MAT 127 Final Exam

December 14, 2009
8:15-10:45am

Name:____________________________ **ID:**__________________

Section:	L01	L02	L03	L04	(circle yours)
	MWF 9:35-10:30am	MW 5:20-6:45pm	TuTh 2:20-3:40pm	TuTh 5:20-6:40pm	

DO NOT OPEN THIS EXAM YET

Instructions

(1) This exam is closed-book and closed-notes; no calculators, no phones.
(2) Please write legibly. Circle or box your final answers.
(3) Show your work. Correct answers only will receive only partial credit.
(4) Simplify your answers as much as possible.
(5) Leave your answers in exact form (e.g. $\sqrt{2}$, not ≈ 1.4).
(6) If you need more blank paper, ask a proctor.
(7) Please write your name and ID number on any additional sheets you'd like to be graded and staple them to the back of the exam (stapler provided); indicate in the exam that the solution continues on the attached sheets.
(8) Anything handed in will be graded; incorrect statements will be penalized even if they are in addition to complete and correct solutions. If you do not want something graded, please erase it or cross it out.

Out of fairness to others, please **stop working and close the exam as soon as the time is called**. A significant number of points will be taken off your exam score if you continue working after the time is called. You will be given a two-minute warning before the end.

Some Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{if} \quad |x| < 1 \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

to receive full credit, justify any other power series expansion you use

1 (10pts)	
2ab+c (20pts)	
3 (20pts)	
4 (20pts)	
Subtotal (70pts)	

5 (20pts)	
6 (20pts)	
7 (20pts)	
8A/B (20pts)	
Subtotal (80pts)	

Total (150pts)	

Problem 1 (10pts)

Determine whether each of the following sequences or series converges or not. In each case, *clearly* circle either **YES** or **NO**, but not both. Each correct answer is worth 2 points. You may use the blank space between the questions to figure out the answer, but no partial credit will be awarded and no justification is expected for your answers on this problem.

(a) the sequence $a_n = 1 - (-1)^n$ **YES** **NO**

(b) the sequence $a_n = 1 + \dfrac{\cos(n)}{n^2}$ **YES** **NO**

(c) the series $\displaystyle\sum_{n=1}^{\infty} \cos(1/n)$ **YES** **NO**

(d) the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ **YES** **NO**

(e) the series $\displaystyle\sum_{n=1}^{\infty} \frac{n + \sin n}{n^2}$ **YES** **NO**

Problem 2 (20pts)

Find Taylor series expansions of the following functions around the given point. In each case, determine the radius of convergence of the resulting power series and its interval of convergence.

(a; 10pts) $f(x) = x^3$ around $x = 2$

(b; 10pts) $f(x) = \dfrac{x}{1 - 4x^2}$ around $x = 0$

(c; bonus 10pts)[1] $f(x) = \dfrac{1}{6 - 5x + x^2}$ around $x = 0$

[1]this part is relatively hard and subject to harsh grading; do and double-check the rest of the exam first

Problem 3 (20pts)

(a; 8pts) Find the radius and interval of convergence of the power series

$$f(x) = \sum_{n=1}^{\infty} \sqrt{n} x^n .$$

(b; 4pts) Find $\lim_{x \longrightarrow 0} \frac{f(x) - x}{x^2}$

(c; 8pts) Find the Taylor series expansion for the function $g = g(x)$ given by

$$g(x) = \int_0^x \frac{f(u) - u}{u^2} \mathrm{d}u$$

around $x = 0$. What are the radius and interval of convergence of this power series?

Problem 4 (20pts)

Show that the following series are convergent and find their sums.

(a; 10pts) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n (\ln 3)^n}{n!}$

(b; 10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$

Problem 5 (20pts)

Explain why each of the following series converges. Then estimate its sum to within 1/18 using the minimal possible number of terms, justifying your estimate; leave your answer as a simple fraction p/q for some integers p and q with no common factor. Is your estimate an under- or over-estimate for the sum? Explain why.

(a; 10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

(b; 10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

Problem 6 (20pts)

Find the general real solution to each of the following differential equations.

(a; 6pts) $9y'' + 4y = 0, \quad y = y(x)$

(b; 7pts) $9y'' - 12y' + 4y = 0, \quad y = y(x)$

(c; 7pts) $9y'' - 12y' = 0, \quad y = y(x)$

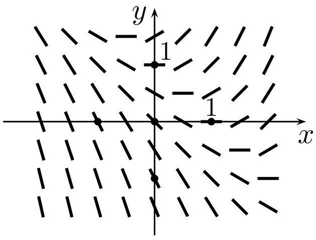
Problem 7 (20pts)

Consider the four differential equations for $y = y(x)$:

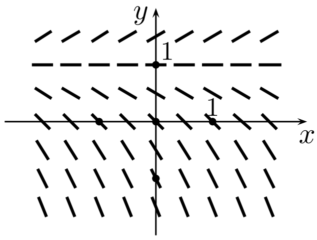
$$\text{(a) } y' = y - 1, \qquad \text{(b) } y' = y^2 - 1, \qquad \text{(c) } y' = x(y^2 - 1) \qquad \text{(d) } y' = x + y - 1.$$

Each of the two diagrams below shows the direction field for one of these equations:

I

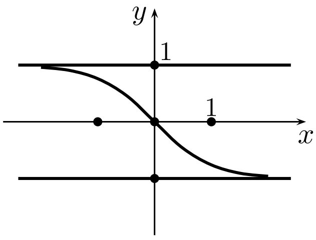


II

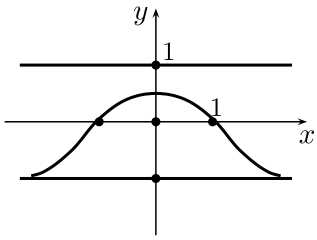


Each of the two diagrams below shows three solution curves for one of these equations:

III



IV



(so ALL three curves in diagram III are solution curves for either (a), or (b), or (c), or (d); same (?) for ALL three curves)

Match each of the diagrams to the corresponding differential equation (the match is one-to-one):

diagram	I	II	III	IV
equation				

Explain your reasoning below.

Problem 8A (20pts)

Only the higher of your scores on Problems 8A and 8B will count toward the total for the exam

A tank contains 100 liters of salt solution with 500 grams of salt dissolved in it. A salt solution containing 2g of salt per liter enters the tank at a rate of 5 liters per minute. The solution is kept thoroughly mixed and drains at a rate of 5L/min (so the volume in the tank stays constant). Let $y(t)$ be the amount of salt in the tank, measured in grams, after t minutes.

(a; 8pts) Explain (based on the above information) why the function $y = y(t)$ solves the initial-value problem

$$y' = 10 - \frac{y}{20}, \quad y = y(t), \qquad y(0) = 500.$$

(b; 8pts) Find the solution $y = y(t)$ to the initial-value problem stated in (a).

(c; 4pts) How long will it take for the amount of salt in the tank to reach 300 grams?

Problem 8B (20pts)

Only the higher of your scores on Problems 8A and 8B will count toward the total for the exam

(a; 8pts) Show that the orthogonal trajectories to the family of curves $xy = k$ are described by the differential equation

$$y' = \frac{x}{y}, \qquad y = y(x).$$

(b; 6pts) Find the general solution to the differential equation stated in (a).

(c; 6pts) Sketch at least 3 representatives of the original family of curves and at least 3 orthogonal trajectories on the same diagram; indicate clearly which is which.

you can write on this page