MAT 127: Calculus C, Spring 2015
Solutions to Some HW9 Problems

Below you will find detailed solutions to six problems from HW9. Since the first five of them were WebAssign problems, your versions of these problems may have had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 8.4, Problem 7 (webassign)

Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}$ *converges or diverges.*

Since the sequence

$$(-1)^n\frac{3n-1}{2n+1}=(-1)^n\frac{(3n-1)/n}{(2n+1)/n}=(-1)^n\frac{3-1/n}{2+1/n}\longrightarrow(-1)^n\frac{3-1/\infty}{2+1/\infty}=(-1)^n\cdot\frac{3}{2}$$

keeps on jumping from near $3/2$ to near $-3/2$ as n approaches ∞, the sequence $(-1)^n\frac{3n-1}{2n+1}$ does not converge to zero and thus the series $\sum_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}$ $\boxed{\text{diverges}}$ by the Test for Divergence.

Section 8.4, Problem 14 (webassign)

Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n5^n}$ *converges. How many terms of the series are needed to approximate the sum with error less than .0001?*

This series is (strictly) alternating, since the odd terms are negative and the even terms are positive. Furthermore, $1/(n5^n)\longrightarrow 0$ as $n\longrightarrow 0$ and $1/((n+1)5^{n+1}) < 1/(n5^n)$ for all n. Thus, the series converges by the Alternating Series Test.

We need to find m so that

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{n5^n}-\sum_{n=1}^{n=m}\frac{(-1)^n}{n5^n}\right|\le 10^{-4}\,.$$

By the previous paragraph, we can use the *Alternating Series Estimation Theorem* on p587, according to which

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{n5^n}-\sum_{n=1}^{n=m}\frac{(-1)^n}{n5^n}\right|\le\frac{1}{(m+1)5^{m+1}}.$$

So, we need m so that $1/(m+1)5^{m+1}\le 1/10^4$, or $(m+1)5^{m+1}>10^4$. The smallest such number m is $\boxed{4}$ (for $m=3$, we get only $4\cdot 5^4=2500$).

Remark: Since the series involves 5^n, we can also use the Ratio Test to show convergence:

$$\frac{|a_{n+1}|}{|a_n|}=\frac{1/((n+1)5^{n+1})}{1/(n5^n)}=\frac{n}{n+1}\cdot\frac{5^n}{5^{n+1}}=\frac{n/n}{(n+1)/n}\cdot\frac{5^n}{5^n\cdot 5^1}=\frac{1}{1+1/n}\cdot\frac{1}{5}\longrightarrow\frac{1}{1+1/\infty}\cdot\frac{1}{5}=\frac{1}{5}\,;$$

since $1/5<1$, the series converges. However, this would not justify the use of the *Alternating Series Estimation Theorem* to answer the second question.

Section 8.4, Problem 31 (webassign)

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$ *converges or diverges.*

Since $\arctan n > 0$ for $n > 0$, we need to see if the series $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$ converges. Since $0 \leq (\arctan n)/n^2 \leq (\pi/2)/n^2$ and the series

$$\sum_{n=1}^{\infty} \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by the p-Series Test with $p = 2 > 1$, the "smaller" series $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^2}$ also converges by the *Comparison Test.* Thus, the original series $\boxed{\text{does}}$ converge absolutely.

Section 8.4, Problem 33 (webassign)

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{(2n-1)!}$ *converges or diverges.*

We need to see if the series $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{(2n-1)!}$ converges. Since there are factorials involved, try the *Ratio Test*:

$$\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} &= \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)(2(n+1)-1)/(2(n+1)-1)!}{1 \cdot 3 \cdot \ldots \cdot (2n-1)/(2n-1)!} \\
&= \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)(2n+1)}{1 \cdot 3 \cdot \ldots \cdot (2n-1)} \cdot \frac{(2n-1)!}{(2n+1)!} = (2n+1)\frac{(2n-1)!}{(2n-1)!2n(2n+1)} = \frac{2n+1}{2n(2n+1)} = \frac{1}{2n} \longrightarrow 0\,.
\end{aligned}$$

So the series $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{(2n-1)!}$ converges, and thus the original series $\boxed{\text{does}}$ converge absolutely.

Note: Since the numerator of the nth summand is the product of the odd integers between 1 and $2n-1$ and the denominator is the product of all integers between 1 and $2n-1$, the nth summand is the reciprocal of the product of all even integers between 1 and $2n-1$ (if $n = 1$, this product is defined to be 1). Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{(2n-1)!} &= \sum_{n=1}^{\infty} \frac{1}{2 \cdot 4 \cdot \ldots \cdot (2n-2)} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^{n-1} \cdot 1 \cdot 2 \cdot \ldots \cdot (n-1)} = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{2^n n!} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)^n.
\end{aligned}$$

We will see in Section 8.7 that this sum is $e^{1/2}$.

Section 8.4, Problem 37 (webassign)

For which of the following series is the Ratio Test inconclusive?

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{1}{n^3}\,, \quad \text{(b)}\ \sum_{n=1}^{\infty} \frac{n}{2^n}\,, \quad \text{(c)}\ \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}\,, \quad \text{(d)}\ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}\,.$$

Compute the limit of the ratio of the absolute values of two consecutive terms:

$$\text{(a)}\quad \frac{|a_{n+1}|}{|a_n|} = \frac{1/(n+1)^3}{1/n^3} = \frac{1}{(n+1)^3/n^3} = \frac{1}{((n+1)/n)^3} = \left(\frac{1}{1+1/n}\right)^3 \longrightarrow \left(\frac{1}{1+1/\infty}\right)^3 = 1;$$

$$\text{(b)}\quad \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)/2^{n+1}}{n/2^n} = \frac{(n+1)}{n} \cdot \frac{2^n}{2^n \cdot 2^1} = \left(1+\frac{1}{n}\right)\frac{1}{2} \longrightarrow \left(1+\frac{1}{\infty}\right)\frac{1}{2} = \frac{1}{2};$$

$$\text{(c)}\quad \frac{|a_{n+1}|}{|a_n|} = \frac{3^{(n+1)-1}/\sqrt{n+1}}{3^{n-1}/\sqrt{n}} = \frac{3^{n-1}\cdot 3^1}{3^{n-1}} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} = 3\frac{1}{\sqrt{n+1}/\sqrt{n}} = \frac{3}{\sqrt{1+1/n}} \longrightarrow \frac{3}{\sqrt{1+1/\infty}} = 3;$$

$$\begin{aligned}\text{(d)}\quad \frac{|a_{n+1}|}{|a_n|} &= \frac{\sqrt{n+1}/(1+(n+1)^2)}{\sqrt{n}/(1+n^2)} = \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1+n^2}{1+(n+1)^2} = \sqrt{\frac{n+1}{n}} \cdot \frac{1/n^2+1}{1/n^2+(n+1)^2/n^2} \\ &= \sqrt{1+\frac{1}{n}} \cdot \frac{1/n^2+1}{1/n^2+((n+1)/n)^2} \\ &= \sqrt{1+\frac{1}{n}} \cdot \frac{1/n^2+1}{1/n^2+(1+1/n)^2} \longrightarrow \sqrt{1+\frac{1}{\infty}} \cdot \frac{1/\infty+1}{1/\infty+(1+1/\infty)^2} = 1.\end{aligned}$$

Thus, the Ratio Test is inconclusive in $\boxed{\text{(a),(d)}}$

Remark: This problem illustrates the principle that the Ratio Test is *not* suitable for series that involve only powers of n, and not something with faster growth such as 2^n, $n!$, or n^n. While the Ratio Test says nothing about the series in (a) and (d), both converge: (a) by the p-series test and (d) because it looks like $\sqrt{n}/n^2 = 1/n^{3/2}$ (so by Limit Comparison and p-series). By the Ratio Test, the series in (b) converges, while the series in (c) diverges. These two examples illustrate the principle that the limit obtained in applying the Ratio Test is not affected by factors of n and is just the absolute value of the common ratio r for a geometric series.

Section 8.4, Problem 42

(a) *Show that the series* $\sum_{n=0}^{\infty} \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}}$ *converges.*

Since all terms are nonzero and involve $n!$ (as well n in the exponent), try the *Ratio Test*:

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(4(n+1))!(1103+26390(n+1))/\big(((n+1)!)^4 396^{4(n+1)}\big)}{(4n)!(1103+26390n)/\big((n!)^4 396^{4n}\big)} \\
&= \frac{(4n+4)!}{(4n)!} \cdot \frac{1103+26390(n+1)}{1103+26390n} \cdot \frac{(n!)^4}{((n+1)!)^4} \cdot \frac{396^{4n}}{396^{4n+4}} \\
&= \frac{(4n+1)(4n+2)(4n+3)(4n+4)}{1} \cdot \frac{1103+26390(n+1)}{1103+26390n} \cdot \frac{1}{(n+1)^4} \cdot \frac{1}{396^4} \\
&= \frac{(4+1/n)(4+2/n)(4+3/n)(4+4/n)(1103/n+26390(1+1/n))}{((n+1)/n)^4 \cdot (1103/+26390)396^4} \\
&\longrightarrow \frac{4^4 \cdot 26390}{1^4 \cdot 26390 \cdot 396^4} = \left(\frac{4}{396}\right)^4 = \left(\frac{1}{99}\right)^4.
\end{aligned}$$

Since $(1/99)^4 < 1$, the sum converges.

(b) *Assuming* $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}}$, *how many correct decimal places of* π *do you get with just one and two terms of this series?*

$$\begin{aligned}
s_0 &= \frac{2\sqrt{2}}{9801} \frac{0! \cdot 1103}{(0!)^4 396^0} = \frac{2206\sqrt{2}}{9801}; \\
s_1 &= \frac{2\sqrt{2}}{9801} \left(\frac{0! \cdot 1103}{(0!)^4 396^0} + \frac{4!(1103+26390)}{(1!)^4 396^4} \right) = \frac{1130173253125\sqrt{2}}{5021227463472}.
\end{aligned}$$

Since $\pi \approx 3.1415926535897932$, $1/s_0 \approx 3.14159273$, and $1/s_1 \approx 3.1415926535897939$, the one-term estimate gives $\boxed{\text{6 or 7}}$ (depending on one's definition) decimal places of π, while the two-term estimate gives $\boxed{15}$ decimal places.

Remark: By the Ratio Test computation in (a), at least for large n the series converges as fast as the geometric series with $r = (1/99)^4$; the latter adds about 8 decimal places with each extra term. So computing the first 17 million digits of π should have required fewer than 2 million terms of the above series.