

MAT 127: Calculus C, Spring 2015

Solutions to Some HW8 Problems

Below you will find detailed solutions to five problems from HW8. Since the first four of them were WebAssign problems, your versions of these problems may have had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 8.3, Problem 4 (webassign)

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent.

- (a) If $a_n > b_n$ for all n , what can you say about $\sum a_n$? Why?
(b) If $a_n < b_n$ for all n , what can you say about $\sum a_n$? Why?

(a) $\sum a_n$ diverges by the Comparison Test

(b) nothing If $a_n = b_n/2$, $\sum a_n$ diverges; if $a_n = \min(1/2^n, b_n/2)$, $\sum a_n$ converges by the Comparison Test.

Section 8.3, Problem 6 (webassign)

Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges or diverges.

Since $f(x) = 1/x^5$ is a positive, decreasing, continuous function for $x \geq 1$, the answer is the same as for

$$\int_1^{\infty} \frac{1}{x^5} dx = \int_1^{\infty} x^{-5} dx = \frac{1}{-4} x^{-4} \Big|_{x=1}^{\infty} = -\frac{1}{4}(0 - 1) = \frac{1}{4}.$$

So the integral converges, and thus the sum converges

Section 8.3, Problem 15 (webassign)

Determine whether the series $\sum_{n=1}^{\infty} ne^{-n}$ converges or diverges.

Note that $0 < e^{-n/2}$, $\sum_{n=1}^{\infty} e^{-n/2}$ converges being geometric series with $r = 1/\sqrt{e} < 1$, and

$$\lim_{n \rightarrow \infty} \frac{ne^{-n}}{e^{-n/2}} = \lim_{n \rightarrow \infty} ne^{-n/2} = 0,$$

since the exponential dominates. Thus, $\sum_{n=1}^{\infty} ne^{-n}$ converges by the *Limit Comparison Test*.

The *Comparison Test* can be used as well. If $f(x) = xe^{-x/2}$,

$$f'(x) = x'e^{-x/2} + x(e^{-x/2})' = e^{-x/2} + xe^{-x/2} \cdot (-1/2) = \frac{1}{2}e^{-x/2}(2-x).$$

So $f(x) \leq f(2) = 2e^{-2/2} < 1$ for $x \geq 2$ and thus $ne^{-n} \leq e^{-n/2}$ for all n . Since $ne^{-n} \geq 0$ and $\sum_{n=1}^{\infty} e^{-n/2}$

converges being geometric series with $r = 1/\sqrt{e} < 1$, $\sum_{n=1}^{\infty} ne^{-n}$ also converges

The *Integral Test* can also be used. The function $f(x) = xe^{-x}$ is positive and continuous for $x \geq 1$. Since

$$f'(x) = x'e^{-x} + x(e^{-x})' = e^{-x} + xe^{-x} \cdot (-1) = e^{-x}(1-x),$$

$f(x)$ is decreasing for $x \geq 1$. So the sum converges if and only if $\int_1^{\infty} xe^{-x} dx$ does. Integration by parts gives

$$\begin{aligned} \int_1^{\infty} xe^{-x} dx &= - \int_1^{\infty} xde^{-x} = - \left(xe^{-x} \Big|_1^{\infty} - \int_1^{\infty} e^{-x} dx \right) = - \left(\lim_{x \rightarrow \infty} xe^{-x} - 1e^{-1} + e^{-x} \Big|_1^{\infty} \right) \\ &= - (0 - e^{-1} + e^{-\infty} - e^{-1}) = 2e^{-1}. \end{aligned}$$

Since the integral is finite, $\sum_{n=1}^{\infty} ne^{-n}$ converges

However, the quickest way here is to use the *Ratio Test* of Section 8.4 (because of e^{-n}):

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)e^{-(n+1)}}{ne^{-n}} = \left(1 + \frac{1}{n}\right) \cdot \frac{e^{-n}e^{-1}}{e^{-n}} = \left(1 + \frac{1}{n}\right)e^{-1} \rightarrow \left(1 + \frac{1}{\infty}\right)e^{-1} = e^{-1}.$$

Since $e^{-1} = 1/e < 1$, the series converges

Section 8.3, Problem 20 (webassign)

Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$ converges or diverges.

The terms in this series look like $n^2/n^4 = 1/n^2$. So we limit-compare it to $\sum 1/n^2$; this is a p -series with $p=2 > 1$ and so converges. This limit-comparison can be made, since both series have positive terms for $n \geq 2$ and

$$\frac{(n^2-1)/(3n^4+1)}{1/n^2} = \frac{(n^2-1)n^2}{3n^4+1} = \frac{(n^2-1)n^2/n^4}{(3n^4+1)/n^4} = \frac{1-1/n^2}{3+1/n^4} \rightarrow \frac{1-1/\infty}{3+1/\infty} = \frac{1}{3}.$$

Since the p -series converges, our series converges as well.

We can also *compare* (as opposed to *limit-compare*) to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{3n^2} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Both series have positive terms for $n \geq 2$ and

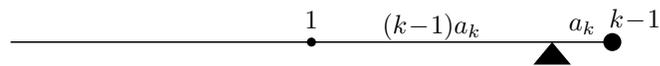
$$\frac{n^2-1}{3n^4+1} \leq \frac{n^2}{3n^4+1} \leq \frac{n^2}{3n^4} = \frac{1}{3n^2}.$$

Since our series is “smaller” than a convergent series, it also converges

Problem p631 6 (10pts)

Suppose you have a large supply of books, all of the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it (see figure on p631). Show that it is possible to do this so that the top book extends entirely beyond the table. In fact, show that the top book can extend any distance at all beyond the edge of the table if the stack is sufficiently high.

We need to stack the books so that the center of mass of the top $k-1$ books is precisely above the edge of the next book down or of the table (for the lowest book); this will keep the stack from collapsing. In this case, the center of mass of the top k books lies above a point on the lowest of these books which is $a_k = 1/2k$ of a book from the edge of this book (this center of mass divides the line segment between the center of mass of the lowest of these books and the center of mass of the top $k-1$ books in the ratio of $(k-1) : 1$; see figure below, which depicts the lowest of these books).



Thus, the k -th book from the top extends by $a_k = 1/2k$ beyond the edge of the next book down. So, if the stack consists of n books, the top one will extend by

$$s_n = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2n}$$

beyond the edge of the table. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n}$$

consists of positive terms and diverges by the p -series test (with $p=1$), its sequence of partial sums $\{s_n\}$, given by the above expression, is increasing to infinity. In particular, for *any* number M we can find n so that $s_n > M$; this says that the far edge of the top book in a stack of n books extends at least distance M beyond the edge of the table (if $M > 1$, then the top book lies entirely beyond the table).

Remark: If you have enough books, you can actually stack them so that they extend any distance M beyond the edge of the table *and* you can walk on top of them without having them collapse. *Do you see why?* The argument is very similar...