MATH 127 Solutions to Final Exam

1. $\boxed{\text{10 points}}$ *According to the poem by Ogden Nash,*

Big fleas have little fleas,
Upon their backs to bite 'em,
And little fleas have lesser fleas,
And so, ad infinitum.

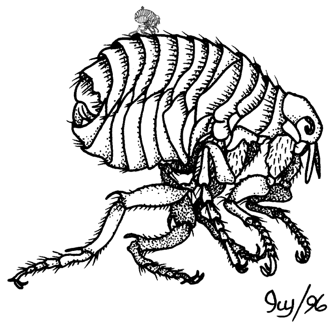


image adapted from
http://biodidac.bio.uottawa.ca

Assume each flea has exactly one flea which bites it. If the largest flea weighs 0.03 *grams, and each flea is* $1/10$ *the weight of the flea it bites, what is the total weight of all the fleas?*

The weight of the first flea is $w_1 = .03$, the weight of the second is $w_2 = .03 \cdot \frac{1}{10}$, the weight of the third is $w_3 = .03 \cdot \frac{1}{100}$, and so on. So the total weight of all the fleas is

$$w = \sum_{n=1}^{\infty} w_n = w_1 + w_2 + w_3 + \ldots = .03 + .03 \cdot \frac{1}{10} + .03 \cdot \frac{1}{100} + \ldots = \sum_{n=0}^{\infty} .03 \frac{1}{10^n}.$$

This is a geometric series with $|r| = 1/10 < 1$ and first term .03. So it converges to

$$w = \sum_{n=0}^{\infty} .03 \frac{1}{10^n} = \frac{.03}{1 - \frac{1}{10}} = \frac{3/100}{9/10} = \boxed{\frac{1}{30} \text{ grams}}$$

2. $\boxed{\textit{12 points}}$ *Find the Maclaurin series of each of the given functions. I suggest you use a familiar power series as your starting point.*

(a) e^{3x^2}

Since $\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$,

$$\mathrm{e}^{3x^2} = \sum_{n=0}^{\infty} \frac{(3x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n (x^2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} \frac{3^n}{n!} x^{2n}}$$

Since the first series converges for all x, so does the second series; i.e. the radius and interval of convergence of the series for e^{3x^2} are ∞ and $(-\infty, \infty)$, respectively.

(b) $\dfrac{1}{1+4x}$

Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ whenever $|x| < 1$,

$$\frac{1}{1+4x} = \frac{1}{1-(-4x)} = \sum_{n=0}^{\infty} (-4x)^n = \boxed{\sum_{n=0}^{\infty} (-4)^n x^n}$$

Since the first series converges whenever x lies in the interval $(-1, 1)$, the second series converges whenever

$$-1 < -4x < 1 \iff 1 > 4x > -1 \iff -1 < 4x < 1 \iff -1/4 < x < 1/4.$$

Thus, the radius and interval of convergence of the series for $\dfrac{1}{1+4x}$ are $1/4$ and $(-1/4, 1/4)$, respectively.

(c) $\sin 2x$

Since $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$,

$$\sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} 2 \cdot 2^{2n} x^{2n+1} = \boxed{2 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n+1)!} x^{2n+1}}$$

Since the first series converges for all x, so does the second series; i.e. the radius and interval of convergence of the series for $\sin 2x$ are ∞ and $(-\infty, \infty)$, respectively.

3. $\boxed{\textit{12 points}}$ *For each of the series below, decide if it converges or diverges.* **You must justify your answer to receive full credit.** *You do not need to find the sum of convergent series.*

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n+3^n} = \frac{1}{4} + \frac{2}{17} + \frac{1}{19} + \frac{8}{353} + \ldots$

This series $\boxed{\text{converges}}$ as $2^n/(5^n+3^n)$ looks like $2^n/5^n = (2/5)^n$ and a geometric series with $r=2/5$ converges. To actually **justify** this, you can use:

- the Ratio Test (note the exponents of n):

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}/(5^{n+1}+3^{n+1})}{2^n/(5^n+3^n)} = \frac{2^{n+1}}{2^n} \cdot \frac{5^n+3^n}{5^{n+1}+3^{n+1}} = 2 \cdot \frac{5^n/5^n + 3^n/5^n}{5 \cdot 5^n/5^n + 3 \cdot 3^n/5^n}$$
$$= 2 \cdot \frac{1+(3/5)^n}{5+3\cdot(3/5)^n} \longrightarrow \frac{2}{5}$$

  Since $2/5 < 1$, the series converges.

- the Limit Comparison Test with $b_n = 2^n/5^n = (2/5)^n > 0$:

$$\frac{a_n}{b_n} = \frac{2^n/(5^n+3^n)}{2^n/5^n} = \frac{5^n}{5^n+3^n} = \frac{5^n/5^n}{5^n/5^n+3^n/5^n} = \frac{1}{1+(3/5)^n} \longrightarrow 1;$$

  since this limits exists and the series $\displaystyle\sum_{n=1}^{\infty}(2/5)^n$ converges (being a geometric series with $|r|<1$), so does the initial series.

- the Comparison Test with $b_n = 2^n/5^n = (2/5)^n > 0$: since

$$0 \le 2^n/(5^n+3^n) \le 2^n/5^n$$

  and the series $\displaystyle\sum_{n=1}^{\infty}(2/5)^n$ converges (being a geometric series with $|r|<1$), so does the initial, "smaller", series.

  Note: unlike the previous two tests, this test would have not worked if $+$ were $-$ in the denominator.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{1}{4\ln 4} + \ldots$

This is a classic example when the Integral Test is most suitable, since the function $f(x) = 1/x(\ln x)$ obtained by replacing n by x makes sense and is rather simple. Since $f(x) > 0$, is continuous and decreasing for $x \ge 2$, the series converges if and only if the integral

$$\int_2^{\infty} f(x)\mathrm{d}x = \int_2^{\infty} \frac{1}{x(\ln x)}\mathrm{d}x = \int_{\ln 2}^{\infty} \frac{1}{u}\mathrm{d}u = \ln u\Big|_{\ln 2}^{\infty}$$

does (in the above $u = \ln x$). Since $\ln u \longrightarrow \infty$ as $u \longrightarrow \infty$, the integral $\boxed{\text{diverges}}$ and so does the series.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{10^n}{n!} = 1 + 10 + 50 + \frac{500}{3} + \frac{1250}{3} + \ldots$

This series $\boxed{\text{converges}}$ as any exponent (with constant base, such as 10^n, not n^n) is negligible compared to $n!$ as $n \longrightarrow \infty$. Given the presence of 10^n and $n!$, the Ratio Test is most suitable to justify this:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{10^{n+1}/(n+1)!}{10^n/n!} = \frac{10^{n+1}}{10^n} \cdot \frac{n!}{(n+1)!} = 10\frac{1}{n+1} \longrightarrow 0.$$

Since $0 < 1$, the series converges.

Note: with some effort, the Comparison Test and the Limit Comparison Test can be used as well.

The sum of this series can be determined as well:

$$\sum_{n=0}^{\infty} \frac{10^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \Big|_{x=10} = \mathrm{e}^x \big|_{x=10} = \mathrm{e}^{10}.$$

Since the power series

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

converges for all x, this also gives another proof that the $x = 10$ series converges.

4. $\boxed{\textit{12 points}}$ *Let* $f(x) = x^{1/3}$.

(a) *Find the Taylor polynomial of degree* 2 *centered at* $a = 8$ *for* $f(x)$.

$$\begin{aligned} f(x) &= x^{1/3} & f(8) &= 8^{1/3} = 2 \\ f'(x) &= \frac{1}{3}x^{-2/3} & f'(8) &= \frac{1}{3}8^{-2/3} = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12} \\ f''(x) &= \frac{1}{3}\cdot\frac{-2}{3}x^{-5/3} = \frac{-2}{9}x^{-5/3} & f''(8) &= -\frac{2}{9}8^{-5/3} = -\frac{2}{9}\cdot\frac{1}{32} = -\frac{1}{144} \end{aligned}$$

Thus, the Taylor polynomial of degree 2 centered at $a = 8$ for $f(x)$ is

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 = \boxed{2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2}$$

(b) *If the result of the previous part is used to estimate* $9^{1/3}$*, how large can the error be (according to Taylor's theorem)?*
According to Taylor's theorem (p608),

$$\left|f(9) - T_2(9)\right| \le \frac{M}{(2+1)!}|9-8|^{2+1}$$

if M is such that $|f'''(x)| \le M$ for all x in the interval $[8, 9]$. Since

$$f'''(x) = \frac{-2}{9}\cdot\frac{-5}{3}\cdot x^{-8/3} = \frac{10}{27}x^{-8/3}$$

and $x^{-8/3}$ is a decreasing function of x, we can take

$$M = |f'''(2)| = \frac{10}{27}\cdot\frac{1}{256} = \frac{5}{3456}.$$

So

$$\left|f(9) - T_2(9)\right| \le \frac{1}{6}\cdot\frac{5}{3456} = \boxed{\frac{5}{20736}}$$

Note: As can be seen by induction, for $n \geq 2$ the n-th derivative is given by

$$f^{\langle n \rangle}(x) = (-1)^{n-1} \cdot \frac{2 \cdot 5 \cdot \ldots (3n-7)(3n-4)}{3^n} x^{-(3n-1)/3}$$

So

$$f^{\langle n \rangle}(8) = (-1)^{n-1} \cdot \frac{2 \cdot 5 \cdot \ldots (3n-7)(3n-4)}{3^n 2^{3n-1}}.$$

The Taylor series expansion of f at $x=8$ is thus given by

$$\sum_{n=0}^{\infty} \frac{f^{\langle n \rangle}(8)}{n!}(x-8)^n = 2 + \frac{x-8}{12} - \sum_{n=2}^{\infty} (-1)^n \frac{2 \cdot 5 \cdot \ldots (3n-7)(3n-4)}{3^n 2^{3n-1} n!}(x-8)^n$$

If b_n denotes the absolute value of the coefficient of $(x-8)^n$ above,

$$\begin{aligned} \frac{b_{n+1}}{b_n} &= \frac{2 \cdot 5 \cdot \ldots (3n-7)(3n-4)(3n-1)/(3^{n+1} 2^{3(n+1)-1}(n+1)!)}{2 \cdot 5 \cdot \ldots (3n-7)(3n-4)/(3^n 2^{3n-1} n!)} \\ &= (3n-1) \cdot \frac{3^n 2^{3n-1} n!}{3^{n+1} 2^{3n-1+3}(n+1)!} = \frac{3n-1}{3 \cdot 2^3 \cdot (n+1)} \longrightarrow \frac{3}{3 \cdot 8} = \frac{1}{8}. \end{aligned}$$

In particular, $b_n \longrightarrow 0$. Since for all $x \geq 8$

$$\frac{|f^{\langle n \rangle}(x)|}{n!} \leq \frac{f^{\langle n \rangle}(8)}{n!} = b_n,$$

by Taylor's Inequality the Taylor series converges to $f(x)$ for all x in $[8, 9]$. Thus,

$$9^{1/3} = 2 + \frac{1}{12} - \sum_{n=2}^{\infty} (-1)^n \frac{2 \cdot 5 \cdot \ldots (3n-7)(3n-4)}{3^n 2^{3n-1} n!} = \frac{25}{12} - \sum_{n=2}^{\infty} (-1)^n b_n.$$

This series is alternating. Since $b_{n+1}/b_n \longrightarrow 1/8$ as $n \longrightarrow \infty$, $b_n > b_{n+1}$ for all $n \geq$ some N and $b_n \longrightarrow 0$. So by the Alternating Series Estimation Theorem (p588),

$$\left| 9^{1/3} - T_2(9) \right| < b_3 = \frac{2}{3^3 2^{3 \cdot 3-1} 3!} = \frac{5}{20736}.$$

So in this case, the Alternating Series Estimation Theorem and Taylor Inequality give essentially (but not quite) the same error bound. The T_2-estimate for $\sqrt[3]{9} \approx 2.08008$ is $599/288 \approx 2.07986$, which is indeed off by less than $5/20736 \approx .00024$. The T_2-estimate is an *under*-estimate because the last term used is negative.

5. $\boxed{\textit{12 points}}$ *Find the interval of convergence for the power series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{(n+1)4^n}$$

To find the **r**adius of convergence R, use the **R**atio Test with $a_n = \dfrac{(-1)^n (x-3)^n}{(n+1)4^n} \neq 0$:

$$\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} &= \frac{|x-3|^{n+1}/(((n+1)+1)4^{n+1})}{|x-3|^n/((n+1)4^n)} \\
&= \frac{|x-3|^{n+1}}{|x-3|^n} \cdot \frac{n+1}{n+2} \cdot \frac{4^n}{4^{n+1}} = |x-3| \cdot \frac{1+1/n}{1+2/n} \cdot \frac{1}{4} \longrightarrow |x-3| \cdot 1 \cdot \frac{1}{4} = \frac{|x-3|}{4}
\end{aligned}$$

So the series converges if $|x-3|/4 < 1$, or $|x-3| < 4$, and diverges if $|x-3|/4 > 1$, or $|x-3| > 4$. It remains to determine what happens for $|x-3| = 4$. If $x-3=4$, we get the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\,.$$

This series converges, since it is alternating (odd terms are positive, even terms are negative), $1/n \longrightarrow 0$ as $n \longrightarrow 0$, and $1/n > 1/(n+1)$. So the end-point $x=7$ is included into the interval of convergence. If $x-3=-4$, we get the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}\,.$$

This series diverges by the p-Series Test. So the end-point $x=-1$ is not included into the interval of convergence. The interval of convergence is thus $\boxed{\text{(-1,7]}}$

6. $\boxed{\textit{12 points}}$ *Solve the initial value problem.*

$$\frac{dy}{dx} = 6x^2y^2 \qquad y(0) = 1$$

This is a separable equation. So move everything involving y to the left-hand side and everything involving x to the right-hand side and integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2y^2 \iff \frac{\mathrm{d}y}{y^2} = 6x^2\mathrm{d}x \iff \int \frac{\mathrm{d}y}{y^2} = \int 6x^2\mathrm{d}x \iff -\frac{1}{y} = 2x^3 + C.$$

Since $y(0) = 1$, $-1/1 = 2 \cdot 0^3 + C$, and so $C = -1$. This gives

$$-\frac{1}{y} = 2x^3 - 1 \iff \frac{1}{y} = -2x^3 + 1 \iff \boxed{y = \frac{1}{1 - 2x^3}}$$

Note 1: you can check that this is correct by plugging $y = y(x)$ into the initial condition and the differential equation:

$$y(0) = \frac{1}{1 - 2 \cdot 0^3} = 1 \ \checkmark$$
$$\frac{dy}{dx} = -\frac{1}{(1 - 2x^3)^2} \cdot (-2 \cdot 3x^2) = \frac{6x^2}{(1 - 2x^3)^2} = 6x^2y^2 \ \checkmark$$

Note 2: If you are asked for the general solutions of a separable equation, you must remember to check for the constant solutions because you might have divided by 0. However, if you find a solution to an initial-value problem which is not constant, then you are done.

Do **any three** of problems 7, 8, 9, and 10. Cross out the one you don't want graded

7. $\boxed{\text{10 points}}$ *Find the solution $y(t)$ to the initial value problem*

$$y'' + 2y' - 3y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

The associated quadratic polynomial is

$$r^2 + 2r - 3 = 0 \quad \Longrightarrow \quad (r+3)(r-1) = 0.$$

So the roots are $r_1 = -3$ and $r_2 = 1$. Since they are real and distinct, the general solution is

$$y(t) = C_1 \mathrm{e}^{-3t} + C_2 \mathrm{e}^{t}$$

We need to find C_1 and C_2 such that

$$y(0) = C_1 \mathrm{e}^0 + C_2 \mathrm{e}^0 = C_1 + C_2 = 0 \quad \text{and} \quad y'(0) = 1.$$

Since $y'(t) = -3C_1 \mathrm{e}^{-3t} + C_2 \mathrm{e}^{t}$, the second condition is $-3C_1 + C_2 = 1$. Thus, we need to solve the system

$$\begin{cases} C_1 + C_2 = 0 \\ -3C_1 + C_2 = 1 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} C_1 + C_2 = 0 \\ -4C_1 = 1 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} C_1 = -1/4 \\ C_2 = 1/4 \end{cases}$$

So the solution to the initial-value problem is $\boxed{y(t) = -\frac{1}{4}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{t}}$

Note: you can check that this is correct by plugging $y = y(t)$ into the initial conditions and the differential equation:

$$\begin{aligned}
y(0) &= -\frac{1}{4}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^0 = -\frac{1}{4} + \frac{1}{4} = 0 \ \checkmark \\
y'(t) &= -\frac{-3}{4}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{t} = \frac{3}{4}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{t}, \quad y'(0) = \frac{3}{4}\mathrm{e}^0 + \frac{1}{4}\mathrm{e}^0 = 1 \ \checkmark \\
y''(t) &= \frac{3(-3)}{4}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{t} = \frac{-9}{4}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{t} \\
y'' + 2y' - 3y &= \left(\frac{-9}{4}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{t}\right) + 2\left(\frac{3}{4}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{t}\right) - 3\left(-\frac{1}{4}\mathrm{e}^{-3t} + \frac{1}{4}\mathrm{e}^{t}\right) \\
&= \left(\frac{-9}{4} + \frac{6}{4} + \frac{3}{4}\right)\mathrm{e}^{-3t} + \left(\frac{1}{4} + \frac{2}{4} - \frac{3}{4}\right)\mathrm{e}^{t} = 0 \ \checkmark
\end{aligned}$$

Do **any three** of problems 7, 8, 9, and 10. Cross out the one you don't want graded

8. $\boxed{\textit{10 points}}$ *A population of armadillos is well modeled by a logistic equation with a carrying capacity of* 1000*. Assume that initially there are 100 armadillos, so the equation is*

$$P(t) = kP\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

where t is in years and k is some constant.

(a) *Determine k if after one year there are 200 armadillos.*
The solution to the logistic model problem is given by

$$P(t) = \frac{K}{1 - \frac{P(0)-K}{P(0)}\mathrm{e}^{-kt}},$$

where $K = 1000$ is the carrying capacity and $P(0) = 100$ is the initial population. So in this case,

$$P(t) = \frac{1000}{1 - \frac{100-1000}{100}\mathrm{e}^{-kt}} = \frac{1000}{1 + 9\mathrm{e}^{-kt}}$$

Since $P(1) = 200$,

$$200 = \frac{1000}{1+9\mathrm{e}^{-k}} \iff 1 = \frac{5}{1+9\mathrm{e}^{-k}} \iff 1 + 9\mathrm{e}^{-k} = 5$$
$$\iff \mathrm{e}^{-k} = 4/9 \iff k = -\ln(4/9) = \boxed{2\ln(3/2) \quad 1/\text{year}}$$

Note: the logistic equation is separable and solved on pp538-539

(b) *When will there be 500 armadillos?* Since $\mathrm{e}^{-k} = 4/9$,

$$P(t) = \frac{1000}{1 + 9(4/9)^t}$$

So we need to find t so that

$$500 = \frac{1000}{1+9(4/9)^t} \iff 1 = \frac{2}{1+9(4/9)^t} \iff 1 + 9(4/9)^t = 2$$
$$\iff (4/9)^t = 1/9 \iff t\ln(4/9) = \ln(1/9) = -2\ln 3$$
$$\iff t = \frac{-2\ln 3}{-2\ln(3/2)} = \boxed{\frac{\ln 3}{\ln(3/2)} \quad \text{years}}$$

Do **any three** of problems 7, 8, 9, and 10. Cross out the one you don't want graded

9. $\boxed{\text{10 points}}$ *Consider the initial value problem*

$$y'' + y' - xy = 0 \qquad\qquad y(0) = 1 \quad y'(0) = 0$$

(a) *Use power series to find a degree 4 polynomial approximation to $y(x)$.*

Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Since $y(0)=1$, $c_0=1$; since $y'(0)=0$, $c_1=0$. Also,

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n \\ y'' &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n \\ y'' + y' - xy &= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= 2c_2 + c_1 + \sum_{n=1}^{\infty} \big((n+2)(n+1)c_{n+2} + (n+1)c_{n+1} - c_{n-1}\big) x^n = 0. \end{aligned}$$

Along with the initial conditions, this gives

$$\begin{aligned} c_0 &= 1, \qquad c_1 = 0, \qquad c_2 = -\frac{1}{2} c_1 = 0 \\ c_{n+2} &= -\frac{1}{n+2} c_{n+1} + \frac{1}{(n+1)(n+2)} c_{n-1} \quad n \geq 1 \\ c_3 &= -\frac{1}{3} c_2 + \frac{1}{3 \cdot 2} c_0 = \frac{1}{6}, \qquad c_4 = -\frac{1}{4} c_3 + \frac{1}{4 \cdot 3} c_1 = -\frac{1}{24} \end{aligned}$$

So the degree 4 polynomial approximation is $\boxed{y(x) = 1 + \frac{1}{6} x^3 - \frac{1}{24} x^4}$

(b) *Is $y(0.5) > 1$? Justify your answer.* (assuming the series converges for $x = .5$)

$\boxed{\text{Yes}}$ Since $y'(0) = y''(0) = 0$ and $y'''(0) > 0$, $y(x)$ is increasing near $x = 0$. If $y(.5) \leq y(0) = 1$, then $y(x)$ reaches a local max at some point b with $0 < b < .5$ and $y(b) > y(0) > 0$. Thus, $y'(b) = 0$ (b is a critical point) and so

$$y''(b) = -y'(b) + by(b) > 0;$$

so b cannot be a local max. Thus, it is impossible that $y(.5) \leq 1$.

Do **any three** of problems 7, 8, 9, and 10. Cross out the one you don't want graded

10. $\boxed{\textit{10 points}}$ *A population of birds and insects is modeled by*

$$\frac{dx}{dt} = 0.4x(1 - 0.000005x) - 0.002xy \qquad \frac{dy}{dt} = -0.2y + 0.000008xy$$

where $x(t)$ is the number of insects and $y(t)$ is the number of birds.

(a) *Find **all** of the equilibrium solutions (also called "fixed points" or "constant solutions"). If there are none, write "None" and justify your answer.*
We need to find pairs of numbers (x, y) such that

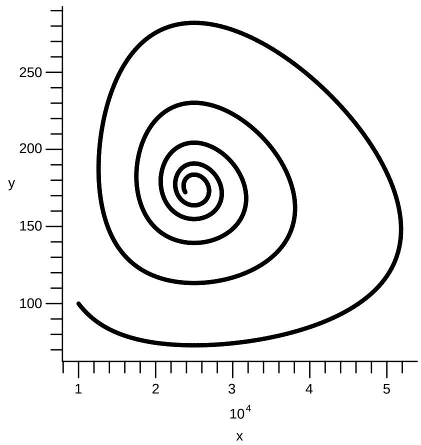
$$\begin{cases} 0.4x(1-0.000005x) - 0.002xy = \frac{2x}{5}\left(1 - \frac{x}{200{,}000}\right) - \frac{xy}{500} = \frac{2x}{5}\left(1 - \frac{x}{200{,}000} - \frac{y}{200}\right) = 0 \\ -0.2y + 0.000008xy = -\frac{y}{5} + \frac{xy}{125{,}000} = -\frac{y}{5}\left(1 - \frac{x}{25{,}000}\right) = 0 \end{cases}$$

$$\iff \qquad \begin{cases} x = 0 \text{ or } 1 - \frac{x}{200{,}000} - \frac{y}{200} = 0 \\ y = 0 \text{ or } 1 - \frac{x}{25{,}000} = 0 \end{cases}$$

We need to choose precisely one condition from each of the two lines. If we choose the first condition from the first line, then we must take the first condition from the second line (since the second condition would give a contradiction); this gives $(x, y) = 0$. If we choose the second condition from the first line and the first condition from the second line, we get $(x, y) = (200000, 0)$. Finally, if we choose the second condition from the first line and the second condition from the second line, we get $(x, y) = (25000, 175)$. So there are 3 equilibrium solutions

$$\boxed{(x, y) = (0, 0), (200000, 0), (25000, 175)}$$

(b) *The figure shows a phase trajectory starting with 10,000 insects and 100 birds. Describe what happens to the bird and insect populations as time passes.*



The insect population first increases, while the bird population first decreases and then increases. The insect population then starts to decrease, as the bird population continues to increase and then starts to decrease. The insect population at first continues to decrease, but then starts to increase. This cycle then keeps on repeating qualitatively. However, all local minima in the insect population increase toward 25,000, while all local maxima in the insect population decrease toward 25,000; all local minima in the bird population increase toward 175, while all local maxima in the bird population decrease toward 175. The populations of the two approach the steady state of 25,000 insects and 175 birds. All minima/maxima in the bird population occur when there are 25,000 insects (as also follows from $\mathrm{d}y/\mathrm{d}t$ equation).