Review Sheet: Final

The final will cover sections 1.1-1.6, 3.1-3.6 as well as 5.1, 5.2, 5.4, and 5.5; in addition to all material covered in the lectures. You may bring two sheets of paper ($8.5\text{in} \times 11\text{in}$) to the test, on which you may record any helpful formulas, problems, etc... However, calculators and other computing devices will not be permitted. The exam will take place on Thursday, May 19, from 2:15pm-4:45pm in the usual classroom. The following list of problems will help in preparing for the exam.

1. Solve

$$a)\ xy' + 2y = x^2 - x + 1, \quad y(1) = \frac{1}{2}, \qquad b)\ y' + (\tan x)y = x \sin 2x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

2. Determine an interval in which the solution is certain to exist,

$$a)\ (x-3)y' + (\log x)y = 2x, \quad y(1) = 2, \qquad b)\ y' + (\tan x)y = \sin x, \quad y(\pi) = 0.$$

3. Solve

$$a)\ y' = \frac{x - e^{-x}}{y + e^{y}}, \qquad b)\ y' = 1 + x + y^2 + xy^2.$$

4. Determine the region in the xy-plane where the hypotheses of the existence and uniqueness theorem apply. Thus there is a unique solution through each given initial point in this region.

$$a)\ y' = \frac{2xy}{1 + y^2}, \qquad b)\ y' = \sqrt{1 - x^2 - y^2}.$$

5. The radioactive isotope thorium-234 disintegrates at a rate proportional to the amount present. If 100mg of this material is reduced to 82mg in one week, find an expression for the amount present at any time. Also, find the time interval that must elapse for the mass to decay to one-half of its original value.

6. At time $t = 0$ a tank contains 10lb of salt dissolved in 100gal of water. Assume that water containing 1/4lb of salt per gallon is entering the tank at a rate of 3gal/min, and that the well-stirred solution is leaving the tank at the same rate. Find an expression for the amount of salt in the tank at time t.

7. A 4lb roast, initially at 50 degrees is placed in a 375 degree oven at 5pm. After 75 minutes it is found that the temperature $T(t)$ of the roast is 125 degrees. When will the roast be 150 degrees (medium rare)?

8. An object of mass m is dropped from rest in a medium that offers resistance proportional to velocity. Assuming the gravitational force to be constant, find the position and velocity of the object at any time t.

9. A body of constant mass m is projected upward from the earth's surface with an initial velocity v_0. Assuming that there is no air resistance, but taking into account the variation of the earth's gravitational field with altitude, find an expression for the velocity during the ensuing motion. Also find the initial velocity that is required to lift the body to a given maximum altitude ξ above the surface of the earth, and the smallest initial velocity for which the body will not return to the earth, ie. the escape velocity.

10. Solve

$$a)\ 2x + y^2 + 2xyy' = 0, \qquad b)\ (3xy + y^2) + (x^2 + xy)y' = 0.$$

11. Solve

$$a)\ y' = \frac{y^2 + 2xy}{x^2}, \qquad b)\ y' = -\frac{4x + 3y}{2x + y}.$$

12. Determine whether the pair of functions are linearly independent or dependent:

$$a)\ f(x) = x^2 + 5x,\ \ g(x) = x^2 - 5x, \qquad b)\ f(x) = e^{3x},\ \ f(x) = e^{3(x-1)}.$$

13. If y_1 and y_2 are zero at the same point of an interval (a, b), can they be a fundamental set of solutions on this interval?

14. Solve

$$a)\ y'' - 3y' - 4y = -8e^x \cos 2x, \qquad b)\ y'' - 3y' - 4y = e^{-x}.$$

15. Solve

$$a)\ y'' + 4y = 3 \csc x, \qquad b)\ x^2y'' + xy' - y = x \log x.$$

Note that $y_1(x) = x$ and $y_2(x) = x^{-1}$ are solutions of the homogeneous equation for part (b).

16. Suppose that a mass weighing 10lb stretches a spring 2in. If the mass is displaced an additional 2in and is then set in motion with an initial upward velocity of one foot per second, determine the position of the mass at any later time. Also determine the period, amplitude, and phase of the motion.

17. If an external force $F_0 \cos \omega t$ is applied to the spring in the previous problem, determine the values of ω for which resonance occurs.

18. Find the general solution of the systems:

$$a)\ \vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}, \qquad b)\ \vec{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{x}, \qquad c)\ \vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}.$$

19. For each of the systems in the previous problem, find a fundamental matrix and calculate the matrix exponential.

20. Suppose that we have a two brine tank system with constant flow rate of 10gal/min going into and out of both tanks. If the volume of tank I is 50gal and that of tank II is 25gal, find the amount of salt in both tanks as a function of time, assuming that the original amount in tank I is 15lbs and that of tank II is 0lbs.