

MAT 127: Calculus C, Spring 2015
Solutions to Final Exam

Problem 1 (10pts)

Determine whether each of the following sequences or series converges or not. In each case, clearly circle either **YES** or **NO**, but not both. Each correct answer is worth 2 points.

(a) the sequence $a_n = 2 - \frac{\cos n}{n^{1/2}}$

(YES)

NO

Since $\cos n/n^{1/2} \rightarrow 0$ (because $|\cos n| \leq 1$), $a_n \rightarrow 2$

(b) the sequence $a_n = n^3 \sin(2/n) - 2n^2$

(YES)

NO

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} \frac{\sin(2/n) - 2/n}{(1/n)^3} = \lim_{x \rightarrow 0^+} \frac{\sin(2x) - 2x}{x^3} = \lim_{x \rightarrow 0^+} \frac{2 \cos(2x) - 2}{3x^2} \\ &= \lim_{x \rightarrow 0^+} \frac{-4 \sin(2x)}{6x} = \lim_{x \rightarrow 0^+} \frac{-8 \cos(2x)}{6} = -\frac{4}{3}. \end{aligned}$$

The 3rd-5th equalities use l'Hospital, which is applicable here because the 3 numerators and the 3 denominators approach 0 as $x \rightarrow 0$. *Alternatively*, using $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, we find that

$$a_n = n^3 \left(\frac{(2/n)^1}{1!} - \frac{(2/n)^3}{3!} + \dots \right) - 2n^2 = -\frac{4}{3} + \dots$$

where ... involve $1/n^2, 1/n^4$, and so on. As $n \rightarrow \infty$, a_n thus approaches $-4/3$.

This question is related to Problem 4.

(c) the series $\sum_{n=1}^{\infty} \frac{2n^2 + (-1)^n}{n^3 + 1}$

YES

(NO)

$$\frac{2n^2 + (-1)^n}{n^3 + 1} \text{ looks like } n^2/n^3 = 1/n: \quad \frac{(2n^2 + (-1)^n)/(n^3 + 1)}{1/n} = \frac{2n^3 + (-1)^n n}{n^3 + 1} = \frac{2 + (-1)^n/n^2}{1 + 1/n^3} \rightarrow 2$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p -series test ($p=1 \leq 1$) and both series are nonnegative, by the Limit

Comparison Test our series also diverges. *Alternatively*,

$$\sum_{n=1}^{\infty} \frac{2n^2 + (-1)^n n}{n^3 + 1} = \sum_{n=1}^{\infty} \frac{2n^2}{n^3 + 1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3 + 1}$$

The last series on RHS converges by the Alternating Series Test (it is alternating and the absolute values of its terms are strictly decreasing to 0). The first series on RHS diverges by Comparison or

Limit Comparison to the p -series $\sum_{n=1}^{\infty} \frac{1}{n}$ or by the Integral Test with $f(x) = 2x^2/(x^3+1)$.

Since the sum of a divergent series and a *convergent* series is divergent, our series diverges.

(d) the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{3n-2}$

YES

(NO)

Since the sequence $(-1)^n \frac{n}{3n-2} = (-1)^n \frac{1}{3-2/n} \rightarrow \pm \frac{1}{3}$ does not approach 0, our series diverges by the Test for Divergence.

(e) the series $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{2^n + 10^n}}$

(YES)

NO

$$\frac{3^n}{\sqrt{2^n + 10^n}} \text{ looks like } \frac{3^n}{\sqrt{10^n}} = \sqrt{\frac{9}{10}} : \quad \frac{3^n/\sqrt{2^n+10^n}}{\sqrt{9/10^n}} = \frac{3^n/\sqrt{9^n}}{\sqrt{2^n+10^n}/\sqrt{10^n}} = \frac{1}{\sqrt{(2/10)^n + (10/10)^n}} \rightarrow 1$$

Since $3 < \sqrt{10}$, the geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{\sqrt{10}}\right)^n$ converges. Since both series are nonnegative, by the

Limit Comparison Test our series also converges.

Problem 2 (15pts)

Answer Only. Put your answers to (a) and (b) below in the corresponding box in the simplest possible form. No credit will be awarded if the answer in the box is wrong; partial credit may be awarded if the answer in the box is correct, but not in the simplest possible form.

(a; 5pts) Write the number $1.1\bar{6} = 1.16666\dots$ as a simple fraction

$\frac{7}{6}$

$$1.1\bar{6} = 1.1 + .06 + .06 \cdot \frac{1}{10} + .06 \cdot \frac{1}{10^2} + \dots = \frac{11}{10} + \frac{6/100}{1 - \frac{1}{10}} = \frac{11}{10} + \frac{3/5}{9} = \frac{11}{10} + \frac{1}{15} = \frac{33+2}{30} = \frac{7}{6}$$

Grading: wrong answer 0pts; as above 5pts; $1\frac{7}{6}$ or fraction not simplified 4pts; both issues 3pts

(b; 5pts) Find the limit of the sequence

5

$$1 + 2\sqrt{2}, \quad 1 + 2\sqrt{2 + \sqrt{2}}, \quad 1 + 2\sqrt{2 + \sqrt{2 + \sqrt{2}}}, \quad 1 + 2\sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}, \quad \dots$$

Assume that this sequence converges.

The limit is $1+2a$, where $a = \lim a_n$ for the sequence recursively defined by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2+a_n}$. Thus, $a = \sqrt{2+a}$ and $a^2 = 2+a$; so $a=2$ or -1 . The latter is impossible, since $a \geq 0$.

Grading: wrong answer 0pts; as above 5pts; not simplified 4pts

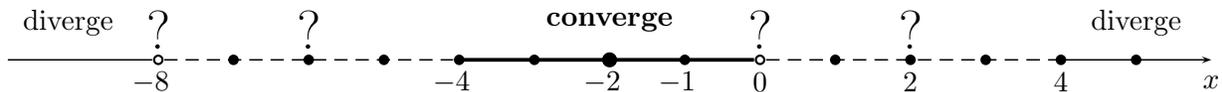
(c; 5pts) Suppose the power series $\sum_{n=0}^{\infty} c_n(x+2)^n$ converges at $x = -4$ and diverges at $x = 4$. What can be said about the convergence of this power series at the following values of x ?

$x = -2 :$	converges	<i>diverges</i>	<i>not enough info to tell</i>
$x = -1 :$	converges	<i>diverges</i>	<i>not enough info to tell</i>
$x = 0 :$	<i>converges</i>	<i>diverges</i>	not enough info to tell
$x = 2 :$	<i>converges</i>	<i>diverges</i>	not enough info to tell
$x = 6 :$	<i>converges</i>	diverges	<i>not enough info to tell</i>

The center of this power series is $a = -2$. Its radius of convergence R satisfies

$$|-2 - (-4)| = 2 \leq R \leq |-2 - 4| = 6.$$

Thus, the interval of convergence, which is centered at $a = -2$, must be of the following form



A power series always converges at its center ($x = -2$ in this case).

Problem 3 (20pts)

Find Taylor series expansions of the following functions around the given point. In each case, determine the radius of convergence of the resulting power series and its interval of convergence. Show your work.

(a; 10pts) $f(x) = x^3 - 6x$ around $x = 1$

In this case, all derivatives can be computed:

$$\begin{aligned} f^{(0)}(x) = x^3 - 6x &\implies f^{(0)}(1) = -5, & f^{(1)}(x) = 3x^2 - 6 &\implies f^{(1)}(1) = -3, \\ f^{(2)}(x) = 6x &\implies f^{(2)}(1) = 6, & f^{(1)}(x) = 6 &\implies f^{(1)}(1) = 6, \end{aligned}$$

and $f^{(n)}(x) = 0$ if $n \geq 4$. So by the Main Taylor Formula:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = \frac{-5}{0!} (x-1)^0 + \frac{-3}{1!} (x-1)^1 + \frac{6}{2!} (x-1)^2 + \frac{6}{3!} (x-1)^3 \\ &= \boxed{-5 - 3(x-1) + 3(x-1)^2 + (x-1)^3} \end{aligned}$$

Since this series is a sum of finitely many (four) terms, it converges for all x . So the interval of convergence is $\boxed{(-\infty, \infty)}$, while the radius is $\boxed{\infty}$

Remark: you can check the Taylor series expansion by expanding the expression in the long box above and getting $x^3 - 6x$.

Grading: statement of general Taylor formula 1pt, with $a=1$ 3pts (*not* in addition to 1pt); vanishing of higher derivatives 1pt and computation of the remaining derivatives 2pts (no separate statement is required if general Taylor formula with $a=1$ is stated and correctly applied in this case); final answer 1pt; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct); at least 2pts off if final answer is not a polynomial in $(x-1)$.

(b; 10pts) $f(x) = \frac{x}{2-x^3}$ around $x = 0$

Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and this power series converges if $|x| < 1$,

$$\frac{x}{2-x^3} = \frac{x}{2} \cdot \frac{1}{1-(x^3/2)} = \frac{x}{2} \sum_{n=0}^{\infty} \left(\frac{x^3}{2}\right)^n = \frac{x}{2} \sum_{n=0}^{\infty} \frac{1}{2^n} (x^3)^n = \boxed{\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^{3n+1}}$$

and this series converges whenever

$$|x^3/2| < 1 \iff |x^3| < 2 \iff -\sqrt[3]{2} < x < \sqrt[3]{2};$$

so the interval of convergence is $\boxed{(-\sqrt[3]{2}, \sqrt[3]{2})}$ and the radius is $\boxed{\sqrt[3]{2}}$

Grading: use of correct standard power series 2pts (no separate statement is required if properly used in the given case); substitution and multiplication 3pts; 2pts for simplifying to a power series in x ; interval of convergence, radius of convergence, and explanation 1pt each (with 3pt loss if the interval of convergence is not correct, except end-points error 1pt off).

Problem 4 (20pts)

(a; 8pts) Find the radius and interval of convergence of the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!} x^{2n}.$$

To find the radius of convergence, use the Ratio Test with $a_n = (-4)^n x^{2n} / (2n+1)! \neq 0$:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{4^{n+1} |x|^{2n+2} / (2n+3)!}{4^n |x|^{2n} / (2n+1)!} = 4|x|^2 \frac{(2n+1)!}{(2n+3)!} = \frac{4|x|^2}{(2n+2)(2n+3)} \rightarrow 0.$$

Since $0 < 1$, the series converges for all x . Thus, the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$

Alternatively,

$$\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!} x^{2n} = \frac{1}{2x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} = \frac{\sin(2x)}{2x},$$

if $x \neq 0$. The series for $\sin(x)$ converges for all x . Thus, the series for $\sin(2x)$ also converges for all x . So, the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$

Grading: radius of convergence 2pt, interval of convergence 1pt, justification 5pts

(b; 4pts) Find $\lim_{x \rightarrow 0} \frac{f(x) - 1}{x^2}$

$$\frac{f(x) - 1}{x^2} = \frac{(1 - \frac{4}{3!}x^2 + \frac{4^2}{5!}x^4 + \dots) - 1}{x^2} = \frac{-\frac{4}{3!}x^2 + \frac{4^2}{5!}x^4 - \dots}{x^2} = -\frac{4}{3!} + \frac{4^2}{5!}x^2 - \dots \rightarrow -\frac{4}{6} + 0 = \boxed{-\frac{2}{3}}$$

Grading: expanding the series 2pts; indication of computation to the answer 2pts; use of l'Hospital rule (twice) is fine, but the 0/0 assumption must be checked each time, with 1pt off for each of the times the check is missing; if the answer is correct, at least 1pt for this part.

(c; 8pts) Find the Taylor series expansion for the function $g = g(x)$ given by

$$g(x) = \int_0^x u^2 f(u) du$$

around $x=0$. What are the radius and interval of convergence of this power series?

$$g(x) = \int_0^x u^2 f(u) du = \int_0^x u^2 \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!} u^{2n} du = \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!} \int_0^x u^{2n+2} du = \boxed{\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+1)!(2n+3)} x^{2n+3}}$$

Since integration does not change the radius of convergence of a power series, the radius of convergence of this power series is still ∞ and so the interval of convergence is still $(-\infty, \infty)$

Grading: integrand as a power series 2pts; power series for g 2pts; radius and interval of convergence 1pt total; explanation 3pts (use of ratio test is ok); no penalty for carry-over errors from (a) or inconsistencies in answers between (a) and (c).

(d; bonus 10pts) Find $g(\pi/3)$.

From the alternative approach in part (a),

$$\begin{aligned}g(\pi/3) &= \frac{1}{2} \int_0^{\pi/3} u \sin(2u) \, du = -\frac{1}{4} \int_0^{\pi/3} u \, d \cos(2u) = -\frac{1}{4} \left(u \cos(2u) \Big|_0^{\pi/3} - \int_0^{\pi/3} \cos(2u) \, du \right) \\&= -\frac{1}{4} \left(u \cos(2u) - \frac{1}{2} \sin(2u) \right) \Big|_0^{\pi/3} = -\frac{1}{4} \left(\left(\frac{\pi}{3} \cos(2\pi/3) - \frac{1}{2} \sin(2\pi/3) \right) \Big|_0^{\pi/3} - \left(0 - \frac{1}{2} \sin(0) \right) \right) \\&= -\frac{1}{4} \left(-\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right) = \boxed{\frac{2\pi+3\sqrt{3}}{48}}\end{aligned}$$

Grading: integrand and limits as above 6pts; integration 2pts; final computation 2pts, with likely penalties for carryover errors

Alternatively, start with (c):

$$\begin{aligned}g(x) &= \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+3)!} (2n+2)x^{2n+3} = \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+3)!} (2n+3)x^{2n+3} - \sum_{n=0}^{\infty} \frac{(-4)^n}{(2n+3)!} x^{2n+3} \\&= \frac{x}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} (2x)^{2n+2} - \frac{1}{8} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+3)!} (2x)^{2n+3} \\&= \frac{x}{4} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n)!} (2x)^{2n} - \frac{1}{8} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!} (2x)^{2n+1} \\&= -\frac{x}{4} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} - 1 \right) + \frac{1}{8} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} - 2x \right) = -\frac{x}{4} \cos(2x) + \frac{1}{8} \sin(2x).\end{aligned}$$

Thus,

$$g(\pi/3) = -\frac{\pi/3}{4} \cos(2\pi/3) + \frac{1}{8} \sin(2\pi/3) = \frac{\pi}{24} + \frac{\sqrt{3}}{16} = \boxed{\frac{2\pi+3\sqrt{3}}{48}}$$

Grading: computation of $g(x)$ with full justification 8pts; final computation 2pts, with likely penalties for carryover errors

Problem 5 (20pts)

All questions in this problem refer to the infinite series

$$\sum_{n=1}^{\infty} n e^{-4n}$$

(a; 3pts) Explain why this series converges.

The quickest way here is to use the *Ratio Test* (because of e^{-4n}):

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)e^{-4(n+1)}}{n e^{-4n}} = \left(1 + \frac{1}{n}\right) \cdot \frac{e^{-4n}e^{-4}}{e^{-4n}} = \left(1 + \frac{1}{n}\right)e^{-4} \longrightarrow \left(1 + \frac{1}{\infty}\right)e^{-4} = e^{-4}.$$

Since $e^{-4} = 1/e^4 < 1$, the series converges.

We can also use the *Limit Comparison Test*. Note that $0 < e^{-2n}$, $\sum_{n=1}^{\infty} e^{-2n}$ converges being a geometric series with $r = 1/e^2 < 1$, and

$$\lim_{n \rightarrow \infty} \frac{n e^{-4n}}{e^{-2n}} = \lim_{n \rightarrow \infty} n e^{-2n} = 0,$$

since the exponential dominates. Thus, our series also converges.

The *Comparison Test* can be used as well. If $f(x) = x e^{-2x}$,

$$f'(x) = x' e^{-2x} + x(-2)e^{-2x} = e^{-2x} - 2x e^{-2x} = e^{-2x}(1 - 2x).$$

So $f(x) \leq f(1) = 1e^{-2} \leq 1$ for $x \geq 1$ and thus $n e^{-4n} \leq e^{-2n}$ for all n . Since $n e^{-4n} \geq 0$ and $\sum_{n=1}^{\infty} e^{-2n}$ converges being a geometric series with $r = 1/e^2 < 1$, our series also converges.

The *Integral Test* can also be used. The function $f(x) = x e^{-4x}$ is positive for $x \geq 1$. Since

$$f'(x) = x' e^{-4x} + x(-4)e^{-4x} = e^{-4x} - 4x e^{-4x} = e^{-4x}(1 - 4x),$$

$f(x)$ is decreasing for $x \geq 1$. So the sum converges if and only if $\int_1^{\infty} x e^{-4x} dx$ does. Integration by parts gives

$$\begin{aligned} \int_1^{\infty} x e^{-4x} dx &= -\frac{1}{4} \int_1^{\infty} x dx e^{-4x} = -\frac{1}{4} \left(x e^{-4x} \Big|_1^{\infty} - \int_1^{\infty} e^{-4x} dx \right) \\ &= -\frac{1}{4} \lim_{x \rightarrow \infty} \left(x e^{-4x} - 1 e^{-4} + \frac{1}{4} (e^{-4x} - e^{-4}) \right) = -\frac{1}{4} \left(0 - e^{-4} + 0 - \frac{1}{4} e^{-4} \right) = \frac{5}{16} e^{-4}. \end{aligned}$$

Since the integral is finite, $\sum_{n=1}^{\infty} n e^{-4n}$ converges. This is the most convenient approach for the purposes of part (b)

We can also view our infinite series as an evaluation of a power series inside of the interval of convergence:

$$\sum_{n=1}^{\infty} n e^{-4n} = \sum_{n=1}^{\infty} n (e^{-4})^n = \sum_{n=1}^{\infty} n x^n \Big|_{x=e^{-4}}.$$

Now use the standard power series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if} \quad |x| < 1.$$

Thus,

$$\sum_{n=1}^{\infty} nx^n = x \sum_{n=0}^{\infty} nx^{n-1} = x \left(\sum_{n=0}^{\infty} x^n \right)' = x \left(\frac{1}{1-x} \right)' = \frac{x}{(1-x)^2} \quad \text{if} \quad |x| < 1.$$

The convergence statement holds because the differentiation of a power series does not change the radius of convergence and may only drop the endpoints from the interval of convergence (and there are no endpoints to drop in this case). Since $|e^{-4}| < 1$, our series also converges. This is the most convenient approach for the purposes of part (d).

Grading: use of Limit Comparison, Comparison, and Integral Tests should include checks of the conditions; the power series approach should include mention of the intervals of convergence; 1-2pts off if these are missing

(b; 4pts) *What is the minimal number of terms required to approximate the sum of this series with error less than 1/100? Justify your answer. You may use that $e \approx 3$ for the purposes of part (b).*

The function $f(x) = xe^{-4x}$ is positive for $x \geq 1$. Since

$$f'(x) = x'e^{-4x} + x(-4)e^{-4x} = e^{-4x} - 4xe^{-4x} = e^{-4x}(1-4x),$$

$f(x)$ is decreasing for $x \geq 1$. Thus, by the *Remainder Estimate for the Integral Test* Theorem

$$\int_{m+1}^{\infty} \frac{1}{x^3} dx < \sum_{n=1}^{\infty} ne^{-4n} - \sum_{n=1}^{n=m} ne^{-4n} = \sum_{n=m+1}^{\infty} ne^{-4n} < \int_m^{\infty} xe^{-4x} dx$$

Since

$$\begin{aligned} \int_m^{\infty} xe^{-4x} dx &= -\frac{1}{4} \int_1^{\infty} xde^{-4x} = -\frac{1}{4} \left(xe^{-4x} \Big|_m^{\infty} - \int_m^{\infty} e^{-4x} dx \right) \\ &= -\frac{1}{4} \lim_{x \rightarrow \infty} \left(xe^{-4x} - me^{-4m} + \frac{1}{4} (e^{-4x} - e^{-4m}) \right) = -\frac{1}{4} \left(0 - me^{-4m} + 0 - \frac{1}{4} e^{-4m} \right) = \frac{4m+1}{16} e^{-4m}. \end{aligned}$$

We find that

$$\frac{4(m+1)+1}{16} e^{-4(m+1)} < \sum_{n=m+1}^{\infty} ne^{-4n} < \frac{4m+1}{16} e^{-4m}.$$

We need the middle term to be less than 1/100. By the second inequality $\boxed{m=1}$ works (because $\frac{5}{16}e^{-4} \leq \frac{1}{100}$; this is equivalent to $500 \leq (2e)^4$, which is the case because $2e > 5$), but $m=0$ does not appear to work (because $\frac{1}{16}e^{-0} > \frac{1}{100}$).

The $m=0$ estimate indeed does *not* work because

$$\sum_{n=1}^{\infty} ne^{-4n} = 1e^{-4} + 2e^{-8} + \dots > e^{-4} > 3^{-4} = \frac{1}{81} > \frac{1}{100}.$$

Grading: remainder bound with integrals 1pt, after integrating 3pts (*not* in addition to 1pt), conclusion that $m=3$ 1pt; the Integral Test conditions must be mentioned; work on (a) taken into account if the Integral Test is used there; *full* justification that $m=0$ does not work bonus 3pts (*no* partial bonus)

(c; 3pts) *Based on your answer in part (b), estimate the sum of this series with error less than 1/100; simply your answer as much as possible, but leave it in terms of e. Is your estimate an under- or over-estimate for the sum? Explain why. (If you do not know how to do (b), take the answer to (b) to be 3).*

Based on the answer in part (b), the required estimate is

$$\sum_{n=1}^{n=1} n e^{-4n} = 1 \cdot e^{-4} = \boxed{e^{-4}}$$

With $m=3$, the estimate would be

$$\sum_{n=1}^{n=1} n e^{-4n} = 1 \cdot e^{-4} + 2 \cdot e^{-8} + 3 \cdot e^{-12} = (e^8 + 2e^4 + 3)e^{-12}.$$

Either estimate is an under-estimate for the infinite sum, because the finite-sum estimate is obtained by dropping only positive terms from the infinite sum.

Grading: correct use of m 1pt; under/over-estimate and justification 1pt each; no penalty for carryover errors from (b)

(d; 10pts) *Find the sum of the infinite series exactly.*

First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=1}^{\infty} n e^{-4n} = \sum_{n=1}^{\infty} n (e^{-4})^n = \sum_{n=1}^{\infty} n x^n \Big|_{x=e^{-4}}.$$

Now use the standard power series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if} \quad |x| < 1.$$

Thus,

$$\sum_{n=1}^{\infty} n x^n = x \sum_{n=0}^{\infty} n x^{n-1} = x \left(\sum_{n=0}^{\infty} x^n \right)' = x \left(\frac{1}{1-x} \right)' = \frac{x}{(1-x)^2} \quad \text{if} \quad |x| < 1.$$

The convergence statement holds because the differentiation of a power series does not change the radius of convergence and may only drop the endpoints from the interval of convergence (and there are no endpoints to drop in this case). Since $|e^{-4}| < 1$, our series also converges and its sum is

$$\frac{x}{(1-x)^2} \Big|_{x=e^{-4}} = \frac{e^{-4}}{(1-e^{-4})^2} = \boxed{\frac{e^4}{(e^4-1)^2}}$$

Grading: correct power series and evaluation point 4pts; sum of power series, $x/(1-x)^2$, and justification 3pts; mention of range of convergence in relation to the evaluation point 1pt, substitution of $x=e^{-4}$, and fully simplified answer 1pt each; work on part (a) taken into account if the power series approach is used there.

Problem 6 (10pts)

All questions in this problem refer to the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+3)3^n}$$

(a; 3pts) Explain why this series converges.

This series converges because it is alternating (odd terms are negative, even terms are positive) and the absolute values of its terms are decreasing to 0:

$$\frac{1}{(2n+3)3^n} \geq \frac{1}{(2(n+1)+3)3^{n+1}} \rightarrow 0.$$

This is the most convenient approach for the purposes of part (b).

We can also use the *Ratio Test* (because of 3^n):

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1/((2(n+1)+3)3^{n+1})}{1/((2n+3)3^n)} = \frac{2n+3}{2n+5} \cdot \frac{3^n}{3^{n+1}} = \frac{2+3/n}{2+5/n} \cdot \frac{1}{3} \rightarrow \frac{2+0}{2+0} \cdot \frac{1}{3} = \frac{1}{3}.$$

Since $1/3 < 1$, the series converges.

The Absolute Convergence Test, followed by Comparison or Limit Comparison Test, can be used as well. By the former, it is sufficient to show that the series $\sum_{n=1}^{\infty} \frac{1}{(2n+3)3^n}$ converges. We compare it to the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$. The latter is a geometric series with positive terms and $r=1/3 < 1$ and thus converges. Since $1/((2n+3)3^n) < 1/3^n$, our series of absolute values also converges. We can also use the Limit Comparison Test with the same geometric series

$$\frac{|a_n|}{b_n} = \frac{1/((2n+3)3^n)}{1/3^n} = \frac{1}{2n+3} \rightarrow 0.$$

Since the limit of the ratios exists and the series $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges and has positive terms, our series of absolute values also converges.

We can also view our infinite series as an evaluation of a power series inside of the interval of convergence:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+3)3^n} &= 3\sqrt{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3} \left(\frac{1}{\sqrt{3}}\right)^{2n+3} = 3\sqrt{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3} x^{2n+3} \Big|_{x=1/\sqrt{3}} \\ &= 3\sqrt{3} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{2n+1} x^{2n+1} \Big|_{x=1/\sqrt{3}} = -3\sqrt{3} \sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \Big|_{x=1/\sqrt{3}}. \end{aligned}$$

Now use the standard power series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{if} \quad |x| < 1.$$

Thus,

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} &= \sum_{n=0}^{\infty} \int_0^x (-1)^n u^{2n} du - \left(x - \frac{x^3}{3}\right) = \int_0^x \sum_{n=0}^{\infty} (-u^2)^n du - \left(x - \frac{x^3}{3}\right) \\ &= \int_0^x \frac{1}{1+u^2} du - \left(x - \frac{x^3}{3}\right) = \arctan u \Big|_0^x - \left(x - \frac{x^3}{3}\right) = \arctan x - \left(x - \frac{x^3}{3}\right).\end{aligned}$$

The third equality above holds if $|x^2| < 1$ and in particular for $x = 1/\sqrt{3}$. Thus, our infinite series converges. This is the most convenient approach for the purposes of part (d).

Grading: use of Alternating Series, Comparison, and Limit Comparison Tests should include checks of the conditions; the power series approach should include mention of the intervals of convergence; 1-2pts off if these are missing

(b; 4pts) *What is the minimal number of terms required to approximate the sum of this series with error less than 1/100? Justify your answer.*

Since the 3 assumptions for the Alternating Series Test hold, by the *Remainder Estimate for the Alternating Series Test* Theorem

$$\begin{aligned}\left| \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+3)3^n} - \sum_{n=1}^{n=m} \frac{(-1)^n}{(2n+3)3^n} \right| &= \left| \sum_{n=m+1}^{\infty} \frac{(-1)^n}{(2n+3)3^n} \right| < |a_{m+1}| = \frac{1}{(2(m+1)+3)3^{m+1}} \\ &= \frac{1}{(2m+5)3^{m+1}}.\end{aligned}$$

Since the left term needs to be less than 1/100, by the inequality $\boxed{m=2}$ works (because $\frac{1}{(2 \cdot 2 + 5)3^{2+1}} \leq \frac{1}{100}$), but $m=1$ does not appear to work (because $\frac{1}{(2 \cdot 1 + 5)3^{1+1}} > \frac{1}{100}$).

The $m=1$ estimate indeed does *not* work because

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+3)3^n} - \sum_{n=1}^{n=1} \frac{(-1)^n}{(2n+3)3^n} &= \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+3)3^n} = \left(\frac{1}{7 \cdot 3^2} - \frac{1}{9 \cdot 3^3}\right) + \left(\frac{1}{11 \cdot 3^4} - \frac{1}{13 \cdot 3^5}\right) + \dots \\ &> \frac{1}{7 \cdot 3^2} - \frac{1}{9 \cdot 3^3} = \frac{3^3 - 7}{7 \cdot 3^5} = \frac{20}{7 \cdot 243} = \frac{20}{1701} > \frac{1}{100}.\end{aligned}$$

Grading: remainder bound with $|a_{m+1}|$ 1pt, after substitution 3pts (*not* in addition to 1pt), conclusion that $m=2$ 1pt; the Alternating Series Test conditions must be mentioned for full credit; work on part (a) taken into account if the Alternating Series Test is used there; *full* justification that $m=1$ does not work bonus 3pts (*no* partial bonus)

(c; 3pts) *Based on your answer in part (b), estimate the sum of this series with error less than 1/100; leave your answer as a simple fraction p/q for some integers p and q with no common factor. Is your estimate an under- or over-estimate for the sum? Explain why. (If you do not know how to do (b), take the answer to (b) to be 3).*

Based on the answer in part (b), the required estimate is

$$\sum_{n=1}^{n=m} \frac{(-1)^n}{(2n+3)3^n} = -\frac{1}{(2 \cdot 1 + 3) \cdot 3^1} + \frac{1}{(2 \cdot 2 + 3) \cdot 3^2} = -\frac{21 - 5}{5 \cdot 7 \cdot 9} = \boxed{-\frac{16}{315}}$$

Since the last term used in estimating this *alternating* series is positive, this estimate is an over-estimate for the infinite sum. Alternatively, the first term of the remainder

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+3)3^n} = -\frac{1}{(2 \cdot 3 + 3) \cdot 3^3} + \dots$$

is negative and thus this *alternating* series is also negative. So, the infinite sum is the $m = 2$ estimate plus a negative number, so the estimate is an over-estimate.

With $m = 3$, the estimate would be

$$\sum_{n=1}^{n=m} \frac{(-1)^n}{(2n+3)3^n} = -\frac{1}{(2 \cdot 1 + 3) \cdot 3^1} + \frac{1}{(2 \cdot 2 + 3) \cdot 3^2} - \frac{1}{(2 \cdot 3 + 3) \cdot 3^3} = -\frac{7 \cdot 3^4 - 5 \cdot 3^3 + 5 \cdot 7}{5 \cdot 7 \cdot 3^5} = -\frac{467}{8505}.$$

Since the last term used in estimating this *alternating* series is negative, this estimate is an under-estimate for the infinite sum. Alternatively, the first term of the remainder

$$\sum_{n=4}^{\infty} \frac{(-1)^n}{(2n+3)3^n} = \frac{1}{(2 \cdot 4 + 3) \cdot 3^4} - \dots$$

is positive and thus this *alternating* series is also positive. So, the infinite sum is the $m = 3$ estimate plus a positive number, so the estimate is an under-estimate.

Grading: correct use of m 1pt; under/over-estimate and justification 1pt each; no penalty for carryover errors from (b)

(d; bonus 10pts) *Find the sum of the infinite series exactly.*

This question is similar to p632, #11, which was part of paper HW11. First, write this infinite series as some power series evaluated at some point:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+3)3^n} = -3\sqrt{3} \sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \Big|_{x=1/\sqrt{3}};$$

see the last approach in part (a). Proceeding as in part (a), we find that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \arctan x - \left(x - \frac{x^3}{3}\right) \quad \text{if } |x| < 1.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+3)3^n} = -3\sqrt{3} \left(\arctan(1/\sqrt{3}) - \frac{1}{\sqrt{3}} + \frac{1}{3 \cdot 3\sqrt{3}} \right) = \boxed{-\frac{3\sqrt{3}\pi - 16}{6}}$$

Grading: 2nd displayed expression above 6pts; correct evaluation point 1pt; mention of range of convergence in relation to the evaluation point 1pt; substitution of $x = 1/\sqrt{3}$ and fully simplified answer 1pt each; work on part (a) taken into account if the power series approach is used there; *no* or very little partial credit if arctan does not appear

Problem 7 (15pts)

Find the general real solution to each of the following differential equations.

(a; 5pts) $y'' - 2y' + y = 0$, $y = y(x)$

The associated polynomial equation is

$$r^2 - 2r + 1 = 0 \quad \iff \quad (r - 1)^2 = 0.$$

Thus, the two roots $r_1 = r_2 = 1$ are the same, and the general real solution is

$$y(x) = C_1 e^{1x} + C_2 x e^{1x} = \boxed{C_1 e^x + C_2 x e^x}$$

(b; 5pts) $y'' + 2y' + 3y = 0$, $y = y(x)$

The associated polynomial equation is

$$r^2 + 2r + 3 = 0 \quad \iff \quad r = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm i\sqrt{2}.$$

Thus, the two roots are complex, and the general real solution is

$$y(x) = \boxed{C_1 e^{-x} \cos(\sqrt{2}x) + C_2 e^{-x} \sin(\sqrt{2}x) = e^{-x} (C_1 \cos(\sqrt{2}x) + C_2 \sin(\sqrt{2}x))}$$

(c; 5pts) $y'' + 2y' - 3y = 0$, $y = y(x)$

The associated polynomial equation is

$$r^2 + 2r - 3 = 0 \quad \iff \quad (r + 3)(r - 1) = 0 \quad \iff \quad r = 1, -3.$$

Since this polynomial has distinct real roots $r_1 = 1$ and $r_2 = -3$, the general solution is

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} = \boxed{C_1 e^x + C_2 e^{-3x}}$$

Grading (each part): associated polynomial and roots 1pt each; general solution based on the roots 3pts; only one C 4pts max; no C 's 3pts max; complex form in (b) 3pts max

Problem 8 (10pts)

Let $y = y(x)$ be the solution to the initial-value problem

$$y' = y, \quad y = y(x), \quad y(0) = 1.$$

Use Euler's method with $n = 2$ steps to estimate the value of $y(1)$. Show your steps clearly and use simple fractions (so $5/4$ or $\frac{5}{4}$, not 1.25).

The step size is $h = (1-0)/2 = 1/2$, so we need to obtain estimates y_1, y_2 for the y -value at $x_1 = 1/2, x_2 = 1$:

$$\begin{aligned} s_0 = y_0 = 1 &\implies y_1 = y_0 + s_0 h = 1 + \frac{1}{2} = \frac{3}{2} \\ s_1 = y_1 = \frac{3}{2} &\implies y_2 = y_1 + s_1 h = \frac{3}{2} + \frac{3}{4} = \boxed{\frac{9}{4}} \end{aligned}$$

Alternatively, this can be done using a table:

i	x_i	y_i	$s_i = y_i$	$y_{i+1} = y_i + \frac{1}{2}s_i$
0	0	1	1	$1 + \frac{1}{2} = \frac{3}{2}$
1	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2} + \frac{3}{4} = \boxed{\frac{9}{4}}$

The first column consists of the numbers i running from 0 to $n-1$, where n is the number of steps (2 in this case). The second column starts with the initial value of x (0 in this case) with subsequent entries in the column obtained by adding the step size h ($\frac{1}{2}$ in this case); it ends just before the final value of x would have been entered ($1 = \frac{1}{2} + \frac{1}{2}$ in this case). Thus, the first two columns can be filled in at the start. The first entry in the third column is the initial y -value (1 in the case). After this, one computes the first entries in the remaining two columns and copies the first entry in the last column to the third column in the next line. The process then repeats across the second row. The estimate for the final value of y is the last entry in the table.

Grading: h, x_0, x_1 1pt each; correct recursive setup 5pts (1pt each for number of steps, slope and change equations, left end points, end of the procedure); 2pts for computations

Problem 9 (10pts)

Answer Only. A two-species interaction is modeled by the following system of differential equations

$$\begin{cases} \frac{dx}{dt} = x - \frac{1}{10}x^2 + \frac{1}{40}xy \\ \frac{dy}{dt} = \frac{1}{2}y - \frac{1}{100}xy \end{cases} \quad (x, y) = (x(t), y(t)),$$

where t denotes time.

(a; 2pts) Which of the following best describes the interaction modeled by this system?

- (i) predator-prey (ii) competition for same resources (iii) cooperation for mutual benefit

Because of the coefficient of $+\frac{1}{40}$ in front of xy in the first equation, the x -species is helped by the presence of the y -species (the growth rate of the former is increased if the population of the latter is nonzero). Because of the coefficient of $-\frac{1}{100}$ in front of xy in the second equation, the y -species is hurt by the presence of the x -species. Thus, x is a predator and y is a prey.

(b; 8pts) This system has 3 equilibrium (constant) solutions; find all of them and explain their significance relative to the interaction the system is modeling. Put one equilibrium solution in each box below and use the space to the right of the box to describe its significance.

(0,0)	no population of either species ever
(10,0)	x -population at its carrying capacity in the absence of y -population
(50, 160)	160 of prey are precisely enough to supplement the carrying capacity for the x -population to support 50 predators and be contained by them

The constant solutions are described by $(x'(t), y'(t)) = 0$. Using the above system this gives

$$\begin{cases} 0 = x \left(1 - \frac{1}{10}x + \frac{1}{40}y\right) \\ 0 = \frac{1}{2}y \left(1 - \frac{1}{50}x\right) \end{cases} \iff \begin{cases} x = 0 \text{ or } 1 - \frac{1}{10}x + \frac{1}{40}y = 0 \\ y = 0 \text{ or } 1 - \frac{1}{50}x = 0 \end{cases}$$

Thus, the constant solutions are the solutions of the following systems:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = 0 \\ 1 - \frac{1}{50}x = 0 \end{cases} \quad \begin{cases} 1 - \frac{1}{10}x + \frac{1}{40}y = 0 \\ y = 0 \end{cases} \quad \begin{cases} 1 - \frac{1}{10}x + \frac{1}{40}y = 0 \\ 1 - \frac{1}{50}x = 0 \end{cases}$$

The first system gives the equilibrium in the first box. The second system has no solutions. The third system gives the equilibrium in the second box, by plugging in $y=0$ into the first equation. The last system gives the third equilibrium point by plugging in $x=50$ into the first equation.

Grading: 1 correct pair 1pt, 2 3pts, 3 5pts, with 1-2pts off for not simplifying; significance 1pt each

Problem 10 (20pts)

(a; 10pts) Show that the orthogonal trajectories to the family of curves $x = ky^2$ are described by the differential equation

$$y' = -\frac{2x}{y}, \quad y = y(x).$$

Differentiate $x = ky^2$ with respect to x , using chain rule and remembering that k is a constant:

$$1 = 2kyy' \iff y' = \frac{1}{2ky}.$$

From the original equation, we find that $k = x/y^2$ and so our curves have slope

$$y' = \frac{1}{2ky} = \frac{1}{2(x/y^2) \cdot y} = \frac{y}{2x}$$

at (x, y) . The slopes of the orthogonal curves are the negative reciprocal of this; so they satisfy

$$y' = -\frac{1}{y/2x} = -\frac{2x}{y}.$$

Grading: computation of slopes of the initial curves 4pts; substitution for k 2pts; the negative reciprocal statement 3pts; conclusion 1pt

(b; 7pts) Find the general solution to the differential equation stated in (a).

This equation is separable, so after writing $y' = dy/dx$, we can move everything involving y to LHS and everything involving x to RHS and then integrate:

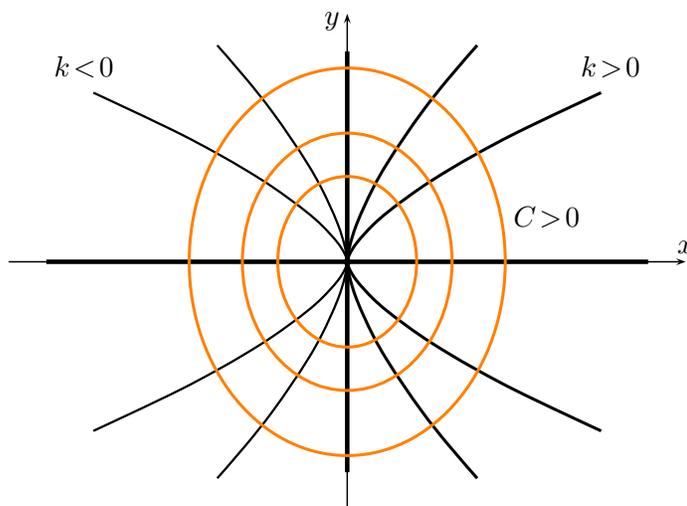
$$\begin{aligned} \frac{dy}{dx} = -\frac{2x}{y} &\iff y \, dy = -2x \, dx &\iff \int y \, dy = -\int 2x \, dx &\iff \frac{1}{2}y^2 = -x^2 + C \\ &&&&\iff \boxed{2x^2 + y^2 = C} \end{aligned}$$

Grading: splitting the variables 3pts; integration 2pts; simplification 2pts (answer with both square roots is ok)

(c; 3pts) Sketch one representative of the original family of curves and one orthogonal trajectory on the same diagram; indicate clearly which is which.

Draw the above curves for different values of k and C :

- $x = 0y^2$ is the y -axis; $x = \infty y^2$ is the x -axis; $x = 1y^2$ is a parabola with the roles of x and y reversed; $x = (-1)y^2$ is the reflection of the last graph about the y -axis;
- the equation $2x^2 + y^2 = C$ has no solutions (x, y) if $C < 0$; if $C = 0$, this “curve” is just the origin $(0, 0)$; the curve $2x^2 + y^2 = 2$ is the ellipse passing through the points $(\pm 1, 0)$ and $(0, \pm\sqrt{2})$, and so it is stretched vertically; the ellipse $2x^2 + y^2 = 2r^2$ is the same ellipse scaled by the factor $|r|$.



Grading: 1pt for a curve in only one of the families; 2pts off if no indication is given which curves are which (k, C values not required); 1pt off if the axes are not labeled