus the series $\sum_{n=1}^{\infty}(b_n - b_{n+m})$) converges if and only if the sequence

$$s_n^- = \sum_{k=n+1}^{k=n+m} b_k = b_{n+1} + b_{n+2} + \ldots + b_{n+m}$$

does. This happens if the sequence $\{b_n\}$ converges, but may happen even if $\{b_n\}$ diverges. For example, all of the series

$$\sum_{n=1}^{\infty}\big(\sin(1/n) - \sin(1/(n+1))\big), \quad \sum_{n=1}^{\infty}\big(\cos(1/n) - \cos(1/(n+2))\big), \quad \sum_{n=1}^{\infty}(-1)^n\big(\ln(n) - \ln(n+2)\big)$$

converge, while the series

$$\sum_{n=1}^{\infty}\big(\cos(n) - \cos(n+1)\big), \quad \sum_{n=1}^{\infty}\big(\ln(n) - \ln(n+1)\big), \quad \sum_{n=1}^{\infty}\big(\mathrm{e}^n - \mathrm{e}^{n+1}\big)$$

diverge.

The simplest possible case occurs when $b_n \longrightarrow 0$, so that the last sum in (I4) disappears as $n \longrightarrow \infty$:

$$\boxed{\sum_{n=1}^{\infty}\big(b_n - b_{n+m}\big) = \sum_{n=1}^{n=m} b_n \quad \text{if } \lim_{n\longrightarrow\infty} b_n = 0,\ m \ge 0} \tag{I5}$$

This is frequently used in conjunction with partial fractions. For example,

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{1}{n^2-1} &= \sum_{n=2}^{\infty}\frac{1}{(n+1)-(n-1)}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)\\ &= \frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots\right) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}\end{aligned}$$

In this case, $b_n = 1/(n-1)$ for $n \ge 2$ and $m = 2$. Generally, re-writing LHS of (I5) as

$$\sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} b_{n+m}$$

will constitute a serious error, since these two sums may not converge. For example,

$$\sum_{n=2}^{\infty}\frac{1}{n-1} = \sum_{n=1}^{\infty}\frac{1}{n}$$

does not converge by the *p-Series Test* in (I9) below. The condition $\lim_{n\longrightarrow\infty} b_n = 0$ in (I5) is absolutely **essential**. For example, the series

$$\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty}\big(\ln(n+1) - \ln n\big)$$

does not converge at all, because the sequence of the partial sums

$$s_n = \sum_{k=1}^{k=n} \big(\ln(k+1) - \ln k \big) = \ln(n+1) - \ln 1 = \ln(n+1)$$

does not converge. The formula (I5) with $b_n = -\ln n$ and $m = 1$ would have produced $b_1 = 0$ for the sum of the infinite series, which is impossible since all terms in the sum are positive. The formula (I5) cannot be applied in this case because the limit condition in (I5) is not satisfied.

I.4 There are many cases when it can be determined whether a series converges, but it is hard to determine its sum (this is relatively rare for sequences). There are several convergence tests for series which, unlike the most important statement for series in (I2) above, can determine convergence. All of the tests in Section 8.3 deal with series that have only positive terms a_n (for $n \geq$ some N); the *Ratio Test for Series* does not care about the signs. Series with terms of different signs are in fact more likely to converge and will be considered in Section 8.4 in more detail after the 2nd midterm. In some cases, different tests can be used to determine whether a series converges.

I.5 The most evident and fundamental convergence test for series with positive terms is the *Comparison Test*:

$$\boxed{\text{if the sequences } \{a_n\} \text{ and } \{b_n\} \text{ have positive terms, } a_n \leq b_n \text{ for all } n \ (\geq \text{some } N)\text{, and the series } \sum_{n=1}^{\infty} b_n \text{ converges, then so does the series } \sum_{n=1}^{\infty} a_n} \tag{I6}$$

This test with the roles of a_n and b_n reversed leads to a divergence test for series:

$$\boxed{\text{if the sequences } \{a_n\} \text{ and } \{b_n\} \text{ have positive terms, } a_n \geq b_n \text{ for all } n \ (\geq \text{some } N)\text{, and the series } \sum_{n=1}^{\infty} b_n \text{ diverges, then so does the series } \sum_{n=1}^{\infty} a_n}$$

While the *Comparison Test* is the basis for most other convergence tests for series, it is often easier to apply one of the other convergence tests instead.

I.6 A close cousin to the *Comparison Test* is the Limit Comparison Test for series states that

$$\boxed{\text{if the sequences } \{a_n\} \text{ and } \{b_n\} \text{ have positive terms, the sequence } a_n/b_n \text{ converges, and the series } \sum_{n=1}^{\infty} b_n \text{ converges, then so does the series } \sum_{n=1}^{\infty} a_n} \tag{I7}$$

For example, to determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$ converges, take $a_n = n/4^n$ and $b_n = 1/2^n$,

$$\lim_{n\longrightarrow\infty} \left(\frac{a_n}{b_n} \right) = \lim_{n\longrightarrow\infty} \left(\frac{n}{2^n} \right) = 0$$

and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges by the geometric series test (I3); so $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges. The same argument applies to $\displaystyle\sum_{n=1}^{\infty} \frac{n^p}{r^n}$ for any $r > 1$ (but the *Ratio Test for Series* is simpler to use here). More typically, the *Limit Comparison Test* is applied to series like

$$\sum_{n=1}^{\infty}\frac{1}{2^n-n}\,, \qquad \sum_{n=2}^{\infty}\frac{1}{n^2-n}\,, \qquad \sum_{n=1}\sin^p(1/n)\,;$$

the summands in these series "asymptotically approximate" $1/2^n$, $1/n^2$, and $1/n^p$, respectively. The *Limit Comparison Test* with the roles of a_n and b_n reversed leads to a divergence test for series:

if the sequences $\{a_n\}$ and $\{b_n\}$ have positive terms, the sequence a_n/b_n converges, $\lim\limits_{n\longrightarrow\infty}(a_n/b_n)\neq 0$, and the series $\sum\limits_{n=1}^{\infty}b_n$ diverges, then so does the series $\sum\limits_{n=1}^{\infty}a_n$

In contrast to the convergence test above, there is the extra condition that a_n/b_n not approach 0; this makes sense since otherwise we could take $a_n=0$, regardless of what b_n is. The *Limit Comparison Test* follows from the *Comparison Test*, but is likely to be more useful in this course.

I.7 Another useful convergence test for series is the Integral Test:

if f is a continuous, positive, and decreasing function on $[1,\infty)$, then the series $\sum\limits_{n=1}^{\infty}f(n)$ converges if and only if the improper integral $\int_1^{\infty}f(x)\mathrm{d}x$ does	(I8)

This test is obtained from the geometric interpretation of the integral as the area under the graph, which can be estimated by rectangles of base one and with heights determined by either left or right end points. A corollary of this test is the p-series Test:

the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ converges if and only if $p>1$	(I9)

I.8 There are also *Rules for Convergent Series*, which are more or less as expected:

if the series $\sum\limits_{n=1}^{\infty}a_n$ and $\sum\limits_{n=1}^{\infty}b_n$ converge and c is any number,
$\sum\limits_{n=1}^{\infty}(a_n\pm b_n)=\sum\limits_{n=1}^{\infty}a_n\pm\sum\limits_{n=1}^{\infty}b_n\,, \qquad \sum\limits_{n=1}^{\infty}ca_n=c\sum\limits_{n=1}^{\infty}a_n$

Note that these rules do not extend to multiplication and division, unlike what is the case for sequences. The series $\sum\limits_{n=1}^{\infty}(a_n\pm b_n)$ can converge even if the series $\sum\limits_{n=1}^{\infty}a_n$ and $\sum\limits_{n=1}^{\infty}b_n$ do not; in such cases, the above rules are useless. Typically these rules are used to compute sums of series; in some cases they could also be used to test for convergence. For example, *if* the series $\sum\limits_{n=1}^{\infty}a_n$ converges, then the series $\sum\limits_{n=1}^{\infty}(a_n\pm b_n)$ converges if and only if $\sum\limits_{n=1}^{\infty}b_n$ does.

I.9 The sum of a convergent series $\sum_{n=1}^{\infty} a_n$ can be estimated by a finite sub-sum: the sum

$$s_m = \sum_{n=1}^{n=m} a_n = a_1 + a_2 + \ldots + a_m$$

of the first m terms; this is the m-th partial sum. As $m \longrightarrow \infty$, s_m approaches the sum of the series, so that

$$\sum_{n=m+1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n \; - s_m \longrightarrow 0.$$

In some cases, the above difference can be estimated. If $f = f(x)$ is positive, decreasing, and continuous on $[1, \infty)$ and $\int_1^{\infty} f(x)\mathrm{d}x$ converges, then

$$\boxed{\int_{m+1}^{\infty} f(x)\mathrm{d}x < \sum_{n=m+1}^{\infty} a_n < \int_m^{\infty} f(x)\mathrm{d}x} \tag{I10}$$

Note that increasing the lower limit (from m to $m+1$ here) makes the integral smaller because $f > 0$. In this case, the finite-sum estimate s_m is an *under*-estimate for the infinite sum because lots of positive terms are dropped from the infinite series.

For example, let's estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to within 1/5. Since $f(x) = 1/x^2 > 0$ is continuous and decreasing on $[1, \infty)$, by (I10) we need to find the smallest integer m such that

$$\int_m^{\infty} f(x)\mathrm{d}x = \int_m^{\infty} \frac{1}{x^2}\mathrm{d}x = \frac{1}{m} \le \frac{1}{5}.$$

So $m = 5$ and the required finite-sum estimate is

$$\sum_{n=1}^{n=5} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} = \frac{3600 + 900 + 400 + 225 + 144}{3600} = \boxed{\frac{5269}{3600}}$$

This is an under-estimate for the infinite sum, as only positive terms are dropped off from the latter.

J: Convergence/Divergence Tests for Sequences and Series (recap)

The two most important things regarding Chapter 8 are

- distinguishing between sequences and series and their convergence/divergence tests;
- realizing that the convergence/divergence issue concerns what happens with "the infinite tail". Thus, dropping the first 159 terms of a sequence or series will not change its convergence/divergence property. If a series does converge, dropping the first 159 terms will however change the sum of the infinite series, precisely by the sum of the first 159 terms.

Confusion about these two points, especially the first one, is likely to be the primary reason for the low scores on the second midterm.

Whether a sequence/series converges or diverges depends primarily on the dominant terms and the presence of any sign-alternating or oscillatory behavior, such as $(-1)^n$ or $\sin n$; factors like $\sin(1/n)$ and $\cos(1/n)$ are *not* oscillatory, since they approach 0 and 1, respectively, as $n \longrightarrow 0$. It is generally helpful to try to isolate the dominant terms, essentially by factoring them out; if the terms are given by a fraction, this usually means dividing top and bottom by the dominant term. The main dominance relations to remember are:

$$\lim_{n\longrightarrow\infty}\frac{(\ln n)^p}{n^q}=0, \qquad \lim_{n\longrightarrow\infty}\frac{n^p}{e^{qn}}=0, \qquad \lim_{n\longrightarrow\infty}\frac{e^{pn}}{(n!)^q}=0, \qquad \lim_{n\longrightarrow\infty}\frac{(n!)}{n^n}=0$$

for any $p,q>0$; you should know how to justify these statements. However, one has to be careful with the dominant terms if there are minus signs between them. For example, while the dominant term of $a_n=\sqrt{9^n+2^n}-3^n$ may appear to be $3^n=\sqrt{9^n}$, in fact

$$a_n=\left(\sqrt{9^n+2^n}-3^n\right)\cdot\frac{\sqrt{9^n+2^n}+3^n}{\sqrt{9^n+2^n}+3^n}=\frac{2^n}{\sqrt{9^n+2^n}+3^n}=\left(\frac{2}{3}\right)^n\cdot\frac{1}{\sqrt{1+(2/9)^n}+1};$$

so the dominant term is $(2/3)^n$ (times 1/2, which does not effect convergence). So the sequence a_n converges to 0, while the series $\sum\limits_{n=1}^{\infty}a_n$ converges to something positive by the *Limit Comparison Test* applied with $b_n=(2/3)^n$.

If the terms of a sequence or series naturally split as a sum of two terms, one of which gives rise to a *convergent* sequence or series, respectively, then you can drop the convergent term in determining whether the entire sequence or series converges. For example, the sequence $a_n=(1+(-1)^n)/n$ converges if and only if the sequence $b_n=(-1)^n/n$ does, because the sequence $c_n=1/n$ converges (to 0, which does not matter in this case); so the sequence a_n does converge (to 0). Since the series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ does not converge, neither does the series $\sum\limits_{n=1}^{\infty}\frac{1+(-1)^n}{n}$. However, be careful *not* to split off a *divergent* sequence or series. For example,

$$\lim_{n\longrightarrow\infty}\left(\sqrt{9^n+2^n}-3^n\right)\neq\lim_{n\longrightarrow\infty}\sqrt{9^n+2^n}-\lim_{n\longrightarrow\infty}3^n;$$
$$\sum_{n=1}^{\infty}\left(\sqrt{9^n+2^n}-3^n\right)\neq\sum_{n=1}^{\infty}\sqrt{9^n+2^n}-\sum_{n=1}^{\infty}3^n;$$
$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)\neq\sum_{n=1}^{\infty}\frac{1}{n}-\sum_{n=1}^{\infty}\frac{1}{n+1}$$

because neither of the two limits on the right-hand side on the first line exists and neither of the four sums on the right-hand sides of the second and third lines exists.

Convergence/divergence of sequences. A sequence is simply an infinite string of numbers described in some way, typically by an explicit formulas, such as $a_n=(-1)^n n^4/(3n^4+1)$, or by a recursive formula, such as $a_{n+1}=\sqrt{6+a_n}$, with some initial condition(s), such as $a_1=\sqrt{6}$. While *sequence* is a longer word than *series*, determining whether a sequence converges or diverges is easier.

(SQ1) If a sequence is given by an explicit formula, it is usually possible to determine whether it converges through a quick inspection. The goal is to plug in $n=\infty$, possibly after some algebraic manipulations. If you get a meaningful number by doing so, the sequence converges to this number ($\infty/\infty, 0\cdot\infty, 0/0, \infty-\infty, 1^\infty$ are *not* meaningful numbers). If it is meaningless to plug in $n\!=\!\infty$ right away, begin by splitting a_n into parts if possible (often not; be careful) and determining the dominant term; see above. For example,

$$a_n = (-1)^n \frac{n^4}{3n^4\!+\!1} = (-1)^n \frac{1}{3+1/n^4}\,;$$

so the dominant term here is $(-1)^n$. If plugging in $n\!=\!\infty$ makes sense then, you are done: the sequence converges; for example, it makes sense to plug in $n\!=\!\infty$ into $1/(3\!+\!1/n^4)$, but not into $n^4/(3n^4\!+\!1)$ or $(-1)^n/(3\!+\!1/n^4)$, because ∞/∞ and $(-1)^\infty$ do not make sense. Typically, a sequence would not converge due to either an oscillatory behavior, which may be exhibited by a factor of $(-1)^n$ or $\sin(n)$, or because it (or part of it) approaches ∞, as $n/(\ln n)$ does. However, the presence of an oscillatory factor does not insure divergence; for example, the sequence

$$(-1)^n \frac{n^3}{3n^4\!+\!1} = \frac{(-1)^n}{n} \cdot \frac{1}{3+1/n^4}$$

converges to 0 because the seemingly oscillatory factor in fact decays to 0. Occasionally (if terms like 2^n, $n!$, or n^n are present), the *Ratio Test for Sequences* can be useful; see **H.9** above.

(SQ2) If a sequence is given by a recursive formula, begin by writing out the first few terms to get an idea whether the sequence converges or diverges. If it appears to converge, the *Monotonic Sequence Theorem* may be useful to justify this (so you may need to use induction to show that either the sequence is bounded above by something and increasing or bounded below and decreasing). If it appears to diverge, this is likely due to some oscillatory behavior or because of going off to infinity; you'll need to justify that this pattern continues as n increases.

(SQ3) The *Squeeze Theorem for Sequences* may be useful in some cases, but is generally avoidable. In some cases, it may be possible to replace n by x and compute the limit as $x\longrightarrow\infty$; this may allow using l'Hospital(if the required conditions are satisfied), but usually this will not be the fastest approach.

Convergence/divergence of series. A series is the sum of terms in a sequence, with the latter typically given by an explicit formula when series are encountered. While *series* is a shorter word than *sequence*, determining whether a series converges is much harder and the concept of a series itself is significantly more abstract.

First, a series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if the *sequence of partial sums* $\{s_n\}$ defined by

$$s_n = a_1 + a_2 + \ldots + a_n$$

does; if this happens, the infinite sum of the a_n's is defined to be the limit of the s_n's. What this means is that you keep on adding more and more terms a_n to the sum and see if the resulting finite sums (with more and more terms) approach anything. However, in practice, it is almost never possible to find an explicit formula for s_n.

Second, there are lots of divergence/convergence tests for series, most with several assumptions that you have to remember to check before applying the test. After trying to split off a convergent part of a series (e.g. $\sum 1/n^2$ from $\sum(1/n^2+(\sin n)/n^3)$) and determining the dominant term, you might want to try doing the following to determine if the series converges.

(SR0) If the sequence $\{a_n\}$ does not converge to 0, the series $\sum a_n$ diverges. For example, the series

$$\sum_{n=1}^{\infty}(-1)^n\,, \quad \sum_{n=1}^{\infty}\frac{n}{2n+1}, \quad \sum_{n=1}^{\infty}\cos(1/n), \quad \sum_{n=1}^{\infty}\sin(n),$$

all diverge. **Note** that even if $\lim\limits_{n\longrightarrow 0} a_n = 0$, the series $\sum a_n$ may still diverge; this is the reason you need the other half-dozen convergence/divergence tests.

(SR1) If the series is a geometric series $\sum cr^n$ or p-series $\sum 1/n^p$, you should know immediately if it converges or diverges; see (I3) and (I9) above. However, be careful not to confuse these with other similarly looking series; these two types of series are very restrictive, but also very important.

(SR2) If the series has **positive terms only**, determine its leading term, such as some power of n, and apply the *Limit Comparison Test* with that power of n; see (I7) above. Remember that $\sin(1/n)$ and $\tan(1/n)$ look like $1/n$ as $n\longrightarrow\infty$, since

$$\lim_{n\longrightarrow\infty}\frac{\tan(1/n)}{1/n} = \lim_{n\longrightarrow\infty}\frac{\sin(1/n)}{1/n}\cdot\lim_{n\longrightarrow\infty}\cos(1/n) = \lim_{x\longrightarrow 0}\frac{\sin(x)}{x}\cdot 1 = 1.$$

So by the *Limit Comparison Test* with $b_n = 1/n^p$, the series

$$\sum_{n=1}^{\infty}\sin^p(1/n), \qquad \sum_{n=1}^{\infty}\tan^p(1/n)$$

converge if and only if $p>1$. However, $\sin(n)$ and $\tan(n)$ do not look like n as $n\longrightarrow\infty$.

(SR3) If the series has **positive terms only**, but the *Limit Comparison Test* is not suitable, try to find a way to use the *Comparison Test*; see (I6) above. So you'll still need to guess b_n, but now the second sequence needs to satisfy different requirements (but still 3 of them). For example, the *Limit Comparison Test* with $b_n = 1/n^2$ cannot be used for the series $\sum\limits_{n=1}^{\infty}\frac{|\sin n|}{n^2}$ because

$$\lim_{n\longrightarrow\infty}\frac{a_n}{b_n} = \lim_{n\longrightarrow\infty}\frac{|\sin n|/n^2}{1/n^2} = \lim_{n\longrightarrow\infty}|\sin n|$$

does not exist. However, we can use the *Comparison Test* with $b_n = 1/n^2$, because

$$0 \le a_n = \frac{|\sin n|}{n^2} \le b_n = \frac{1}{n^2}$$

and the series $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ converges by the *p-Series Test* with $p=2$; see (I9) above. This implies that so does the "smaller" series $\sum\limits_{n=1}^{\infty}\frac{|\sin n|}{n^2}$. This argument cannot be used to directly conclude that the series $\sum\limits_{n=1}^{\infty}\frac{|\sin n|}{n}$ diverges[1], because the divergence of the series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$ does not imply that the "smaller" series $\sum\limits_{n=1}^{\infty}\frac{|\sin n|}{n}$ also diverges.

[1]this series does indeed diverge because $|\sin x|+|\sin(x+1)|\ge 1/2$ for all x

(SR4) For some series with **positive terms only**, the *Integral Test* can be used; see (I8) above. For this, the function f obtained from the terms of the series by replacing n by x must make sense for all $x \geq 1$ (or at least for $x \geq N$ for some N); for example, $x!$ does not make sense. You also have to check that the function f obtained in this way is positive, continuous, and decreasing for $x \geq 1$ (or at least for $x \geq N$ for some N). For example, while the function $f(x) = |\sin x|/x$ makes sense for $x \geq 1$ and is continuous, it is not decreasing (and or even positive); so the fact that the integral $\int \frac{|\sin x|}{x} dx$ diverges does not say anything directly about the infinite series. The most important use of the *Integral Test* has been to obtain the *p-Series Test*; see (I9) above. It has also been used in the present of $\ln n$. The *Integral Test* can be used to show that all of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}, \quad \sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)^p}, \quad \ldots, \qquad \sum_{n=1}^{\infty} \sin^p(1/n), \quad \sum_{n=1}^{\infty} \tan^p(1/n)$$

converge if and only if $p > 1$. Except for the last 2 series, the relevant integral can actually be computed fairly easy. In the case of the last 2 series, the integral is much harder to compute, but it can be shown to be finite if and only if $p > 1$, which suffices; however, it is simpler to apply the *Limit Comparison Test* to the last 2 series with $b_n = 1/n^p$.

(SR5) In rare cases, it is possible to determine whether a series converges or diverges by **computing the corresponding sequence of partial sums** (so directly from the definition of convergence for series). This can be done when the series has the form

$$\sum_{n=1}^{\infty} \left(b_n - b_{n+m}\right)$$

for some sequence $\{b_n\}$; see **I.3** above. This approach is also useful for computing sums of series like $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ via partial fractions and partial sums. However, for showing that this series converges, it is much simpler to use the *Limit Comparison Test*.

Each of the convergence tests works only for some series, and the convergence of some series can be determined using more than one of the convergence tests (but one of them may still be easier to use). Most importantly, try to see what a given series looks like, in terms of the leading terms and oscillatory behavior if any; in most cases, you may be able to guess whether it converges or diverges rather quickly based on these. If you are supposed to justify your answer, make sure you check that all of the conditions of the test you want to use hold; often this will mean stating the required properties, but sometimes additional justification may be required. For example, it is sufficient to state that $1/n \geq 0$, but some explanation is required to justify that $1/(n^2 - n + 1) \geq 0$.

In order to do well on the second midterm, you will need to decide fairly quickly which convergence test to use for each given series. It will not be possible to do so without *a lot of* practice. You should go through all of the sequences and series in the exercises in 8.1-8.3 and determine whether each converges or diverges and why; with some practice, each of them should take you only 30 seconds or so.

Good luck on the midterm!