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THREE QUASI-LOCAL MASSES

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We propose a definition of quasi-local mass based on the Penrose Inequality. Two further definitions are given by measuring distortions of the exponential map.

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1. Quasi-Local Mass from the Penrose Inequality

General Relativity differs from most classical field theories in that there is no well-defined notion of energy density for the gravitation field, as can be seen from Einstein's principle of equivalence. Thus at best one can only hope to calculate the mass/energy contained within a domain, as opposed to at a point. Such a concept is referred to as quasi-local mass, that is, a functional which assigns a real number to each compact spacelike hypersurface in a spacetime. Of course there are many properties that any such functional should satisfy under appropriate conditions; most notable among these are the properties of non-negativity and rigidity (for an expanded list see Ref. 4). Although numerous definitions of quasi-local mass have been proposed, most seem to possess undesirable properties, in fact most fail the crucial test of non-negativity. However, there is one definition which appears to satisfy most of the required properties, namely the mass proposed by Bartnik.[2]

Bartnik's idea is to localize the ADM (or total) mass in the following way. Here we restrict attention to the time symmetric case. Let (Ω, h) be a compact three-manifold with boundary, and define an *admissible extension* to be an asymptotically flat three-manifold (M, g) with (or without) boundary satisfying the following conditions: (M, g) has non-negative scalar curvature, the boundary (if nonempty) is

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minimal and no other minimal surfaces exist within (M, g), lastly (Ω, h) embeds isometrically into (M, g). Then Bartnik's mass is given by

$$\mathcal{M}_B(\Omega) = \inf\{M_{\mathrm{ADM}}(M, g) | (M, g) \text{ an admissible extension of } (\Omega, h)\}\,,$$

where M_{ADM} is the ADM mass. A primary benefit of this construction is that non-negativity is achieved for free, from the positive mass theorem. However, it is not *a priori* clear that this definition is nontrivial, in the sense that the mass is nonzero whenever Ω is nonflat. That this is the case,[8] is a consequence of (and reason for) the no horizons (minimal surfaces) assumption in the class of admissible extensions. Although this mass satisfies many other desired properties, it suffers from the apparent deficiency of being difficult to compute. In order to remedy this problem, Bartnik[2] has proposed the following solution. Namely, he conjectures that the infimum $\mathcal{M}_B(\Omega)$ is realized by an admissible extension (M_0, g_0) which is smooth and static vacuum outside of Ω, is $C^{0,1}$ across $\partial\Omega$, and has non-negative scalar curvature (in the distributional sense). Thus Bartnik's mass may be computed as the ADM mass of the given static vacuum extension.

Recall that a Riemannian manifold (M, g) is static vacuum if there exists a potential function $u(x) > 0$ which satisfies the static vacuum Einstein equations (equivalently, the spacetime metric $-u^2dt^2 + g$ is Ricci flat):

$$\mathrm{Ric}(g) = u^{-1}\nabla^2 u\,, \qquad \Delta u = 0\,. \tag{1.1}$$

Here ∇^2 is the Hessian and Δ is the Laplacian with respect to the metric g. The physical reasoning behind this conjecture is that once all mass/energy has been squeezed out, there is nothing left to support matter fields (vacuum) or gravitational dynamics (static). Moreover, significant mathematical motivation exists as well. For instance, if the mass minimizing extension is not static then Corvino[5] has shown that there exist compactly supported metric variations which increase the scalar curvature, so that one may then perform an appropriate conformal deformation to lower the ADM mass.

Ideally there should be a unique static vacuum extension, and so one must append boundary conditions to the static vacuum equations. The choice of boundary conditions is dictated by the desire for the positive mass theorem to remain valid for the complete manifold $(\tilde{M}_0 \cup \Omega, g_0 \cup h)$, where $\tilde{M}_0 = M_0 - \Omega$. This will in fact be the case[10,12] if the metric g_0 remains $C^{0,1}$ across the divide between the static and nonstatic parts, and if the mean curvatures agree:

$$g_0|_{\partial\tilde{M}_0} = h|_{\partial\Omega}\,, \qquad H_{\partial\tilde{M}_0} = H_{\partial\Omega}\,. \tag{1.2}$$

Furthermore it follows from the Riccati and Gauss equations that these Bartnik boundary conditions imply that the scalar curvature is in fact non-negative in the distributional sense, as demanded by the conjecture.

The conjecture may be divided into two distinct parts. The first step is to establish a basic existence result for the static vacuum equations (1.1) with boundary conditions (1.2). Then with this mass minimizer in hand, the second step entails showing that the given static solution actually realizes the infimum. While there has been some progress on part one of the conjecture,[1,11] the second part remains essentially uninvestigated.

It should be noted that the static metric extension conjecture (if true) shows that the Bartnik mass depends only on the boundary geometry. Namely, let $\Sigma = \partial\Omega$, then $\mathcal{M}_B(\Omega)$ depends only on the induced metric and mean curvature on Σ. For this reason, we may write $\mathcal{M}_B(\Omega) = \mathcal{M}_B(\Sigma)$.

Although the Bartnik mass has many desirable properties it has been criticized for potentially overestimating the true "physical" mass contained in a domain.[13] In this section we propose a modification of the Bartnik mass with the aim of alleviating this difficulty. In order to state the definition, we must first recall the Penrose inequality. Let (M, g) be an asymptotically flat Riemannian three-manifold, with non-negative scalar curvature, and outermost minimal surface boundary ∂M. A minimal surface is called outermost if no other minimal surface encloses it. If $|\partial M|$ denotes the area of the boundary, then the Riemannian Penrose inequality asserts that

$$M_{\mathrm{ADM}}(M, g) \geq \sqrt{\frac{|\partial M|}{16\pi}},$$

and equality holds if and only if (M, g) is isometric to the $t = 0$ slice of the Schwarzschild spacetime. This theorem has been proven by Huisken and Ilmanen[8] for one boundary component, and by Bray[3] for multiple boundary components. Just as Bartnik's mass is based on the positive mass theorem, we may use the Penrose inequality to construct a quasi-local mass in an analogous way. Namely, with the same notion of an admissible extension, we propose the following definition:

$$\mathcal{M}_{QL}(\Omega) = \inf\left\{ M_{\mathrm{ADM}}(M, g) - \sqrt{\frac{|\partial M|}{16\pi}} \,|\, (M, g) \text{ an admissible extension of } (\Omega, h) \right\}.$$

Notice that an admissible extension has minimal surface boundary and that no other minimal surfaces exist within the extension, hence the boundary is in fact an outermost minimal surface and the Penrose inequality applies to show that $\mathcal{M}_{QL}(\Omega) \geq 0$ for any (Ω, h) with non-negative scalar curvature. Moreover, it is immediate from the definition that monotonicity holds, in that if $\Omega_1 \subset \Omega_2$, then $\mathcal{M}_{QL}(\Omega_1) \leq \mathcal{M}_{QL}(\Omega_2)$. When compared with the Bartnik mass, it is apparent that $\mathcal{M}_{QL} \leq \mathcal{M}_B$, which suggests that this new definition may remedy the potential tendency of the Bartnik mass to overestimate the true value of the physical mass. It should also be observed that $\mathcal{M}_{QL}(\Omega) = 0$ for any domain Ω contained within the $t = 0$ slice of the Schwarzschild spacetime and lying outside of the horizon. Although this differs in spirit from the Bartnik mass, it conforms with other masses such as the well-known Komar mass[13] that also shares this property. We conjecture that $\mathcal{M}_{QL}(\Omega) = 0$ if and only if Ω is a domain exterior to the horizon in the $t = 0$ slice of the Schwarzschild spacetime. This generalizes the rigidity statement for the Bartnik mass, in that Minkowski space is replaced by the Schwarzschild solution.

Furthermore, in analogy with Bartnik's conjecture, we conjecture that the infimum $\mathcal{M}_{QL}(\Omega)$ is realized by an admissible extension (M_0, g_0) which is smooth and static outside of Ω, is $C^{0,1}$ across $\partial\Omega$, and has non-negative scalar curvature (in the distributional sense). If true, $\mathcal{M}_{QL}(\Omega)$ may then be computed as the ADM mass minus the mass of the boundary, of the given static extension. The extension should be unique, and should depend only on the induced metric and mean curvature of the boundary $\Sigma = \partial\Omega$. Thus, as in the case with the Bartnik mass, we should be able to write $\mathcal{M}_{QL}(\Omega) = \mathcal{M}_{QL}(\Sigma)$.

2. Quasi-Local Mass from Volume Distortion

In the remainder of this paper, two further definitions of quasi-local mass will be proposed; both definitions will be restricted to the time symmetric case only. Although unrelated to the Bartnik mass, they share a common theme in that each measures a distortion of the exponential map. The first, denoted $\underline{\mathcal{M}}$, measures volume distortion and is discussed in the present section. The second, denoted $\overline{\mathcal{M}}$, is discussed in the next section and measures the distortion of a modulus of curves. In addition to sharing a common theme, these two definitions are highly related in a local manner, but differ globally. More precisely, it turns out that inside strictly convex balls the modulus of curves is given by the volume (Lemma 3.2). Moreover, volume is an upper bound for the modulus. This means that the two definitions agree over small scales, but most likely differ in the large with $\underline{\mathcal{M}} \leq \overline{\mathcal{M}}$.

We will use the following notation. Let (M, g) be a complete, time symmetric (zero extrinsic curvature), spacelike hypersurface, of a spacetime satisfying the dominant energy condition. That is, (M, g) is a Riemannian three-manifold with non-negative scalar curvature. The tangent space at $p \in M$ will be denoted by T_pM, and S_pM will denote its unit tangent space. All curves will be parametrized over the unit interval unless otherwise specified. For $v \in S_pM$, we denote by $\tau(v)$ the (possibly infinite) distance to the tangential cut locus of p in the direction v. We also denote

$$E_pM = \{tv \mid v \in S_pM\,,\, 0 \leq t < \tau(v)\}\,.$$

For $U \subset M$ we denote by $\mathrm{Vol}_g(U)$ its Riemannian volume, and for $V \subset T_pM$ we denote by $\mathrm{Vol}(V)$ its Euclidean volume induced by the inner product given by g restricted to T_pM. The metric ball in (M, g) of radius r centered at p will be written $B_r(p)$. The ball of radius r centered at $u \in T_pM$ will be denoted $B(r, u)$.

We make frequent use of the asymptotic expansion from Theorem 3.1 in Ref. 7, for the volume of metric balls in a Riemannian n-manifold:

$$\mathrm{Vol}_g(B_r(p)) = a_0 r^n - a_1 \,\mathrm{Scal}_g(p) r^{n+2} + a_2 \Lambda(g, p) r^{n+4} + O(r^{n+6})\,, \tag{2.1}$$

where $a_0, a_1, a_2 > 0$ are constants depending only on n, and $\Lambda(g, p)$ can be expressed in terms of coordinate fields $\{X_1, \ldots, X_n\}$, chosen to be orthonormal at p, as follows:

$$\Lambda(g,p) = -3\sum_{i,j,k,l}\mathrm{Riem}_g(X_i,X_j,X_k,X_l)^2 + 8\sum_{i,j}\mathrm{Ric}_g(X_i,X_j)^2$$
$$+ 5\,\mathrm{Scal}_g(p)^2 - 18\Delta_g\,\mathrm{Scal}_g(p)\,. \tag{2.2}$$

For any path connected, precompact, open set $\Omega \subset M$ and any $p \in M$ we let

$$\mathcal{A}_p(\Omega) = \{U \subset E_pM \text{ open } |\overline{\exp_p U} \subset \Omega\,,\ tu \in U \text{ for all } u \in U \text{ and } t \in [0,1]\}\,. \tag{2.3}$$

Now we define the (outer) volume distortion of the exponential map at the point $p \in \Omega$, by

$$\underline{\mathcal{K}}_p(\Omega) = \sup_{U\in\mathcal{A}_p(\Omega)} \frac{\mathrm{Vol}(U)}{\mathrm{Vol}_g(\exp_p U)}\,.$$

A quasi-local mass (in the time symmetric case) should depend only on the boundary $\Sigma = \partial\Omega$, its induced metric $g|_\Sigma$, and mean curvature H_g; the triple $(\Sigma, g|_\Sigma, H_g)$ will be referred to as Bartnik boundary data. With this in mind we then propose the following definition of quasi-local mass

$$\underline{\mathcal{M}}(\Sigma) = \inf_h \sup_{p\in\Omega} \log \underline{\mathcal{K}}_p(\Omega)\,,$$

where in analogy with the Bartnik mass the infimum is taken over all metrics h with non-negative scalar, whose Bartnik boundary data $(\partial\Omega, h|_{\partial\Omega}, H_h)$ agrees with the given boundary data $(\Sigma, g|_\Sigma, H_g)$. We conjecture that these boundary conditions guarantee that this definition of mass is nontrivial. That is, it should hold that $\underline{\mathcal{M}}(\Sigma) > 0$ unless Σ embeds isometrically into Euclidean space in such a way that the mean curvature from the Euclidean embedding agrees with H_g, the mean curvature from the embedding in (M,g). Here we shall prove the following.

Theorem 2.1. *Let (M,g) be a complete Riemannian three-manifold with non-negative scalar curvature. For any closed surface $\Sigma \subset M$ bounding a path connected precompact domain, we have $\underline{\mathcal{M}}(\Sigma) \geq 0$. Equality holds and the infimum is realized by a metric h on a domain Ω, if and only if (Ω,h) is locally isometric to Euclidean space. If in addition there is a set $U \in \mathcal{A}_p(\Omega)$ for some $p \in \Omega$ such that $\exp_p(U) = \Omega$, then Ω is isometric to a subset of Euclidean space.*

Proof. We proceed by contradiction. Suppose that $\underline{\mathcal{M}}(\Sigma) < 0$. This implies that there exists a metric h on Ω having the given Bartnik boundary data and with $\mathrm{Scal}_h \geq 0$, such that $\log \underline{\mathcal{K}}_p(\Omega) < 0$ for each $p \in \Omega$. Pick a point $p \in \Omega$ and consider balls of radius $r > 0$ about the origin in T_pM. Note that for r sufficiently small these balls are in $\mathcal{A}_p(\Omega)$, and that their images under the exponential map are also a sequence of metric balls. We then have

$$\frac{\mathrm{Vol}(B(r,0))}{\mathrm{Vol}_h(\exp_p B(r,0))} = \frac{\mathrm{Vol}(B(r,0))}{\mathrm{Vol}_h(B_r(p))} < 1\,. \tag{2.4}$$

If $\mathrm{Scal}_h(p) > 0$ then the expansion (2.1) contradicts (2.4). Thus, either $\underline{\mathcal{M}}(\Sigma) \geq 0$ or $\mathrm{Scal}_h(p) = 0$ for all $p \in \Omega$. We proceed with the zero scalar curvature assumption.

Comparing the expansion (2.1) and the inequality (2.4) again, yields $\Lambda(h,p) \geq 0$. Since Ω has dimension 3 the curvature tensor can be expressed entirely in terms of the Ricci tensor:

$$\mathrm{Riem}_h(X_i,X_j,X_i,X_j) = \frac{1}{2}[\mathrm{Ric}_h(X_i,X_i) + \mathrm{Ric}_h(X_j,X_j) - \mathrm{Ric}_h(X_k,X_k)]\,, \quad (2.5)$$

$$\mathrm{Riem}_h(X_i,X_j,X_i,X_k) = \mathrm{Ric}_h(X_j,X_k)\,, \quad (2.6)$$

where $\{X_1,X_2,X_3\}$ are coordinate fields orthonormal at p, and $i,j,k \in \{1,2,3\}$ are distinct. Since $\mathrm{Scal}_h \equiv 0$ on Ω, (2.2) gives the following expression for Λ:

$$\begin{aligned}\Lambda(h,p) = 8&\left[\sum_i \mathrm{Ric}_h(X_i,X_i)^2 + 2\sum_{i<j}\mathrm{Ric}_h(X_i,X_j)^2\right]\\ &- 3\left[\sum_{[ijk]}(\mathrm{Ric}_h(X_i,X_i) + \mathrm{Ric}_h(X_j,X_j) - \mathrm{Ric}_h(X_k,X_k))^2\right.\\ &\left.+8\sum_{j<k}\mathrm{Ric}_h(X_j,X_k)^2\right].\end{aligned}$$

Here $[ijk]$ denotes summation over the three cyclic permutations of $(1,2,3)$. Since $\Lambda(h,p) \geq 0$ for all $p \in \Omega$, it follows that

$$\begin{aligned}&-\sum_i \mathrm{Ric}_h(X_i,X_i)^2 + 6\sum_{i<j}\mathrm{Ric}_h(X_i,X_i)\,\mathrm{Ric}_h(X_j,X_j)\\ &\quad - 8\sum_{i<j}\mathrm{Ric}_h(X_i,X_j)^2 \geq 0\,. \quad (2.7)\end{aligned}$$

Furthermore, $\mathrm{Scal}_h(p) = 0$ for all $p \in \Omega$ gives that

$$0 = \left(\sum_i \mathrm{Ric}_h(X_i,X_i)\right)^2 = \sum_i \mathrm{Ric}_h(X_i,X_i)^2 + 2\sum_{i<j}\mathrm{Ric}_h(X_i,X_i)\,\mathrm{Ric}_h(X_j,X_j)\,.$$

Combining this and (2.7) produces

$$-4\sum_i \mathrm{Ric}_h(X_i,X_i)^2 - 8\sum_{i<j}\mathrm{Ric}_h(X_i,X_j)^2 \geq 0\,.$$

Since p was arbitrary, the Ricci curvature is identically zero on Ω. Applying (2.5) and (2.6) gives that the full Riemann tensor is also identically zero. Thus, (Ω,h) is locally isometric to Euclidean space and $\log \underline{\mathcal{K}}_p(\Omega) = 0$ for all $p \in \Omega$. The conclusion of the theorem now follows, except perhaps for the last sentence. To complete the proof note that since $\exp_p|_U$ is injective for any $U \in \mathcal{A}_p(\Omega)$, if $\exp_p U = \Omega$ then it is surjective as well. $\square$

3. Quasi-Local Mass from Distortion of a Modulus of Curves

First we introduce some notation. As above (M, g) will denote a complete Riemannian three-manifold of non-negative scalar curvature. For any set of curves $A \subset C^0([0,1], M)$, consider the set of Borel functions

$$\mathcal{F}(A) = \left\{ f : M \to [0, \infty] \,\middle|\, \int_\gamma f \geq \mathrm{dist}_g(\gamma(0), \gamma(1)) \text{ for all } \gamma \in A \right\},$$

and define the modulus of curves in A,

$$\mathrm{mod}(A) = \inf\{\|f\|_g \mid f \in \mathcal{F}(A)\},$$

where $\|f\|_g$ is the L^3 norm induced by the Riemannian measure (i.e. it is the volume of the possibly singular conformal metric $f^2 g$).

Similarly, a set of curves B in T_pM takes the set of Borel functions

$$\mathcal{F}_p(B) = \left\{ f : T_pM \to [0, \infty] \,\middle|\, \int_\gamma f \geq \|\gamma(0) - \gamma(1)\|_g \text{ for all } \gamma \in B \right\}$$

and define the modulus of curves in B,

$$\mathrm{mod}_p(B) = \inf\{\|f\|_{(p)} \mid f \in \mathcal{F}_p(B)\},$$

where $\|f\|_{(p)}$ is the L^3 norm of f on T_pM induced by the inner product g restricted to T_pM. Note that this modulus is the same as that on curves in $\mathbb{R}^3$ under the Euclidean metric and the subscript p is there only to denote in which tangent space the curves reside.

For any set $U \subset (M, g)$ with nonempty interior, Γ_U will be the set of all continuous curves γ parametrized on the unit interval with $\gamma(t)$ in the interior of U for all $0 < t < 1$ and $\gamma(0), \gamma(1) \in \partial U$. Estimates of the modulus of sets of curves Γ_U were given in Lemma 2.4, Lemma 2.5, and Theorem 2.6 of Ref. 9. For the convenience of the reader we summarize relevant results here in the following two lemmas.

Lemma 3.1. *If $U = B_r(p)$ is a metric ball in (M, g) and r is less than the convexity radius at p, then* $\mathrm{mod}(\Gamma_U) = \mathrm{Vol}_g(U)$.

Lemma 3.2. *Let V be an open set with compact closure.*

(i) *If $V \subset B_r(p)$ with r less than the convexity radius at p, then* $\mathrm{mod}(\Gamma_V) = \mathrm{Vol}_g(V)$.
(ii) *If $V \subset T_pM$, then* $\mathrm{mod}_p(\Gamma_V) = \mathrm{Vol}(V)$.

Now we define a quasi-local mass from distortion of the exponential map on the modulus on curves given above. Let Σ be a closed surface embedded inside M and bounding a region, with Bartnik data $(\Sigma, g|_\Sigma, H_g)$. For any compact, path connected Riemannian three-manifold (Ω, h) with boundary Σ, and any $p \in \Omega$ we let $\mathcal{A}_p(\Omega)$ be defined as in (2.3). For each of these collections let the (outer) modulus distortion $\overline{\mathcal{K}}_p$ of the exponential map at the point $p \in \Omega$ be given by

$$\overline{\mathcal{K}}_p(\Omega) = \sup_{U \in \mathcal{A}_p(\Omega)} \frac{\mathrm{mod}_p(\Gamma_U)}{\mathrm{mod}(\Gamma_{\overline{\exp_p U}})}.$$

We then define the quasi-local mass for these distortions in the following way

$$\overline{\mathcal{M}}(\Sigma) = \inf_{h} \sup_{p\in\Omega} \log \overline{\mathcal{K}}_p(\Omega)\,,$$

where the infimum is taken over all metrics h with non-negative scalar curvature, whose Bartnik boundary data $(\partial\Omega, h|_{\partial\Omega}, H_h)$ agrees with the given boundary data $(\Sigma, g|_{\Sigma}, H_g)$. We conjecture that these boundary conditions guarantee that this definition of mass is nontrivial. That is, it should hold that $\overline{\mathcal{M}}(\Sigma) > 0$ unless Σ embeds isometrically into Euclidean space in such a way that the mean curvature from the Euclidean embedding agrees with H_g, the mean curvature from the embedding in (M, g). Here we shall prove the following.

Theorem 3.3. *Let (M, g) be a complete Riemannian three-manifold with non-negative scalar curvature. For any closed surface $\Sigma \subset M$ bounding a path connected precompact domain, we have $\overline{\mathcal{M}}(\Sigma) \geq 0$. Equality holds and the infimum is realized by a metric h on a domain Ω, if and only if (Ω, h) is locally isometric to Euclidean space. Furthermore, $\underline{\mathcal{M}}(\Sigma) \leq \overline{\mathcal{M}}(\Sigma)$.*

Proof. The proof is similar to that of Theorem 2.1, so we only give an outline. We proceed by contradiction. Suppose that $\overline{\mathcal{M}}(\Sigma) < 0$. This implies that there exists a metric h on Ω having the given Bartnik boundary data and with $\mathrm{Scal}_h \geq 0$, such that $\log \overline{\mathcal{K}}_p(\Omega) < 0$ for each $p \in \Omega$. Pick a point $p \in \Omega$ and consider $U = B(r, 0) \subset T_pM$. By Lemma 3.1,

$$1 > \overline{\mathcal{K}}_p(\Omega) \geq \frac{\mathrm{mod}_p(\Gamma_U)}{\mathrm{mod}(\Gamma_{\overline{\exp_p U}})} = \frac{\mathrm{Vol}(B(r,0))}{\mathrm{Vol}_g(B_r(p))}\,. \tag{3.1}$$

If $\mathrm{Scal}_h(p) > 0$, this contradicts expansion (2.1). Thus, either $\overline{\mathcal{M}}(\Sigma) \geq 0$ or $\mathrm{Scal}_h(p) = 0$ for all $p \in \Omega$. We proceed with the zero scalar curvature assumption. Comparing the expansion (2.1) and the inequality (3.1) again, yields $\Lambda(h, p) \geq 0$, where $\Lambda(h, p)$ is given in (2.2). As in the proof of Theorem 2.1, this implies that the full Riemann tensor vanishes, and hence (Ω, h) is locally isometric to Euclidean space. In this case, the exponential map at each $p \in \Omega$ restricted to any $U \in \mathcal{A}_p(\Omega)$ is an isometry, and so applying Lemma 3.2 shows that $\overline{\mathcal{M}}(\Omega) = 0$. The conclusions of the theorem now follow, except perhaps for the last sentence. To complete the proof it is sufficient to show that if a set U has nonempty interior, then

$$\mathrm{mod}(\Gamma_U) \leq \mathrm{Vol}_h(U)\,.$$

Since the integral of 1 along any curve γ is greater than or equal to the Riemannian distance between the endpoints of γ, we have that $1 \in \mathcal{F}(\Gamma_U)$ and the claim follows immediately. $\square$

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