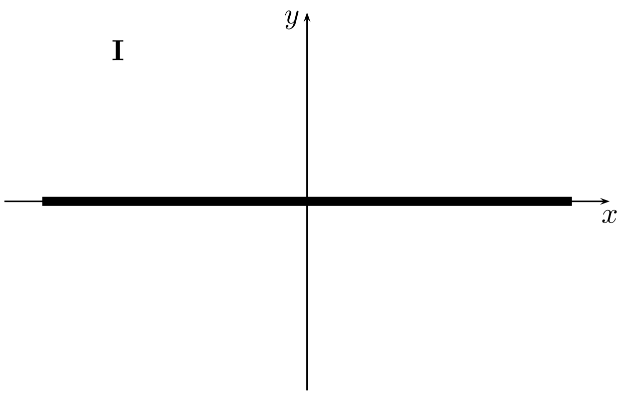
MAT 127: Calculus C, Fall 2010
Solutions to Midterm I

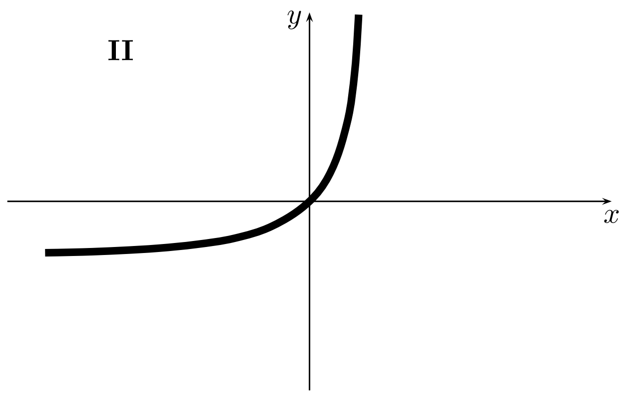
Problem 1 (10pts)

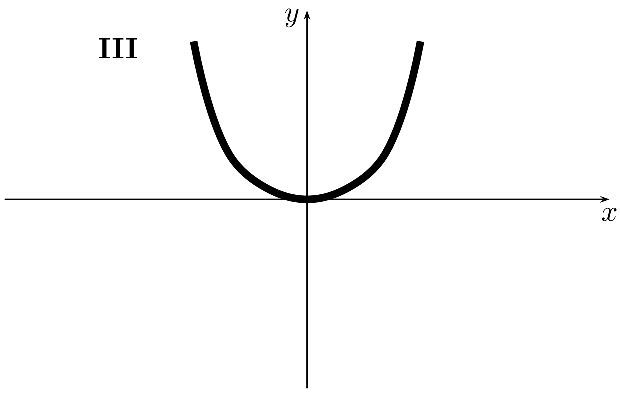
Consider the four differential equations for $y = y(x)$:

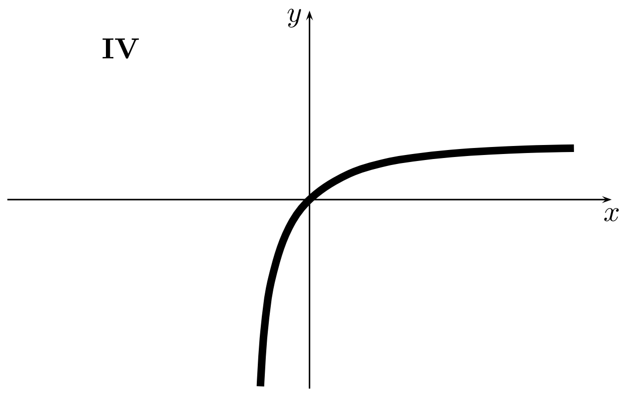
(a) $y' = x(1+y^2)$ (b) $y' = y(1+x^2)$ (c) $y' = e^{x+y}$ (d) $y' = e^{-x-y}$.

Each of the four diagrams below shows a solution curve for one of these equations:









Match each of the diagrams to the corresponding differential equation (the match is one-to-one):

diagram	I	II	III	IV
equation	b	c	a	d

Answer Only: *no explanation is required.*

Explanation is on the next page.

Of the four diagrams, the most distinctive is **I**; it shows the graph of the constant function $y(x) = 0$. Plugging in this function into the four equations, we get

$$\text{(a)}\ \ 0 \overset{?}{=} x(1+0^2) \qquad \text{(b)}\ \ 0 \overset{?}{=} 0(1+x^2) \qquad \text{(c)}\ \ 0 \overset{?}{=} \mathrm{e}^{x+0} \qquad \text{(d)}\ \ 0 \overset{?}{=} \mathrm{e}^{-x-0}\,.$$

Of these four potential equalities, only (b) is satisfied for all x; so diagram **I** must correspond to (b).

Of the three remaining graphs, the most distinctive is **III**; the slope of the graph at $(0,0)$ there is 0. This is the case for the slope of equation (a) at (0,0) (because $0(1+0^2) = 0$), but the slopes for the other two remaining equations are always positive (because e^x is always positive). So diagram **III** must correspond to (a).

The two remaining graphs look rather similar. One distinguishing feature is that the graph in **II** becomes steeper as x, y increase, while the graph in **IV** less steep as x, y increase. Considering the two remaining equations, e^{x+y} becomes larger as x, y increase (thus making the slope of the graph steeper), while e^{-x-y} becomes smaller as x, y increase (thus making the slope of the graph less steep). So, diagram **II** must correspond to (c), while diagram **IV** must correspond to (d).

Alternatively, diagram **II** depicts the graph of a function with $y'' > 0$, while diagram **IV** depicts the graph of a function with $y'' < 0$. If $y = y(x)$ is any function satisfying equation (c), then by the *Chain Rule*

$$y'' = (y')' = \big(\mathrm{e}^{x+y}\big)' = \mathrm{e}^{x+y} \cdot (x+y)' = \mathrm{e}^{x+y} \cdot (1+y') = \mathrm{e}^{x+y} \cdot (1+\mathrm{e}^{x+y}) > 0,$$

because $\mathrm{e}^{x+y} > 0$. On the other hand, if $y = y(x)$ is any function satisfying equation (d), then by the *Chain Rule*

$$y'' = (y')' = \big(\mathrm{e}^{-x-y}\big)' = \mathrm{e}^{-x-y} \cdot (-x-y)' = -\mathrm{e}^{-x-y} \cdot (1+y') = -\mathrm{e}^{-x-y} \cdot (1+\mathrm{e}^{-x-y}) < 0,$$

because $\mathrm{e}^{-x-y} > 0$. So, **II** must correspond to (c), while **IV** must correspond to (d).

Grading:

correct − repeats	0-	1	2	3	4
points	0	2	5	9	10

Problem 2 (15pts)

(a; 7pts) *Show that the function* $y(x)=x\mathrm{e}^{-2x}$ *is a solution to the initial-value problem*

$$y''+4y'+4y=0, \quad y=y(x), \qquad y(0)=0, \qquad y'(0)=1.$$

Show your work and/or explain your reasoning.

Compute $y'(x)$ and $y''(x)$ to check that the differential equation is satisfied:

$$\begin{aligned} y(x)=x\mathrm{e}^{-2x} \quad &\Longrightarrow \quad y'(x)=\mathrm{e}^{-2x}+x\mathrm{e}^{-2x}\cdot(-2)=\mathrm{e}^{-2x}-2x\mathrm{e}^{-2x} \\ &\Longrightarrow \quad y''(x)=\mathrm{e}^{-2x}\cdot(-2)-2\big(\mathrm{e}^{-2x}+x\mathrm{e}^{-2x}\cdot(-2)\big)=-4\mathrm{e}^{-2x}+4x\mathrm{e}^{-2x} \\ &\Longrightarrow \quad y''+4y'+4y=\big(-4\mathrm{e}^{-2x}+4x\mathrm{e}^{-2x}\big)+4\big(\mathrm{e}^{-2x}-2x\mathrm{e}^{-2x}\big)+4x\mathrm{e}^{-2x} \\ &\qquad\qquad\qquad\quad =(-4+4)\mathrm{e}^{-2x}+(4-8+4)x\mathrm{e}^{-2x}=0. \checkmark \end{aligned}$$

It remains to check that the initial condition are satisfied:

$$y(0)=0\cdot\mathrm{e}^{-2\cdot0}=0, \checkmark \qquad y'(0)=\mathrm{e}^{-2\cdot0}-2\cdot0\mathrm{e}^{-2\cdot0}=1. \checkmark$$

Grading: $y'(x)$ 1pt; $y''(x)$ from $y'(x)$ 2pts; plug in into DE and simplify 1pt each; check initial conditions 1pt each; no carry-over penalties

(b; 8pts) *Find the general solution of the differential equation*

$$y''+4y'+4y=0, \quad y=y(x).$$

Show your work and/or explain your reasoning.

Since $y(x)=x\mathrm{e}^{-2x}$ is a solution of this equation (by above), $y(x)=\mathrm{e}^{-2x}$ is also a solution (by the structure theorem for solutions of such equations) and the general solution of the differential equation is

$$\boxed{y(x)=C_1\mathrm{e}^{-2x}+C_2x\mathrm{e}^{-2x}}$$

Alternatively, the associated polynomial equation for this differential equation is

$$r^2+4r+4=0 \qquad \Longleftrightarrow \qquad (r+2)^2=0.$$

So there is a double root of $r=-2$, which again gives the above general solution.

Grading: in the 2nd approach, 3pts for the roots, 1pt each for $y(x)$ containing e^{-2x} and $x\mathrm{e}^{-2x}$, with remainder for the correct formula; if the roots are obtained incorrectly, 1pt for setting up the polynomial and up to 4 additional points for converting the roots to the corresponding equation for $y(x)$; in the 1st approach, a little bit of explanation is required (anything in parentheses above is not required); if no explanation is given, 4pts for correct answer, 2pts if $y(x)$ contains only one constant *or* missing e^{-2x} or $x\mathrm{e}^{-2x}$; 1pt for $y(x)=C\mathrm{e}^{-2x}$ or $y(x)=Cx\mathrm{e}^{-2x}$.

Solution to this problem continues on the next page.

Alternatively, one could first use the second approach in (b) to find that the general solution to the differential equation is

$$y(x) = C_1 e^{-2x} + C_2 x e^{-2x}$$

and then find C_1 and C_2 so that the two initial conditions in (a) hold. In order to do this, compute $y'(x)$, $y(0)$, and $y'(0)$ and set them equal to the given initial condition:

$$\begin{aligned} y'(x) &= C_1 e^{-2x} \cdot (-2) + C_2\big(e^{-2x} + x e^{-2x} \cdot (-2)\big) = (C_2 - 2C_1)e^{-2x} - 2C_2 x e^{-2x}\,, \\ y(0) &= C_1 e^{-2\cdot 0} + C_2 \cdot 0 \cdot e^{-2\cdot 0} = C_1 \qquad y'(0) = (C_2 - 2C_1)e^{-2\cdot 0} - 2C_2 \cdot 0 \cdot e^{-2\cdot 0} = (C_2 - 2C_1)\,, \\ &\implies \quad \begin{cases} y(0) = C_1 = 0 \\ y'(0) = C_2 - 2C_1 = 1 \end{cases} \qquad \implies \qquad C_1 = 0,\ C_2 = 1. \end{aligned}$$

So the solution to the initial-value problem in (a) is

$$y(x) = 0 \cdot e^{-2x} + 1 \cdot x e^{-2x} = x e^{-2x}\,,$$

which is the function given in (a).

Grading: finding the general solution as in the second approach to part (b) on the previous page; after that, finding $y'(x)$ 2pts, setting up the system 2pts; finding C_1 and C_2 2pts, and concluding the argument 1pt.

Remark: using the last approach indicates solid understanding of how to solve initial-value problems, but also misunderstanding of the basic concept of how to check that a given function solves a given initial-value problem.

Problem 3 (15pts)

A sample of tritium-3 (a radioactive substance) decayed to 90% of its original amount in 2 years.

(a; 10pts) *Let $y = y(t)$ be the ratio of the amount of tritium-3 remaining after t years to the original amount. Find a formula for $y(t)$. Show your work and/or explain your reasoning.*

Since the decay rate is proportional to y, $y(t)$ satisfies the exponential decay equation:

$$y(t) = y(0)\mathrm{e}^{-rt} = 1 \cdot \mathrm{e}^{-rt},$$

where r is the relative decay rate. By the last assumption,

$$y(2) = \mathrm{e}^{-r\cdot 2} = .9 \quad \Longleftrightarrow \quad -2r = \ln .9 \quad \Longleftrightarrow \quad -r = (\ln .9)/2.$$

Thus, $\boxed{y(t) = \mathrm{e}^{(\ln .9)t/2} = .9^{t/2}}$

Grading: $y(t) = \mathrm{e}^{-rt}$ 3pts; $\mathrm{e}^{-2r} = .9$ 2pts; finding r 2pts; final answer 3pts (either form acceptable, as well %, but there should be no units otherwise); finding r not necessary for full credit (something like $\mathrm{e}^{-rt} = (\mathrm{e}^{-2r})^{t/2}$ would do instead); correct answer with no explanation 4pts.

(b; 5pts) *If the initial weight of the sample was 100 mg (at time $t = 0$), what is the weight of the sample after 6 years (from $t = 0$)? Show your work and/or explain your reasoning.*

Since every 2 years the amount drops by a factor of .9, in $6 = 3 \cdot 2$ years this happens 3 times; so the amount drops by a factor of $.9 \cdot .9 \cdot .9 = .729$. Since the original amount was 100 mg, the amount after 6 years is then

$$100 \cdot .729 = \boxed{72.9 \text{ mg}}$$

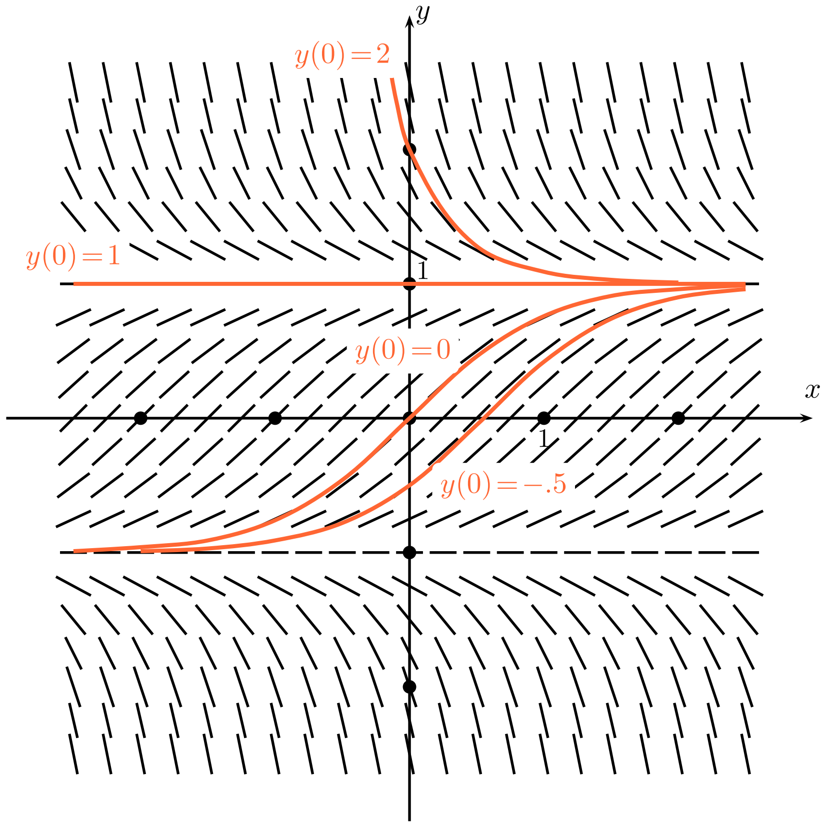
Alternatively, plug in $t = 6$ in $y(t)$ in part (a) and multiply by 100 mg:

$$100 \cdot y(t) = 100 \cdot .9^{6/2} = \boxed{72.9 \text{ mg}}$$

Grading: correct answer in exactly the same form as above (or 729/10 mg) with no explanation is 3pts; 2pts off if the units are missing or incorrect; plugging in $t = 6$ and multiplying by 100 in the second approach is 1 point each; if the final answer involves $\mathrm{e}^{3(\ln .9)}$, then 0pts for part (b); if $y(t)$ is found incorrectly in (a) and is used in the second approach to (b), then the same grading principles apply to the function $y(t)$ found in (a).

Problem 4 (20pts)

The direction field for a differential equation is shown below.



(a; 16pts) *On the direction field, sketch and clearly label the graphs of the four solutions with the initial conditions* $y(0) = -.5$, $y(0) = 0$, $y(0) = 1$, *and* $y(0) = 2$ *(each of these four conditions determines a solution to the differential equation). No explanation is required.*

The four curves must pass through the points $(0, -.5)$, $(0, 0)$, $(0, 1)$, and $(0, 2)$, respectively. These curves must be tangent to the slope lines at those points and roughly approximate the slopes everywhere else. They should never intersect.

The solution curves with $y(0) = -.5$ and $y(0) = 0$ ascend toward the line $x = 1$ as $x \longrightarrow \infty$ and descend toward the line $x = -1$ as $x \longrightarrow -\infty$. The solution curve with $y(0) = 1$ is the horizontal line $y = 1$. The solution curve $y(0) = 2$ descends toward the line $x = 1$ as $x \longrightarrow \infty$ and ascends rapidly as x decreases.

Grading: 4pts for each curve (passing through specified point and roughly tangent there 1pt each; general shape ok 2pts; the 3rd curve must be the correct line to receive the last 2pts); curves clearly intersect (as opposed to being asymptotic) 4pts off; one curve not labeled 1pt off, 2-4 curves not labeled 2pts off.

Solution to this problem continues on the next page.

(b; 4pts) *The direction field above is for one of the following differential equations for $y = y(x)$:*

$$\boxed{(i)}\ y' = 1 - y^2\,, \qquad (ii)\ y' = y^2 - 1\,, \qquad (iii)\ y' = y^4 - 1\,.$$

Which of these three equations does the direction field correspond to and why?

The slopes for $y \in (-1, 1)$ are positive in the diagram, while the slopes for $y < -1$ and for $y > 1$ are negative. Each of these three properties holds only for the first of the three equations.

Grading: wrong answer 0pts regardless of explanation; correct answer 2pts; justification clearly ruling out the other answers up to 2pts (one of several possible reasons would suffice).

Problem 5 (20pts)

(a; 15pts) *Let $y = y(x)$ be the solution to the initial-value problem*

$$y' = x^2 - \frac{1}{3}y, \quad y = y(x), \qquad y(1) = 0.$$

Use Euler's method with $n = 3$ steps to estimate the value of $y(2)$. Show your steps clearly and use simple fractions (so 5/4 or $\frac{5}{4}$, not 1.25).

The step size is $h = (2-1)/3 = 1/3$, so we need to obtain estimates y_1, y_2, y_3 for the y-value at $x_1 = 4/3$, $x_2 = 5/3$, and $x_3 = 2$ starting with the value $y_0 = 0$ of y at $x_0 = 1$:

$$\begin{aligned}
s_0 &= x_0^2 - \frac{1}{3}y_0 = 1^2 - 0 = 1 && \Longrightarrow && y_1 = y_0 + s_0 h = 0 + \frac{1}{3} = \frac{1}{3}\\
s_1 &= x_1^2 - \frac{1}{3}y_1 = \frac{16}{9} - \frac{1}{9} = \frac{5}{3} && \Longrightarrow && y_2 = y_1 + s_1 h = \frac{1}{3} + \frac{5}{9} = \frac{8}{9}\\
s_2 &= x_2^2 - \frac{1}{3}y_2 = \frac{25}{9} - \frac{8}{27} = \frac{67}{27} && \Longrightarrow && y_3 = y_2 + s_2 h = \frac{8}{9} + \frac{67}{81} = \frac{72+67}{81} = \boxed{\frac{139}{81}}
\end{aligned}$$

Alternatively, this can be done using a table:

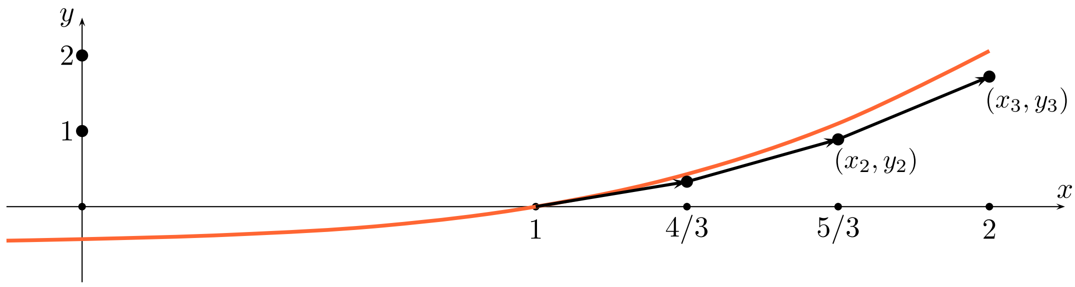
i	x_i	y_i	$s_i = x_i^2 - \frac{1}{3}y_i$	$y_{i+1} = y_i + \frac{1}{3}s_i$
0	1	0	$1^2 - 0 = 1$	$0 + \frac{1}{3} = \frac{1}{3}$
1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{16}{9} - \frac{1}{9} = \frac{5}{3}$	$\frac{1}{3} + \frac{5}{9} = \frac{8}{9}$
2	$\frac{5}{3}$	$\frac{8}{9}$	$\frac{25}{9} - \frac{8}{27} = \frac{67}{27}$	$\frac{8}{9} + \frac{67}{81} = \frac{72+67}{81} = \boxed{\frac{139}{81}}$

The first column consists of the numbers i running from 0 to $n-1$, where n is the number of steps (3 in this case). The second column starts with the initial value of x (1 in this case) with subsequent entries in the column obtained by adding the step size h ($\frac{1}{3}$ in this case); it ends just before the final value of x would have been entered ($2 = \frac{5}{3} + \frac{1}{3}$ in this case). Thus, the first two columns can be filled in at the start. The first entry in the third column is the initial y-value (0 in the case). After this, one computes the first entries in the remaining two columns and copies the first entry in the last column to the third column in the next line. The process then repeats across the second and third rows. The estimate for the final value of y is the last entry in the table.

Grading: h, x_0, x_1, x_2 1 pt each; correct recursive setup 5pts (1pt each for number of steps, slope and change equations, left end points, end of the procedure); each of 2 steps in each of 3 computations 1pt each

Solution to this problem continues on the next page.

(b; 5pts) *Sketch the path in the xy-plane that represents the approximation carried out in part (a) and indicate its (path's) primary relation to the graph of the actual solution $y = y(x)$ of the initial-value problem in (a).*



This path consists of the three connected line segments from (x_i, y_i) to (x_{i+1}, y_{i+1}) with $i = 0, 1, 2$. The first of these is tangent to the solution curve at $(1, 0)$.

Grading: 3 line segments 1pt, correct end points 2pts, solution curve through (1,0) 1pt, tangent to the 1st line segment 1pt; axes not labeled 1pt off; comments not required

(c; bonus 5pts, all or nothing) *Is your estimate for $y(2)$ in part (a) an over-estimate (larger than $y(2)$) or an under-estimate? Justify your answer; no credit for correct answer only.*

If $y = y(x)$ is a solution of the differential equation in (a), then

$$y'' = (y')' = \left(x^2 - \frac{1}{3}y\right)' = 2x - \frac{1}{3}y' = 2x - \frac{1}{3}\left(x^2 - \frac{1}{3}y\right) = \frac{x}{3}(6 - x) + \frac{1}{9}y.$$

Thus, $y'' > 0$ if $x \in (0, 6)$ and $y > 0$. Furthermore, if $y(x_0) = 0$ for some $x_0 \neq 0$, then

$$y'(x_0) = x_0^2 - \frac{1}{3} \cdot 0 > 0;$$

so if $y(x_0) \geq 0$ for some $x_0 > 0$, then $y(x) \geq 0$ for all $x \geq 0$. It follows that if $y(x_0) \geq 0$ for some $x_0 > 0$, then $y''(x) > 0$ for all $x \in (x_0, 6)$.

By the previous paragraph, the first line segment, which is tangent to the solution curve at (x_0, y_0), lies below this solution curve. Similarly, the second line segment, which is tangent to the solution curve at (x_1, y_1), lies below this new solution curve. Finally, the third line segment, which is tangent to the solution curve at (x_2, y_2), lies below this last solution curve. Since solution curves do not intersect, it follows that the entire 3-segment approximation path lies below the solution curve through (x_0, y_0) and so $y_3 < y(x_3)$, i.e. our estimate y_3 for $y(2)$ is an $\boxed{\text{under-estimate}}$

Grading: no partial credit; explanation must be complete on the substance in order to receive the 5 bonus points.

Problem 6 (20pts)

(a; 5pts) What are the constant solutions of the differential equation

$$y' = 2y(y-2), \qquad y = y(x) \ \ ?$$

Show your work and/or explain your reasoning.

The constant solutions correspond to the values of y so that $2y(y-2) = 0$ (for all x in general, but there is no x in this case to worry about). Thus, the constant solutions are $\boxed{y(x) = 0 \text{ and } y(x) = 2}$

Grading: correct answer 3pts (can be written as $y = 0, 2$, etc.); minimal explanation, such as setting RHS=0, 2pts; listing $y = 0$ or $y = 2$ only 1pt total, regardless of explanation.

(b; 15pts) *Find the general solution to the differential equation*

$$y' = 2y(y-2), \qquad y = y(x).$$

Simplify your answer as much as possible. Show your work and/or explain your reasoning.

This differential equation is separable, so write y' as $\mathrm{d}y/\mathrm{d}x$ and move all terms involving y to LHS and all terms involving x to RHS, and integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y(y-2) \iff \frac{\mathrm{d}y}{y(y-2)} = 2\mathrm{d}x \iff \int \frac{\mathrm{d}y}{y(y-2)} = \int 2\mathrm{d}x = 2x + C.$$

In order to do the y-integral, we need to use partial fractions:

$$\frac{1}{y(y-2)} = \frac{1}{((-2)) - ((-0))}\left(\frac{1}{y(-0)} - \frac{1}{y(-2)}\right) = \frac{1}{-2}\left(\frac{1}{y} - \frac{1}{y-2}\right) = \frac{1}{2}\left(\frac{1}{y-2} - \frac{1}{y}\right).$$

Thus,

$$\int \frac{\mathrm{d}y}{y(y-2)} = \frac{1}{2}\int\left(\frac{1}{y-2} - \frac{1}{y}\right)\mathrm{d}y = \frac{1}{2}\big(\ln|y-2| - \ln|y|\big) + C' = \frac{1}{2}\ln\left|\frac{y-2}{y}\right| + C'.$$

Combining this with the first line gives

$$\ln\left|\frac{y-2}{y}\right| = 4x + C \iff \left|\frac{y-2}{y}\right| = \mathrm{e}^{4x+C} = \mathrm{e}^{4x}\mathrm{e}^{C} \iff \frac{y-2}{y} = \pm\mathrm{e}^{4C}\mathrm{e}^{4x} = A\mathrm{e}^{4x},$$

where A is any nonzero constant. This equation defines y implicitly as a function of x, but it can be simplified further:

$$1 - \frac{2}{y} = A\mathrm{e}^{4x} \iff 1 - A\mathrm{e}^{4x} = \frac{2}{y} \iff y = \frac{2}{1 - A\mathrm{e}^{4x}}.$$

We now have to remember the constant solutions found in (a): $y = 0$ and $y = 2$. The latter corresponds to $A = 0$ in the above formula. So the general solution of the differential equation is:

$$\boxed{y(x) = 0, \quad y(x) = \frac{2}{1 - A\mathrm{e}^{4x}}}$$

Solution to this problem continues on the next page.

Grading: "integrating" the equation to something like $y = (2/3)y^3 - 2y^2$ or $y = 2y(y-2)x$ 0pts for the entire question; separation of variables 1pt; RHS integral 1pt; LHS integral 5pts with penalties for computational errors; missing absolute value -1pt; RHS is "integrated" to something like $\ln|y(y-2)|$ loss of entire 5pts; simplification from anti-derivatives to the final form 5pts; "simplifying" e^{4x+C} to $\mathrm{e}^{4x}+\mathrm{e}^{C}$, etc. loss of entire 5pts; incorporating the constant solutions into the final answer 3pts (if they are listed twice or $y = 2$ is not absorbed into the formula -2pts); if separation of variables is done incorrectly in a significant way (as opposed to minor copying errors, such as factors of 2) 5pts max for the entire question (with reductions for missing constant solutions, etc.)

Alternatively, this differential equation can be re-written as a "logistic growth equation",

$$y' = ry\left(1 - \frac{y}{K}\right), \qquad y = y(x),$$

the general solution of which is

$$y(x) = \frac{K}{1 - A\mathrm{e}^{-rx}}, \qquad y(x) = 0.$$

In this case,

$$y' = 2y(y-2) = -2y(2-y) = -4y\left(1 - \frac{y}{2}\right),$$

and so $r = -4$ and $K = 2$. Putting these into the general solution again gives

$$\boxed{y(x) = \frac{2}{1 - A\mathrm{e}^{4x}}, \qquad y(x) = 0}$$

While in an actual logistic growth equation $r, K > 0$, this does not matter in terms of finding the general solution of the equation ($K \neq 0$ is needed, since there is division by K).

Grading: restating DE as "logistic" equation with clear indication that this is the purpose 5pts; formula with correct r and K 7pts (5pts off if the sign in exponent is wrong); 3pts for the zero constant solution (2pts off if $y = 2$ is also listed separately)