MAT 127: Calculus C, Spring 2015
Solutions to Some HW11 Problems

Below you will find detailed solutions to four problems from HW11. Since the first three of them were WebAssign problems, your versions of these problems may have had different numerical coefficients. However, the principles behind the solutions and their structure are as described below.

Section 8.6, Problems 13

Find a power series representation for the function $f(x)=\ln(5-x)$ *and its interval of convergence.*

The geometric series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n\,, \tag{1}$$

converges if and only if $|x|<1$. Integrating this series from $x=0$, we obtain

$$\ln(1-x)=\int_0^x\frac{-1}{1-t}\mathrm{d}t=-\int_0^x\sum_{n=0}^{\infty}t^n\mathrm{d}t=-\sum_{n=0}^{\infty}\int_0^x t^n\mathrm{d}t=-\sum_{n=0}^{\infty}\frac{t^{n+1}}{n+1}\Big|_{t=0}^{t=x}=-\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}=-\sum_{n=1}^{\infty}\frac{x^n}{n}\,.$$

For the last equality, we can renumber the summands, replacing $n+1$ with n, so that the sum begins with $n=1$ instead of $n=0$.

We will relate our function to the power series representation

$$\ln(1-x)=-\sum_{n=1}^{\infty}\frac{x^n}{n}\,. \tag{2}$$

By the *Ratio Test* (which is not effected by powers of n), the radius of convergence of this series is still $R=1$. This series diverges for $x=1$ by the *p-Series Test* and converges for $x=-1$ by the *Alternating Series Test*. So, the series (2) converges if and only if $-1\le x<1$.

In order to make our series look like LHS of (2), use log rules to pull out 5

$$\ln(5-x)=\ln\big(5\cdot(1-x/5)\big)=\ln 5+\ln(1-x/5)=\ln 5-\sum_{n=1}^{\infty}\frac{(x/5)^n}{n}=\boxed{\ln 5-\sum_{n=1}^{\infty}\frac{x^n}{n5^n}}$$

Since series (2) converges if and only if $-1\le x<1$ and we substitute $x/5$ for x, our series converges if and only if $-1\le x/5<1$, i.e. of $-5\le x<5$; so the interval of convergence is $\boxed{[-5,5)}$

Remark 1: More generally, the *radius* of convergence is not affected by differentiation/integration of power series, but the *interval* of convergence may be effected. However, differentiation can only *drop* both or either of the endpoints from the interval of convergence, but in this case there are no endpoints to drop since the interval of convergence for the series (1) does not have any.

Remark 2: The answer to the first part of this question must be a power series in x, and not in $(x/5)$, etc.; thus, the expressions preceding the box cannot be the final answers.

Section 8.6, Problem 23

Evaluate the indefinite integral $\int \frac{t}{1-t^8}\mathrm{d}t$ *as a power series and find its radius of convergence.*

Put $x = t^8$ into series (1) and integrate:

$$\int \frac{t}{1-t^8}\mathrm{d}t = \int t\sum_{n=0}^{\infty}(t^8)^n\mathrm{d}t = \sum_{n=0}^{\infty}\int t^{8n+1}\mathrm{d}t = \boxed{C + \sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2}}$$

Since integration does not change the radius of convergence, the radius of convergence of the last series is the same as the radius of convergence of the series

$$\frac{1}{1-t^8} = \sum_{n=0}^{\infty} t^{8n}\,.$$

Since series (1) converges if and only if $|x| < 1$ and we substitute t^8 for x, the last series converges if and only if $|t^8| < 1$, i.e. for $|t| < 1$. Thus, the radius of convergence of the last series and of the integrated series is $\boxed{1}$ For $t = \pm 1$, the integrated series becomes

$$\sum_{n=0}^{\infty}\frac{(\pm 1)^{8n+2}}{8n+2} = \sum_{n=0}^{\infty}\frac{1}{8n+2}\,;$$

this series diverges by Limit Comparison the p-Series $\sum_{n=1}^{\infty}\frac{1}{n}$ or Comparison with $\frac{1}{10}\sum_{n=1}^{\infty}\frac{1}{n}$. So the interval of convergence of the integrated series is still $(-1, 1)$.

Section 8.6, Problem 38

(a) *Starting the geometric series* $f(x) = \sum_{n=0}^{\infty} x^n$, *find the sum of the series*

$$\sum_{n=1}^{\infty} nx^{n-1} \qquad |x| < 1.$$

Since $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$,

$$\sum_{n=1}^{\infty} nx^{n-1} = \left(\sum_{n=0}^{\infty} x^n\right)' = \left(\frac{1}{1-x}\right)' = \boxed{\frac{1}{(1-x)^2} \quad \text{if } |x| < 1}$$

(b) *Find the sum of each of the following series:*

$$\text{(i)} \quad \sum_{n=1}^{\infty} nx^n \quad |x| < 1, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{n}{2^n}$$

By part (a),

$$\sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = x \cdot \frac{1}{(1-x)^2} = \boxed{\frac{x}{(1-x)^2}}$$

Since $|1/2| < 1$,

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} nx^n\Big|_{x=1/2} = \frac{x}{(1-x)^2}\Big|_{x=1/2} = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = \boxed{2}$$

(c) *Find the sum of each of the following series:*

$$\text{(i)} \quad \sum_{n=2}^{\infty} n(n-1)x^n \quad |x| < 1, \qquad \text{(ii)} \quad \sum_{n=2}^{\infty} \frac{n^2-n}{2^n}, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Similarly to parts (a) and (b),

$$\sum_{n=2}^{\infty} n(n-1)x^n = x^2 \sum_{n=2}^{\infty} n(n-1)x^{n-2} = x^2\left(\sum_{n=0}^{\infty} x^n\right)'' = x^2\left(\frac{1}{1-x}\right)'' = x^2 \frac{2}{(1-x)^3} = \boxed{\frac{2x^2}{(1-x)^3}}$$

Since $|1/2| < 1$,

$$\sum_{n=2}^{\infty} \frac{n^2-n}{2^n} = \sum_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} n(n-1)x^n\Big|_{x=1/2} = \frac{2x^2}{(1-x)^3}\Big|_{x=1/2} = \frac{2(1/2)^2}{(1-1/2)^3} = \frac{1/2}{1/8} = \boxed{4}$$

Combining this with (b) gives

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=2}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n} = 4 + 2 = \boxed{6}$$

Section 8.6, Problem 40

(a) *By completing the square, show that*

$$\int_0^{1/2} \frac{\mathrm{d}x}{x^2 - x + 1} = \frac{\pi}{3\sqrt{3}}$$

Since

$$x^2 - x + 1 = \frac{3}{4} + \left(x - 1/2\right)^2 = \frac{3}{4}\left(1 + \frac{(x-1/2)^2}{(\sqrt{3}/2)^2}\right) = \frac{3}{4}\left(1 + \left(\frac{x-1/2}{\sqrt{3}/2}\right)^2\right) = \frac{3}{4}\left(1 + \left((2x-1)/\sqrt{3}\right)^2\right),$$

we obtain

$$\int_0^{1/2} \frac{\mathrm{d}x}{x^2 - x + 1} = \frac{4}{3}\int_0^{1/2} \frac{\mathrm{d}x}{1 + ((2x-1)/\sqrt{3})^2} = \frac{2}{\sqrt{3}}\int_{-1/\sqrt{3}}^{0} \frac{\mathrm{d}u}{1+u^2} = \frac{2}{\sqrt{3}} \arctan\Big|_{-1/\sqrt{3}}^{0}$$
$$= \frac{2}{\sqrt{3}}\left(\arctan 0 - \arctan(-1/\sqrt{3})\right) = \frac{2}{\sqrt{3}}\left(0 + \arctan(1/\sqrt{3})\right) = \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}},$$

where $u = (2x-1)/\sqrt{3}$.

(b) *By factoring* $x^3 + 1$ *as a sum of cubes, rewrite the integral in (a). Then express* $1/(x^3+1)$ *as the sum of a power series and use it to show that*

$$\pi = \frac{3\sqrt{3}}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{8^n}\left(\frac{2}{3n+1} + \frac{1}{3n+2}\right).$$

Since $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$,

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty}(-x^3)^n = \sum_{n=0}^{\infty}(-1)^n(x^3)^n = \sum_{n=0}^{\infty}(-1)^n x^{3n}$$

if $|x^3| < 1$, or equivalently $|x| < 1$. Since $x^3 + 1^3 = (x+1)(x^2 - x + 1)$,

$$\frac{1}{x^2 - x + 1} = \frac{1+x}{1+x^3} = (1+x)\sum_{n=0}^{\infty}(-1)^n x^{3n} = \sum_{n=0}^{\infty}(-1)^n\left(x^{3n} + x^{3n+1}\right)$$

if $|x| < 1$. Since $|x| < 1$ whenever $0 < x < 1/2$,

$$\int_0^{1/2} \frac{\mathrm{d}x}{x^2 - x + 1} = \int_0^{1/2}\left(\sum_{n=0}^{\infty}(-1)^n\left(x^{3n} + x^{3n+1}\right)\right)\mathrm{d}x = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x^{3n+1}}{3n+1} + \frac{x^{3n+2}}{3n+2}\right)\Big|_0^{1/2}$$
$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{(1/2)^{3n+1}}{3n+1} + \frac{(1/2)^{3n+2}}{3n+2}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{3n}2^2}\left(\frac{2}{3n+1} + \frac{1}{3n+2}\right)$$
$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{8^n}\left(\frac{2}{3n+1} + \frac{1}{3n+2}\right)$$

Comparing this result with the statement in (a), we obtain

$$\pi = \frac{3\sqrt{3}}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{8^n}\left(\frac{2}{3n+1} + \frac{1}{3n+2}\right).$$

Problem p632, 11

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$.

The principle with such problems is to *guess* a function $f(x)$ with a simple power series representation,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

so that the given power series is obtained by replacing x with some number a. If this a lies in the interval of convergence for the power series, then the sum of the given series is simply $f(a)$. The hard part is usually to guess f correctly.

In the given case, the coefficients in the power series are reciprocals of odd integers $1/(2n+1)$. This is similar to the series in Example 7 in Section 8.6:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{if} \quad -1 < x \leq 1.$$

So, we relate our series to this series:

$$\sum_{n=1}^{\infty} (-1)^n \frac{(1/\sqrt{3})^{2n}}{2n+1} = \sqrt{3} \sum_{n=1}^{\infty} (-1)^n \frac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sqrt{3}\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} - x \right)\Bigg|_{x=1/\sqrt{3}}$$

$$= \sqrt{3}\left(\arctan\left(\frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{3}} \right) = \sqrt{3}\left(\frac{\pi}{6} - \frac{1}{\sqrt{3}} \right) = \boxed{-\frac{6 - \pi\sqrt{3}}{6}}$$