MAT 127: Calculus C, Spring 2015
Solutions to Some HW2 Problems

Below you will find detailed solutions to two problems from HW2. Since they were part of the WebAssign portion of this homework, your versions of these problems likely had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 7.2, Problems 3-6 (webassign)

Match the differential equation with its direction field (labeled I-IV on p506 in the book). Give reasons for your answer.

$$(3)\ y' = 2 - y, \qquad (4)\ y' = x(2-y), \qquad (5)\ y' = x + y - 1, \qquad (6)\ y' = \sin x \sin y.$$

The slopes y' in (3) are independent of x, so do not change under horizontal shifts of the picture. This is the case only in III.

The slopes y' in (4) are 0 for $x = 0$ (the y-axis) and $y = 2$, and so are represented by horizontal line segments on the y-axis and the line $y = 2$. This is the case only in I.

The slope y' in (5) at the origin $(0, 0)$ is negative, represented by a downward-sloping segment. This is the case only in IV.

This leaves only II for (6). As a check, note that the slopes of y' in (6) vanish for $x = 0$ and $y = 0$ (the two axes). This is the case only in II.

(3) III; (4) I; (5) IV; (6) II

Section 7.2, Problem 23 (webassign)

Use Euler's method with step size .1 to estimate $y(.5)$, where $y(x)$ is the solution of the initial-value problem

$$y' = y + xy, \quad y = y(x), \qquad y(0) = 1.$$

Since the initial-value of x is 0 and the final is .5, while the step size is $h = .1$, there are

$$n = \frac{x_{\text{final}} - x_{\text{initial}}}{h} = \frac{.5 - 0}{.1} = 5$$

steps to make. Thus, we need to find estimates y_1, y_2, y_3, y_4, y_5 for $y(x_i)$ with

$$x_i = x_{\text{initial}} + hi = 0 + \frac{1}{10} i = \frac{i}{10}\,.$$

They are obtained inductively by computing the slope $s_i = y'(x_i, y_i)$ at (x_i, y_i) and moving along the slope line from x_i to $x_{i+1} = x_i + h$ so that $y_{i+1} = y_i + s_i h$. The results of this computation are:

i	x_i	y_i	$s_i = y_i(1 + x_i)$	$y_{i+1} = y_i + \frac{1}{10} s_i$
0	0	1	$1(1 + 0) = 1$	$1 + \frac{1}{10} = \frac{11}{10}$
1	$\frac{1}{10}$	$\frac{11}{10}$	$\frac{11}{10}\left(1 + \frac{1}{10}\right) = \frac{121}{100}$	$\frac{11}{10} + \frac{121}{1000} = \frac{1221}{1000}$
2	$\frac{1}{5}$	$\frac{1221}{1000}$	$\frac{1221}{1000}\left(1 + \frac{1}{5}\right) = \frac{3663}{2500}$	$\frac{1221}{1000} + \frac{3663}{25{,}000} = \frac{8547}{6250}$
3	$\frac{3}{10}$	$\frac{8547}{6250}$	$\frac{8547}{6250}\left(1 + \frac{3}{10}\right) = \frac{111{,}111}{62{,}500}$	$\frac{8547}{6250} + \frac{111{,}111}{625{,}000} = \frac{965{,}811}{625{,}000}$
4	$\frac{2}{5}$	$\frac{965{,}811}{625{,}000}$	$\frac{965{,}811}{625{,}000}\left(1 + \frac{2}{5}\right) = \frac{6{,}760{,}677}{3{,}125{,}000}$	$\frac{965{,}811}{625{,}000} + \frac{6{,}760{,}677}{31{,}250{,}000} = \boxed{\frac{55,051,227}{31,250,000}}$

See the previous problem for comments.

Note 1: Our estimate y_5 is an *under*-estimate for $y(.5)$ because $y'' < 0$ for all $x \in (0, .5)$ and for all solutions $y = y(x)$ used to estimate $y(.5)$, and so the tangent lines lie below the graphs. In order to see this, note that $y' = y(1+x) > 0$ if $y > 0$ and $x \geq 0$. Thus, a solution $y = y(x)$ of the differential equation is increasing for all $x \geq x_0 \geq 0$ if $y(x_0) > 0$. For such a solution y,

$$y'' = (y + xy)' = y' + y + xy' = (y+xy) + y + x(y+xy) = y(2+2x+x^2) > 0 \quad \text{for all} \quad x \geq x_0,$$

as claimed above.

Note 2: Since the above initial-value problem involves a separable differential equation, it can be solved (please do so!) to find that the actual solution is

$$y(x) = \mathrm{e}^{x + \frac{x^2}{2}} .$$

Once this function is given, one can check directly that it solves the initial-value problem (please do this also!). The above formula for the solution gives

$$y(.5) = \mathrm{e}^{\frac{1}{2} + \frac{1}{8}} = \mathrm{e}^{\frac{5}{8}} \approx 1.868.$$

So the above estimate

$$\frac{55,051,227}{31,250,000} = 1.761639264$$

is not great despite of the relatively small step size; this is because the second derivative is large.