Midterm Exam II: MAT 310

Instructions: Complete all problems below. You may not use calculators or other electronic devices, including cell phones. Show all of your work. Be sure to write your name and student ID on each page that you hand in.

1.(14pts) Determine the values of \( \alpha \) for which the following system of linear equations is consistent, and write down the general solution.

\[
\begin{align*}
3x_1 + 2x_2 + 3x_3 - 2x_4 &= 1 \\
x_1 + x_2 + x_3 &= \alpha \\
x_1 + 2x_2 + x_3 - x_4 &= 2
\end{align*}
\]

2.(15pts) Let

\[
A = \begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 9 & 0 & 2 \\ 8 & 3 & 2 & -2 \\ 4 & 3 & 1 & 1 \end{pmatrix}.
\]

a) Calculate the determinant of \( A \) using any method that you know.

b) What is the determinant of \( -2A \)?

3.(13pts) Let \( A \) and \( B \) be \( n \times n \) matrices such that \( AB = -BA \). Prove that if \( n \) is odd then \( A \) or \( B \) is not invertible.

4.(13pts) Determine if the following matrix is diagonalizable and justify your answer. If so, find an invertible matrix \( Q \) and a diagonal matrix \( D \) such that \( A = QDQ^{-1} \).

\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

5.(15pts) Solve the following system of differential equations with initial condition:

\[
\begin{align*}
x_1' &= 3x_1 - 2x_2, \quad x_1(0) = 1, \\
x_2' &= 2x_1 - 2x_2, \quad x_2(0) = -1.
\end{align*}
\]

6.(15pts) Let \( A \) be an \( n \times n \) matrix. Prove that the set \( \{ I, A, \ldots, A^{n-1}, A^k \} \) is linearly dependent for every integer \( k \geq n \).

7.(15pts) Compute the Jordan canonical form \( J \) of the matrix \( A \) in problem 4 above. In addition, find an invertible matrix \( Q \) such that \( A = QJQ^{-1} \).