Midterm Exam I: MAT 310

Instructions: Complete all problems below. You may not use calculators or other electronic devices, including cell phones. Show all of your work. Be sure to write your name and student ID on each page that you hand in.

1. (15pts) If $U$ and $W$ are subspaces of a vector space $V$, show that $U \cup W$ need not be a subspace. However, if $U \cup W$ is a subspace, show that either $U \subset W$ or $W \subset U$.

2. (14pts) Solve the following system of linear equations.

\[
\begin{align*}
    x_1 + 2x_2 - x_3 + x_4 &= 5 \\
    x_1 + 4x_2 - 3x_3 - 3x_4 &= 6 \\
    2x_1 + 3x_2 - x_3 + 4x_4 &= 8
\end{align*}
\]

3. (13pts) Determine whether or not \{$(1, 1, 0), (2, 0, -1), (-3, 1, 1)$\} is a basis for $\mathbb{R}^3$. Justify your answer.

4. (15pts) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ denote a linear transformation such that $T((1, 0, 0, 0)) = (3, -1, 0)$, $T((1, 1, 1, 1)) = (-2, 1, 3)$, and $T((0, 0, 1, 1)) = (0, 1, 1)$. Compute the dimension of the null space $\text{dim}(N(T))$.

5. (15pts) Let $\alpha = \{1, x, x^2\}$ and $\beta = \{x^2+x, x-1, x^2+x+1\}$ be two different bases for the vector space of polynomials $P_2(\mathbb{R})$. Compute the change of coordinates matrix $Q$ which changes $\alpha$-coordinates to $\beta$-coordinates. Next, compute the coordinates $[1 + x]_\beta$.

6. (15pts) Let $T : V \rightarrow W$ and $L : W \rightarrow Z$ be linear transformations between vector spaces $V$, $W$, and $Z$. Prove that if $LT$ is onto then $L$ is onto. Must $T$ also be onto?

7. (13pts) Let $C^\infty$ denote the vector space (over the complex numbers) of infinitely differentiable complex valued functions defined on the real line. Find the general solution to the homogeneous differential equation $p(D)y = 0$ where

\[ p(x) = (x^2 + 1)^2(x^2 + 4x + 3). \]