MAT 127: Calculus C, Spring 2015
Solutions to Some HW7 Problems

Below you will find detailed solutions to six problems from HW7. Since five of them were WebAssign problems, your versions of these problems may have had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 8.1, Problem 8 (webassign)

Find a formula for the general term a_n of the sequence

$$-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots$$

assuming the pattern continues.

First, the signs alternate beginning with -1; so the sign of a_n is $(-1)^n$. The numerator of a_n is n; the denominator is $(n+1)^2$. So $\boxed{a_n = (-1)^n \frac{n}{(n+1)^2}}$

Section 8.1, Problems 22 (webassign)

Determines whether the following sequence

$$a_n = \cos(2/n)$$

converges; if so, find its limit.

$$a_n = \ln\frac{2n^2+1}{n^2+1} = \ln\frac{2n^2/n^2+1/n^2}{n^2/n^2+1/n^2} = \ln\frac{2+1/n^2}{1+1/n^2} \longrightarrow \ln\frac{2+1/\infty^2}{1+1/\infty^2} = \ln\frac{2+0}{1+0} = \ln 2;$$

So the sequence $\boxed{\text{converges to } \ln 2}$

Section 8.1, Problems 27 (webassign)

Determine whether the sequence

$$a_n = \left(1+\frac{2}{n}\right)^n.$$

converges; if so, find its limit.

Replacing $n \longrightarrow \infty$ with $x \longrightarrow \infty$ makes sense in this case and

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty}\left(1+\frac{1}{n/2}\right)^n = \lim_{x\longrightarrow\infty}\left(1+\frac{1}{x/2}\right)^x = \lim_{x\longrightarrow\infty}\left(\left(1+\frac{1}{x/2}\right)^{(x/2)}\right)^2$$

$$= \left(\lim_{x\longrightarrow\infty}\left(1+\frac{1}{x}\right)^x\right)^2 = e^2\,;$$

the last equality uses box 5 on p225.

Here is another argument. Let $b_n = \ln a_n$, so that

$$b_n = n\cdot\ln\left(1+\frac{2}{n}\right) = \frac{\ln\left(1+2\frac{1}{n}\right)}{1/n} \quad\Longrightarrow\quad \lim_{n\longrightarrow\infty} b_n = \lim_{x\longrightarrow 0}\frac{\ln(1+2x)}{x} = \lim_{x\longrightarrow 0}\frac{\frac{1}{1+2x}\cdot 2}{1} = \frac{\frac{1}{1+2\cdot 0}\cdot 2}{1} = 2.$$

The above limit computation uses l'Hospital. It is applicable here, since $\ln(1+2x), x \longrightarrow 0$ as $x\longrightarrow 0$ (the top and bottom of a fraction must *both* approach 0 or $\pm\infty$ for l'Hospital to apply). Since $b_n \longrightarrow 2$, $a_n = \mathrm{e}^{b_n} \longrightarrow \mathrm{e}^2$.

Section 8.1, Problem 60

Define $\{a_n\}$ *by* $a_1 = 1$, $a_{n+1} = 1 + 1/(1+a_n)$.

(a) *Find the first eight terms of this sequence. What do you notice about the odd terms and the even terms.*

$$1,\ 1+\frac{1}{1+1} = \frac{3}{2} = 1.5,\ 1+\frac{1}{1+\frac{3}{2}} = 1+\frac{2}{5} = \frac{7}{5} = 1.4,\ 1+\frac{1}{1+\frac{7}{5}} = 1+\frac{5}{12} = \frac{17}{12} \approx 1.41667$$
$$1+\frac{1}{1+\frac{17}{12}} = 1+\frac{12}{29} = \frac{41}{29} \approx 1.41379,\ 1+\frac{1}{1+\frac{41}{29}} = 1+\frac{29}{70} = \frac{99}{70} \approx 1.41429$$
$$1+\frac{1}{1+\frac{99}{70}} = 1+\frac{70}{169} = \frac{239}{169} \approx 1.41420,\ 1+\frac{1}{1+\frac{239}{169}} = 1+\frac{169}{408} = \frac{577}{408} \approx 1.41422$$

The odd terms are increasing, the even terms are decreasing and are greater than the odd terms; the odd terms are smaller than $\sqrt{2} \approx 1.41421$, while the even ones are larger. Both sets of terms appear to approach $\sqrt{2}$.

(b) *By considering the odd and even terms separately, show that the sequence converges and its limit is* $\sqrt{2}$. *This gives the continued fraction expansion*

$$\sqrt{2} = 1 + \frac{1}{2+\frac{1}{2+\ldots}}$$

Since the odd and even terms are to be considered separately, we need to express a_{n+2} in terms of a_n:

$$\begin{aligned} a_{n+2} &= 1+\frac{1}{1+a_{n+1}} = 1+\frac{1}{1+1+\frac{1}{1+a_n}} = 1+\frac{1}{2+\frac{1}{1+a_n}} = 1+\frac{1+a_n}{3+2a_n} \\ &= \frac{4+3a_n}{3+2a_n} = \frac{3}{2} - \frac{1/2}{3+2a_n}. \end{aligned}$$

If $x>0$ and $x = (4+3x)/(3+2x)$, then $2x^2 - 4 = 0$, so that $x=\sqrt{2}$. Thus,

$$f(x) = \frac{4+3x}{3+2x} = \frac{3}{2} - \frac{1/2}{3+2x}$$

is a strictly increasing function of $x>0$ such that $f(\sqrt{2})=\sqrt{2}$ (so $f(x)<\sqrt{2}$ if and only if $x<\sqrt{2}$). Furthermore,

$$f(x) - x = \frac{4+3x}{3+2x} - x = \frac{4-2x^2}{3+2x}.$$

So $x < f(x) < \sqrt{2}$ if $x < \sqrt{2}$ and $x > f(x) > \sqrt{2}$ if $x > \sqrt{2}$. Since $a_{n+2} = f(a_n)$ and $a_1 < \sqrt{2}$, $a_n < a_{n+2} < \sqrt{2}$ for all n odd; since $a_2 > \sqrt{2}$, $a_n > a_{n+2} > \sqrt{2}$ for all n even. So, the sequence $\{a_{2n-1}\}_{n\ge1}$ is increasing and bounded above and thus converges by the Monotonic Sequence Theorem; the sequence $\{a_{2n}\}_{n\ge1}$ is decreasing and bounded below and thus converges by the Monotonic Sequence Theorem. Each of the sequences must converge to a non-negative number a such that $a = f(a)$; the only such number is $a = \sqrt{2}$. Since the even and the odd terms converge to $\sqrt{2}$, the entire sequence converges to $\sqrt{2}$.

Note: until it is established that the entire sequence converges, it is wrong to assume that the limit is a number a such that $a = 1 + 1/(1+a)$. If $a_{n+1} = f(a_n)$, it could be the case that the odd and even terms converge to different numbers a_o and a_e such that $a_o = f(a_e)$ and $a_e = f(a_0)$; this cannot happen in this case because f is a strictly increasing function for $x > 0$.

Section 8.2, Problem 32 (webassign)

Determine if the series

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$$

converges and find its sum by expressing s_n as a telescoping sum if converges.

By (quick) partial fractions,

$$\frac{1}{n^2+4n+3} = \frac{1}{(n+1)(n+3)} = \frac{1}{(+3)-((+1))}\left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$

Thus, for $n \ge 2$

$$s_n = \sum_{k=1}^{k=n}\left(\frac{1}{k+1} - \frac{1}{k+3}\right) = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{n+1} - \frac{1}{n+3}\right)$$
$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$

since the second term in each pair with $k \le n-2$ gets canceled by the first term in the pair $k+2$, leaving the first terms in the first two pairs and the second terms in the last two pairs. Since $1/n \longrightarrow 0$ as $n \longrightarrow \infty$, the sequence s_n converges; thus the series also $\boxed{\text{converges}}$ and

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{n\longrightarrow\infty} s_n = \frac{1}{2} + \frac{1}{3} = \boxed{\frac{5}{6}}$$

Section 8.2, Problems 36 (webassign)

Express the numbers $0.\overline{73} = 0.737373\ldots$

$$.\overline{73} = .73 + \frac{.73}{100} + \frac{.73}{100^2} + \ldots = \frac{73/100}{1-\frac{1}{100}} = \boxed{\frac{73}{99}}$$