MAT 127: Calculus C, Spring 2015
Solutions to Some HW4 Problems

Below you will find detailed solutions to two problems from HW4. Since they were part of the WebAssign portion of this homework, your versions of these problems likely had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 7.4, Problem 9 (webassign)

The half-life of cesium-137 is 30 years. Suppose we have a 100 mg sample.
(a) *Find the mass that remains after t years.*
(b) *How much of the sample remains after 100 years?*
(c) *After how long will only 1 mg remain?*

Let $y(t)$ be the portion of cesium-137 remaining after t years. Since we assume exponential decay,

$$y(t) = y(0)\mathrm{e}^{rt} = \mathrm{e}^{rt}\,,$$

where r is the relative decay rate; it is a constant. Since $y(30) = .5$,

$$\mathrm{e}^{30r} = \frac{1}{2} \qquad \Longleftrightarrow \qquad 30r = \ln 2^{-1} = -\ln 2 \qquad \Longleftrightarrow \qquad r = -(\ln 2)/30.$$

Thus,
$$y(t) = \mathrm{e}^{-(\ln 2)t/30} = \left(\mathrm{e}^{\ln 2}\right)^{(-t/30)} = 2^{-t/30}\,.$$

Since we start with 100 mg, the amount left after t years is $\boxed{100 \cdot 2^{-t/30}\ \text{mg}}$. In particular, the amount remaining after 100 years is

$$100 \cdot 2^{-100/30} = \boxed{100 \cdot 2^{-10/3} \approx 9.92\ \text{mg}}$$

The number of years t it will take for the sample to decline to 1 mg is given by

$$y(t) = 100 \cdot 2^{-t/30} = 1 \quad \Longleftrightarrow \quad 2^{-t/30} = 1/100 \quad \Longleftrightarrow \quad -t/30 = \log_2 10^{-2} = -2\log_2 10$$
$$\Longleftrightarrow \quad \boxed{t = 60\log_2 10 = 60\frac{\ln 10}{\ln 2} \approx 199.32\ \text{years}}$$

Section 7.4, Problem 20 (webassign)

(a) *How long will it take an investment to double in value if the interest rate is 6% compounded continuously?*
(b) *What is the equivalent annual interest rate?*

(a) Let $y(t)$ be the ratio of the balance after t years and of the original investment. Since $y(t)$ grows exponentially,

$$y(t) = y(0)\mathrm{e}^{rt} = \mathrm{e}^{rt}\,,$$

where r is the relative growth rate. In this case, $r = 6\% = .06$, so $y(t) = \mathrm{e}^{.06t}$. We need to find t so that $y(t) = 2$, i.e.

$$2 = \mathrm{e}^{.06t} \qquad \Longleftrightarrow \qquad \ln 2 = .06t \qquad \Longleftrightarrow \qquad \boxed{t = \tfrac{\ln 2}{.06} \approx 11.55 \text{ years}}$$

(b) Since $y(1) = \mathrm{e}^{.06} \approx 1.062$, the annual interest rate is $\boxed{\approx 6.2\%}$