PROOF OF THE RIEMANNIAN PENROSE INEQUALITY WITH CHARGE FOR MULTIPLE BLACK HOLES

MARCUS KHURI, GILBERT WEINSTEIN, AND SUMIO YAMADA

ABSTRACT. We present a proof of the Riemannian Penrose inequality with charge in the context of asymptotically flat initial data sets for the Einstein-Maxwell equations, having possibly multiple black holes with no charged matter outside the horizon, and satisfying the relevant dominant energy condition. The proof is based on a generalization of Hubert Bray's conformal flow of metrics adapted to this setting.

1. INTRODUCTION

In a seminal paper [26] (see also [27]), in which he proposed the celebrated cosmic censorhip conjecture, R. Penrose also proposed a related inequality, now referred to as the Penrose Inequality. The inequality is derived from cosmic censorship via a heuristic argument relying on Hawking's area theorem [14]. Consider an asymptotically flat Cauchy surface in a spacetime satisfying the dominant energy condition, having ADM mass m, and containing an event horizon of area $A = 4\pi\rho^2$, which undergoes gravitational collapse and settles to a Kerr-Newman solution. Since the ADM mass m_∞ of the final state is no greater than m, the area radius ρ_∞ is no less than ρ, and the final state must satisfy $m_\infty \geq \frac{1}{2}\rho_\infty$ in order to avoid naked singularities, it must have been the case that $m \geq \frac{1}{2}\rho$ also at the beginning of the evolution. A counterexample to the Penrose inequality would therefore suggest data which leads under the Einstein evolution to naked singularities, and a proof of the Penrose inequality may be viewed as evidence in support of cosmic censorship.

The event horizon is indiscernible in the original slice without knowing the full evolution, however one may, without disturbing this inequality, replace the event horizon by the outermost minimal area enclosure of the apparent horizon (the boundary of the region admitting trapped surfaces). The inequality further simplifies in the time-symmetric case, in which the outermost minimal area enclosure of the apparent horizon coincides with the outermost minimal surface, and the dominant energy condition reduces simply to nonnegative scalar curvature. This leads to the Riemannian version of the inequality: the ADM mass m and the area radius r of the outermost minimal surface in an asymptotically flat 3-manifold of nonnegative scalar curvature, satisfy

$$(1.1) \qquad\qquad m \geq \frac{\rho}{2}$$

with equality if and only if the manifold is isometric to the canonical slice of the Schwarzschild spacetime. Note that this characterizes the canonical slice of Schwarzschild as the unique minimizer of m among all such 3-manifolds admitting an outermost horizon of area $A = 4\pi\rho^2$.

This inequality was first proved in the special case where the horizon is connected by Huisken and Ilmanen [15] using the inverse mean curvature flow, an approach proposed by Jang and Wald [17], following Geroch [10] who had shown that the Hawking mass is nondecreasing under the flow. The

inequality was proven in full generality by Bray [1] using a conformal flow of the initial Riemannian metric, and the positive mass theorem [28], [31].

We now turn to the charged case which is somewhat more subtle. It is natural to conjecture as above that the Reissner-Nordström spacetime, the charged analog of Schwarzschild, is the unique minimizer of m, given ρ and q. Since Reissner-Nordström satisfies $m = \frac{1}{2}(\rho + q^2/\rho)$ where q is the total charge, one is thus lead to conjecture that in any asymptotically flat data satisfying an appropriate energy condition it holds

$$(1.2) \qquad m \geq \frac{1}{2}\left(\rho + \frac{q^2}{\rho}\right),$$

with equality if and only if the initial data is the canonical slice of Reissner-Nordström. This follows from [15], and is based on Jang [16], but only for a connected horizon, since the proof relies on inverse mean curvature flow. In fact (1.2) can fail if the horizon is not connected, and a counterexample based on the Majumdar-Papapetrou spacetime with two black holes was constructed in [30]. This counterexample nonetheless does not suggest a counterexample to cosmic censorship. This is because the right-hand side of (1.2) is not monotone increasing in ρ. Indeed, already Jang observed that (1.2) is equivalent to two inequalities:

$$(1.3) \qquad m - \sqrt{m^2 - q^2} \leq \rho \leq m + \sqrt{m^2 - q^2}.$$

Cosmic censorship suggests that the upper bound always holds, while the counterexample in [30] violates the lower bound. It turns out, however, that the lower bound also holds, and furthermore is motivated by cosmic censorship in the case of a single black hole, or more generally when $\rho \geq |q|$ (see [7]).

In this paper, we prove the upper bound in (1.3) for multiple black holes. By the positive mass theorem with charge, $m \geq |q|$ with equality if and only if the data is Majumbdar-Papapetrou [12]; see [5], [19] for the rigidity result. Hence if $\rho \leq |q|$, the upper bound in (1.3) follows immediately

$$(1.4) \qquad \rho \leq |q| \leq m \leq m + \sqrt{m^2 - q^2}.$$

It thus only remains to prove the upper bound under the additional hypothesis $|q| < \rho$. Under this hypothesis, it is the lower bound that follows immediately

$$(1.5) \qquad m \leq |q| + \sqrt{m^2 - q^2} < \rho + \sqrt{m^2 - q^2}.$$

In fact the condition $|q| \leq \rho$ is always valid for a single horizon, in light of its stability [11], [20], however for multiple horizons this inequality is indeed a nontrivial restriction. In view of all the above, the upper bound in (1.3) is equivalent to (1.2) under the additional hypothesis $|q| \leq \rho$. The proof of this latter statement will be based on a generalization of Bray's conformal flow. It should also be noted that the right-hand side of (1.2) is nondecreasing as a function of ρ (with fixed q), precisely when $|q| \leq \rho$. Thus, (1.2) with the auxiliary area-charge inequality may also be derived using the heuristic Penrose argument.

The inequalities discussed in the previous paragraphs are most easily visualized in Figure 1. The white area is the positive mass theorem $m \geq 0$. The light shaded area is the Penrose inequality, and the other two darker shaded areas are the charged Penrose inequality. The inequality represented by the region to the left of the dashed vertical line $\rho = |q|$, and above the solid horizontal line $m = |q|$, follows from the charged positive mass theorem. Moreover, the dotted curve in this region is the lower bound in (1.3), or a continuation of the equality curve from (1.2); the black dot represents the counterexample in [30]. This paper deals with the proof of the inequality represented by the darkest shaded region, to the right of $\rho = |q|$ and above $m = \frac{1}{2}(\rho + q^2/\rho)$, which is the upper bound
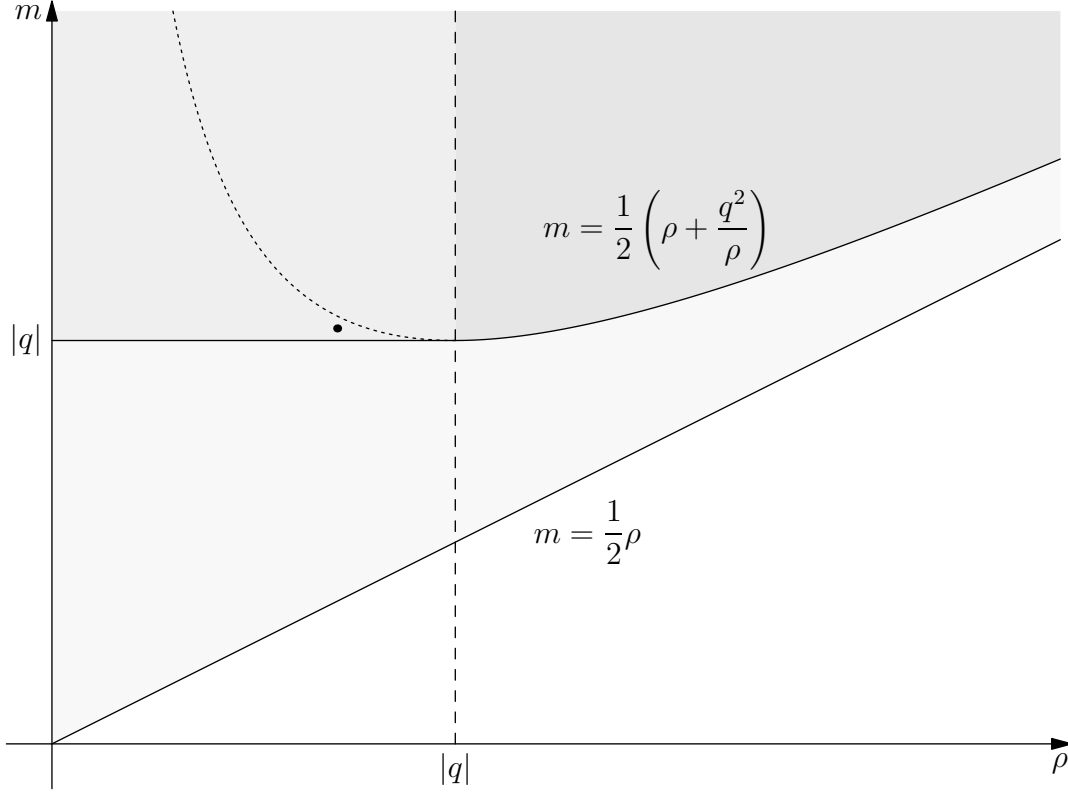
FIGURE 1. Graphical representation of geometric inequalities

in (1.3) with $|q| \leq \rho$, or (1.2) with $|q| \leq \rho$. Lastly, it should be noted that every configuration with one black hole component lies to the right of the vertical dashed line $\rho = |q|$, or equivalently, every configuration to the left of the vertical line has multiple black hole components.

We end the introduction with a few definitions and the statement of our main theorem and its corollaries. An initial data set (M, g, E, B) consists of a 3-manifold M, a Riemannian metric g, and vector fields E and B. It will be assumed that the data satisfy the Maxwell constraints with no charges outside the horizon $\operatorname{div}_g E = \operatorname{div}_g B = 0$, and that the charged dominant energy condition

(1.6) $$16\pi\mu_{EM} = R_g - 2(|E|_g^2 + |B|_g^2) \geq 0$$

is valid, where R_g is the scalar curvature of g and μ_{EM} is the energy density of the matter fields after contributions from the electromagnetic field have been removed. It should be noted that typically the charged dominant energy condition is given by the slightly stronger statement $\mu_{EM} \geq |J_{EM}|_g$, where $4\pi J_{EM} = E \times B$ is minus one half the momentum density of the electromagnetic field. It turns out, however, that for the results of the current paper the hypothesis (1.6) is sufficient. Moreover in the case of equality for (1.2), it will be shown that E and B are linearly dependent so that $J_{EM} = 0$. Typically when Penrose-type inequalities are saturated, the vanishing of the momentum density arises at least in part due to the stronger version of the charged dominant energy condition. Nevertheless, the same result holds here under the weaker form of the energy condition (1.6).

We assume further that the data is strongly asymptotically flat, meaning that there is a compact set K such that $M \setminus K$ is the finite union of disjoint ends, and in the coordinates given on each end

the fields decay according to

$$(1.7) \qquad g_{ij} = \delta_{ij} + O_2(|x|^{-1}), \quad E_i = O_1(|x|^{-2}), \quad B_i = O_1(|x|^{-2}).$$

and R_g is integrable. This guarantees that the ADM mass and the total electric and magnetic charges

$$(1.8) \qquad m = \frac{1}{16\pi} \int_{S_\infty} (g_{ij,j} - g_{jj,i})\nu^i \, dA,$$

$$(1.9) \qquad q_e = \frac{1}{4\pi} \int_{S_\infty} E_i \nu^i \, dA, \quad q_b = \frac{1}{4\pi} \int_{S_\infty} B_i \nu^i \, dA,$$

are well defined, with squared total charge $q^2 = q_e^2 + q_b^2$. Here ν is the outer unit normal, and the limit is taken in a designated end. Without loss of generality, we assume that the magnetic charge $q_b = 0$, and so from now on $q = q_e$. This can always be achieved by a fixed rotation in (E, B) space. Conformally compactifying all but the designated end, we can now restrict our attention to surfaces which bound compact regions, and define S_2 to enclose S_1 to mean $S_1 = \partial K_1$, $S_2 = \partial K_2$ and $K_1 \subset K_2$. An outermost horizon is a compact minimal surface not enclosed in any other compact minimal surface. The following results were first discussed in the announcement [21].

Theorem 1.1. *Let (M, g, E, B) be a strongly asymptotically flat initial data set with outermost minimal surface boundary of area $A = 4\pi\rho^2$, satisfying the charged dominant energy condition and the Maxwell constraints without charged matter. If $|q| \le \rho$, then (1.2) holds with equality if and only if the data set arises as the canonical slice of the Reissner-Nordström spacetime.*

Corollary 1.2. *Let (M, g, E, B) be a strongly asymptotically flat initial data set with outermost minimal surface boundary of area $A = 4\pi\rho^2$, satisfying the charged dominant energy condition and the Maxwell constraints without charged matter. Then the upper bound in (1.3) holds with equality if and only if the data set arises as the canonical slice of the Reissner-Nordström spacetime.*

Corollary 1.3. *Assume that the above hypotheses hold. If q and ρ are fixed with $|q| \le \rho$, then the canonical Reissner-Nordström slice is the unique minimizer of m. Moreover, if m and q are fixed with $m \ge |q|$, then the canonical Reissner-Nordström slice is the unique maximizer of ρ.*

In the case when charged matter is present, and in particular is not compactly supported, counterexamples exist [22]. The full version of the inequality in the non-time-symmetric case remains an open problem. A reduction argument similar to that proposed by Bray and the first author in [3], [4], has been given in [9] (see also [18]). However it only applies to the case of a single black hole, as it is based on a coupling of the static Jang equation with inverse mean curvature flow. Coupling the static Jang equation to Bray's conformal flow is possible and also leads to a reduction argument for the Penrose inequality; this was briefly discussed in [4]. It seems likely then that a coupling to the charged conformal flow presented in this paper, should reduce the general charged Penrose inequality to the time-symmetric case as well. Whether the coupled system admits a solution with the appropriate boundary and asymptotic behavior is then an important open question.

This paper is organized as follows. In the next two sections, a generalized version of Bray's conformal flow will be defined, and its existence will be established. In Section 4, it will be shown that the flowing outermost minimal surfaces move out into the asymptotic end and eventually exhaust the manifold. In Bray's original flow this exhaustion always occurs, however for the charged conformal flow, the exhaustion can only happen when $|q| \le \rho$, and is one of the most interesting and surprising differences between this flow and the original. Section 5 is dedicated to monotonicity of the mass, which follows from a modified doubling argument in analogy to the original flow. In Section 6 we solve

a quasi-linear elliptic equation, whose solution plays an important part in the proof of monotonicity, and in Section 7 proofs of the main theorem will be given. Lastly, two appendices are added which include an auxiliary a priori estimate, and the model example for the new flow.

2. The Charged Conformal Flow

The goal here is to construct a flow (M_t, g_t, E_t, B_t) of asymptotically flat initial data for the Einstein-Maxwell equations, starting from the given initial data (M, g, E, B) at $t = 0$, and which preserves the boundary area $|\partial M_t|_{g_t}$, total charge q_t, Maxwell constraints, the charged dominant energy condition, and exhibits a nonincreasing ADM mass $m(t)$. Moreover, this flow should reduce to Bray's conformal flow when $|E|_g = |B|_g = 0$, and should proceed by coordinate rescalings in the standard initial data for Reissner-Nordström. This flow, defined below, will be referred to as the *charged conformal flow*.

Consider the conformal flow of metrics defined by $g_t = u_t^4 g$, with $u_0 \equiv 1$. Given the metric g_t, define ∂M_t to be the outermost minimal area enclosure of ∂M in (M, g_t), and denote the region enclosed by ∂M_t and spatial infinity by M_t. It will turn out that ∂M_t does not intersect ∂M, and hence it is an outermost minimal surface. Also set $E_t^i = u_t^{-6} E^i$ and $B_t^i = u_t^{-6} B^i$. Given g_t, E_t, B_t, and ∂M_t, define v_t to be the unique solution of the Dirichlet problem

$$(2.1) \qquad \Delta_{g_t} v_t - \left(|E_t|_{g_t}^2 + |B_t|_{g_t}^2 \right) v_t = 0, \quad \text{on} \quad M_t,$$

$$(2.2) \qquad v_t = 0 \quad \text{on} \quad \partial M_t, \qquad v_t \to -1 \quad \text{as} \quad r \to \infty.$$

By expanding the solution in spherical harmonics, it follows that

$$(2.3) \qquad v_t = -1 + \frac{\gamma_t}{r} + O\left(\frac{1}{r^2} \right) \quad \text{as} \quad r \to \infty,$$

for some constant $\gamma_t > 0$. On $M \setminus M_t$ set $v_t \equiv 0$. The function v_t will act as the logarithmic velocity of the flow $\frac{d}{dt} u_t = v_t u_t$. Thus given v_t, define $u_t = \exp\left(\int_0^t v_s ds \right)$.

The existence and regularity of this flow is similar to that of the original conformal flow, and will be discussed in the next section. Moreover, it is clear that it reduces to Bray's flow when the electromagnetic field vanishes, and indeed is trivial in the Reissner-Nordström solution as is shown in Appendix B. We now prove that it satisfies the other desired properties.

Theorem 2.1. *For all $t \geq 0$ it holds that $q_t = q$, $|\partial_t M|_{g_t} = |\partial M|_g$, $\mathrm{div}_{g_t} E_t = \mathrm{div}_{g_t} B_t = 0$ and*

$$(2.4) \qquad R_{g_t} \geq 2 \left(|E_t|_{g_t}^2 + |B_t|_{g_t}^2 \right).$$

Proof. The same arguments used by Bray [1] apply to show that the area remains constant throughout the flow (see Section 3 below). In order to show that the charge remains constant, observe that $|E_t|_{g_t}^2 = u_t^{-8} |E|_g^2$, and hence

$$(2.5) \qquad 4\pi q_t = \int_{S_\infty} g_t(E_t, \nu_t) dA_{g_t} = \int_{S_\infty} g(E, \nu) dA_g = 4\pi q.$$

Furthermore

$$(2.6) \qquad \mathrm{div}_{g_t} E_t = \frac{1}{\sqrt{\det g_t}} \partial_i \left(\sqrt{\det g_t} E_t^i \right) = \frac{u_t^{-6}}{\sqrt{\det g}} \partial_i \left(\sqrt{\det g} E^i \right) = u_t^{-6} \mathrm{div}_g E = 0,$$

and similarly for the magnetic field B.

It remains to show that the charged dominant energy condition remains preserved throughout the flow. Let L_g denotes the conformal Laplacian, then by a standard formula

$$(2.7) \qquad u_t^5 R_{g_t} = -8L_g u_t = -8\left(\Delta_g u_t - \frac{1}{8}R_g u_t\right),$$

so that with help from the conformal covariance of L_g it follows that

$$
\begin{aligned}
\frac{d}{dt}(u_t^8 R_{g_t}) &= \frac{d}{dt}[u_t^3(u_t^5 R_{g_t})] \\
&= 3u_t^2\left(\frac{d}{dt}u_t\right)u_t^5 R_{g_t} + u_t^3 \frac{d}{dt}(-8L_g u_t) \\
&= 3v_t u_t^8 R_{g_t} - 8u_t^3 L_g(u_t v_t) \\
&= 3v_t u_t^8 R_{g_t} - 8u_t^8 L_{g_t} v_t \\
&= 3v_t u_t^8 R_{g_t} - 8u_t^8\left(\Delta_{g_t} v_t - \frac{1}{8}R_{g_t} v_t\right) \\
&= 4v_t u_t^8 R_{g_t} - 8v_t u_t^8(|E_t|_{g_t}^2 + |B_t|_{g_t}^2).
\end{aligned}
$$
(2.8)

Then since $\frac{d}{dt}(u_t^8|E_t|_{g_t}^2) = \frac{d}{dt}(u_t^8|B_t|_{g_t}^2) = 0$, we have

$$(2.9) \qquad \frac{d}{dt}[u_t^8\left(R_{g_t} - 2|E_t|_{g_t}^2 - 2|B_t|_{g_t}^2\right)] = 4v_t u_t^8\left(R_{g_t} - 2|E_t|_{g_t}^2 - 2|B_t|_{g_t}^2\right),$$

so that

$$(2.10) \quad u_t^8\left(R_{g_t} - 2|E_t|_{g_t}^2 - 2|B_t|_{g_t}^2\right) = e^{\int_0^t 4v_s ds}\left(R_g - 2|E|_g^2 - 2|B|_g^2\right) = u_t^4\left(R_g - 2|E|_g^2 - 2|B|_g^2\right) \geq 0.$$

$\square$

Monotonicity of the mass is of course more difficult and relegated to its own section, Section 5. Notice also that we do not prove that the flow converges to the canonical Reissner-Nordström data, in analogy with the fact that the original conformal flow converges to the canonical Schwarzschild data. While we strongly believe that this result holds for the charged conformal flow, it is not needed to prove the main theorem and is hence left for future investigation.

3. Existence of the Flow

In this section we prove that the charged conformal flow exists, by employing the same discretization procedure developed Bray. The presentation will closely follow that in [1]. For each $\epsilon \in (0, \frac{1}{2})$ a family of approximate solutions $u_t^\epsilon(x)$ will easily be constructed, and the solution shall arise from the limit

$$(3.1) \qquad u_t(x) = \lim_{\epsilon \to 0} u_t^\epsilon(x).$$

Given the metric $g_t^\epsilon = (u_t^\epsilon)^4 g$ (with $u_0^\epsilon \equiv 1$), define for $t \geq 0$

$$(3.2) \qquad \partial M_t^\epsilon = \begin{cases} \partial M & \text{if } t = 0, \\ \text{the outermost minimal area enclosure} \\ \text{of } \partial M_{t-\epsilon}^\epsilon \text{ in } (M, g_t^\epsilon) & \text{if } t = k\epsilon \text{ with } k \in \mathbb{Z}_+, \\ \partial M_{\lfloor t \rfloor_\epsilon}^\epsilon & \text{otherwise,} \end{cases}$$

where

$$(3.3) \qquad \lfloor t \rfloor_\epsilon := \epsilon \left\lfloor \frac{t}{\epsilon} \right\rfloor.$$

Let M_t^ϵ denote the region enclosed between ∂M_t^ϵ and spatial infinity. Moreover, given ∂M_t^ϵ we may define

$$(3.4) \qquad u_t^\epsilon(x) = \exp\left(\int_0^t v_s^\epsilon(x) \, ds \right),$$

where v_t^ϵ is the solution of the Dirichlet problem

$$(3.5) \qquad \begin{cases} \Delta_{g_{\lfloor t \rfloor_\epsilon}^\epsilon} v_t^\epsilon - \left(|E_{\lfloor t \rfloor_\epsilon}^\epsilon|^2_{g_{\lfloor t \rfloor_\epsilon}^\epsilon} + |B_{\lfloor t \rfloor_\epsilon}^\epsilon|^2_{g_{\lfloor t \rfloor_\epsilon}^\epsilon} \right) v_t^\epsilon &=& 0 & \text{on } M_t^\epsilon, \\ v_t^\epsilon &=& 0 & \text{on } \partial M_t^\epsilon, \\ v_t^\epsilon &\to& -1 & \text{as } |x| \to \infty, \end{cases}$$

with $v_t^\epsilon(x) \equiv 0$ on $M \setminus M_t^\epsilon$ and $(E_t^\epsilon)^j = (u_t^\epsilon)^{-6} E^j$, $(B_t^\epsilon)^j = (u_t^\epsilon)^{-6} B^j$. Note that (3.4) directly implies that $u_t^\epsilon(x) \to e^{-t}$ as $|x| \to \infty$.

Now observe that ∂M_t^ϵ and hence $v_t^\epsilon(x)$ are fixed for $t \in [k\epsilon, (k+1)\epsilon)$. Furthermore, for $t = k\epsilon$ with $k \in \mathbb{Z}_+$, ∂M_t^ϵ does not touch $\partial M_{t-\epsilon}^\epsilon$ because $\partial M_{t-\epsilon}^\epsilon$ has negative mean curvature in (M, g_t^ϵ). This follows from the fact that $\partial_\nu u_t^\epsilon|_{\partial M_{t-\epsilon}^\epsilon} < 0$, where ν is the unit outer normal pointing to spatial infinity. To see that this is in fact the case, first observe that $\partial_\nu u_{t-\epsilon}^\epsilon|_{\partial M_{t-\epsilon}^\epsilon} = 0$ since $\partial M_{(k-1)\epsilon}^\epsilon$ is minimal in $(M, g_{(k-1)\epsilon}^\epsilon)$, and $\partial_\nu v_{t-\epsilon}^\epsilon|_{\partial M_{t-\epsilon}^\epsilon} < 0$ from the Hopf lemma. Therefore, using $u_t^\epsilon = u_{t-\epsilon}^\epsilon \exp\left(\epsilon v_{(k-1)\epsilon}^\epsilon \right)$ we find that

$$(3.6) \qquad \partial_\nu u_t^\epsilon \Big|_{\partial M_{t-\epsilon}^\epsilon} = u_{t-\epsilon}^\epsilon \partial_\nu \exp\left(\epsilon v_{(k-1)\epsilon}^\epsilon(x) \right) \Big|_{\partial M_{t-\epsilon}^\epsilon} = \epsilon u_t^\epsilon \partial_\nu v_{t-\epsilon}^\epsilon \Big|_{\partial M_{t-\epsilon}^\epsilon} < 0.$$

This inequality says that by pushing the surface $\partial M_{t-\epsilon}^\epsilon$ outwards, the area can be reduced in (M, g_t^ϵ). Hence, $\partial M_{t-\epsilon}^\epsilon$ acts as a barrier in (M, g_t^ϵ). As the outermost condition implies the outer-minimizing condition, ∂M_t^ϵ is actually a strictly outer minimizing horizon of (M, g_t^ϵ), and is smooth since g_t^ϵ is smooth outside $\partial M_{t-\epsilon}^\epsilon$.

The same arguments presented in [1] yield the following facts. Not only are the surfaces ∂M_t^ϵ smooth, but any limits of these surfaces are smooth. Furthermore from the definition of ∂_t^ϵ, it is apparent that for $\epsilon > 0$ the horizon $\partial M_{t_2}^\epsilon$ encloses $\partial M_{t_1}^\epsilon$ for all $t_2 \geq t_1 \geq 0$. Also, the horizon ∂M_t^ϵ is the outermost minimal area enclosure of ∂M in (M, g_t^ϵ) when $t = k\epsilon$ with $k \in \mathbb{Z}_+$.

Lemma 3.1. *The functions $u_t^\epsilon(x)$ are positive, bounded, locally Lipschitz functions (in x and t) with uniform Lipschitz constants independent of ϵ.*

Proof. Positivity is obvious from the definition of u_t^ϵ. By the maximum principle, v_t^ϵ cannot achieve a nonnegative maximum. This then implies that $u_t^\epsilon(x) \leq 1$. That $u_t^\epsilon(x)$ is Lipschitz in t follows from its definition and the fact that $-1 < v_t^\epsilon(x) \leq 0$. That $u_t^\epsilon(x)$ is Lipschitz in x follows from the fact that $v_t^\epsilon(x)$ is Lipschitz in x (with Lipschitz constant depending on t), which follows from Corollary 15 of [1]. $\qquad \square$

Corollary 3.2. *There exists a subsequence $\{\epsilon_i\}$ converging to zero such that*

$$(3.7) \qquad u_t(x) = \lim_{\epsilon_i \to 0} u_t^{\epsilon_i}(x)$$

exists, is locally Lipschitz in x and t, and the convergence is locally uniform. Hence we may define

$$(3.8) \qquad g_t = \lim_{\epsilon_i \to 0} g_t^{\epsilon_i} = u_t^4(x) g$$

for $t \geq 0$.

Define $\{\widetilde{\Sigma}_\gamma(t)\}$ to be the collections of limit surfaces of $\partial M_t^{\epsilon_i}$ in the limit as ϵ_i approaches 0. As discussed in [1], the limiting surfaces $\{\widetilde{\Sigma}_\gamma(t)\}$ are all smooth.

Proposition 3.3. *The surface $\widetilde{\Sigma}_{\gamma_2}(t_2)$ encloses $\widetilde{\Sigma}_{\gamma_1}(t_1)$ for all $t_2 > t_1 \geq 0$ and for any γ_1 and γ_2.*

Proof. The same arguments as in the proof of Theorem 5 in [1] apply here, except for one technical point that needs to be addressed. Namely, in [1], it is used that v_t^ϵ is a harmonic function so that the maximum principle applies. In our setting, this function should be replaced by $u_t^\epsilon v_t^\epsilon$, since here v_t^ϵ represents the logarithmic velocity $\frac{d}{dt}u_t^\epsilon = v_t^\epsilon u_t^\epsilon$, while in [1] v_t^ϵ represents the velocity $\frac{d}{dt}u_t^\epsilon = v_t^\epsilon$. Thus it remains to show that $u_t^\epsilon v_t^\epsilon$ satisfies an equation outside ∂M_t^ϵ, to which the maximum principle applies. To see this, note that (2.10) holds with ϵ, and use a standard property for the conformal Laplacian to obtain

$$
\begin{aligned}
\Delta_g(u_t^\epsilon v_t^\epsilon) &= (u_t^\epsilon)^5 \Delta_{g_t^\epsilon} v_t^\epsilon - \frac{1}{8} R_{g_t^\epsilon}(u_t^\epsilon)^5 v_t^\epsilon + \frac{1}{8} R_g u_t^\epsilon v_t^\epsilon \\
(3.9) \qquad &= v_t^\epsilon (u_t^\epsilon)^5 \left(|E_t^\epsilon|_{g_t^\epsilon}^2 + |B_t^\epsilon|_{g_t^\epsilon}^2 \right) - \frac{1}{8} R_{g_t^\epsilon}(u_t^\epsilon)^5 v_t^\epsilon + \frac{1}{8} R_g u_t^\epsilon v_t^\epsilon \\
&= \left[\frac{3}{4}(u_t^\epsilon)^4 \left(|E_t^\epsilon|_{g_t^\epsilon}^2 + |B_t^\epsilon|_{g_t^\epsilon}^2 \right) + \frac{1}{4} \left(|E|_g^2 + |B|_g^2 \right) \right] (u_t^\epsilon v_t^\epsilon).
\end{aligned}
$$

Since the term in brackets on the right-hand side is nonnegative, it follows that the resulting equation for $u_t^\epsilon v_t^\epsilon$ admits a maximum principle. $\qquad\square$

Define ∂M_t to be the outermost minimal area enclosure of the original horizon ∂M in (M, g_t). Apart from the proposition above, the rest of the proof of existence of the flow is identical to the arguments in [1]. In particular, we have the following result.

Theorem 3.4. *The surface ∂M_{t_2} encloses ∂M_{t_1} for all $t_2 > t_1 \geq 0$, and the areas remain constant $|\partial M_t|_{g_t} = |\partial M|_g$ for all $t \geq 0$. Furthermore, the set J of $t(\geq 0)$ at which point the surface "jumps", namely when*

$$(3.10) \qquad\qquad \lim_{s \to t^-} \partial M_s \neq \lim_{s \to t^+} \partial M_s,$$

is countable, and for $t \notin J$, $\widetilde{\Sigma}_\gamma(t)$ is single valued. Given the horizon ∂M_t, v_t may be defined as in Section 2, and serves as the logarithmic velocity of the flow $\frac{d}{dt}u_t = v_t u_t$.

4. Exhaustion

The existence of the charged conformal flow, and its properties listed in Sections 2 and 3, are independent of the area/charge inequality $|\partial M|_g \geq 4\pi q^2$, or equivalently $\rho \geq |q|$ as expressed in the introduction. It is then noteworthy and perhaps surprising, that the property of exhaustion, which states that the flowing surfaces ∂M_t eventually enclose any bounded set, essentially holds[1] if and only if the area/charge inequality is valid. In fact, this section is the only place in the paper where the area/charge inequality plays a role. As in [1], the proof will follow two basic steps. The first consists of showing that ∂M_t cannot, for all $t \geq 0$, be enclosed by any fixed large coordinate sphere in the asymptotic end, and the second entails showing that it is not possible for ∂M_t to be only partially contained, for all $t \geq 0$, in a large coordinate sphere. It turns out that the second step may

---

[1]It is proven that the strict area/charge inequality is sufficient for exhaustion, and that the nonstrict area/charge inequality is necessary for exhaustion.

be proved directly from the same arguments in [1], and does not require the area/charge inequality. Thus, we will focus here on the first step in which the area/charge inequality is needed.

Before proceeding, we show that the area/charge inequality is a necessary condition for exhaustion. Note that if exhaustion occurs, then eventually the surfaces ∂M_t become connected.

Lemma 4.1. *If for some $t \geq 0$, ∂M_t is connected, then $|\partial M|_g \geq 4\pi q^2$.*

Proof. Since the areas and charges are preserved throughout the flow, it suffices to prove the conclusion at time t. Observe that by the second variation of area formula

$$(4.1) \quad 0 \leq \int_{\partial M_t} \left[-\psi \Delta_{\partial M_t} \psi - (|\,\mathrm{II}_t\,|^2 + \mathrm{Ric}_{g_t}(\nu, \nu))\psi^2 + H_t^2 \psi^2 \right] dA_t, \quad \text{for any} \quad \psi \in C^\infty(\partial M_t),$$

where II_t is the second fundamental form and $\mathrm{Ric}_{g_t}(\nu, \nu)$ is the Ricci curvature in the normal direction. Since ∂M_t is a minimal surface, the Gauss equations yield

$$(4.2) \quad |\,\mathrm{II}_t\,|^2 + \mathrm{Ric}_{g_t}(\nu, \nu) = |\,\mathrm{II}_t\,|^2 + \frac{1}{2}R_{g_t} - K_t + \frac{1}{2}H_t^2 - \frac{1}{2}|\,\mathrm{II}_t\,|^2 = \frac{1}{2}|\,\mathrm{II}_t\,|^2 + \frac{1}{2}R_{g_t} - K_t,$$

where K_t is Gaussian curvature. It follows that

$$(4.3) \quad 0 \leq \int_{\partial M_t} \left(|\nabla \psi|^2 - \frac{1}{2}|\,\mathrm{II}_t\,|^2 \psi^2 - \frac{1}{2}R_{g_t}\psi^2 + K_t\psi^2 \right) dA_t.$$

Choose $\psi \equiv 1$, and note that since ∂M_t has spherical topology, the Gauss-Bonnet theorem and (2.4) imply that

(4.4)

$$
\begin{aligned}
4\pi \geq \int_{\partial M_t} \frac{1}{2} \left(|\,\mathrm{II}_t\,|^2 + R_{g_t} \right) dA_t &\geq \int_{\partial M_t} \left(|E_t|_{g_t}^2 + |B_t|_{g_t}^2 \right) dA_t \\
&\geq \int_{\partial M_t} \left(|E_t \cdot \nu|^2 + |B_t \cdot \nu|^2 \right) dA_t \\
&\geq |\partial M_t|_{g_t}^{-1} \left[\left(\int_{\partial M_t} E_t \cdot \nu dA_t \right)^2 + \left(\int_{\partial M_t} B_t \cdot \nu dA_t \right)^2 \right] = \frac{(4\pi)^2 q_t^2}{|\partial M_t|_{g_t}},
\end{aligned}
$$

where we have used Jensen's inequality and the fact that the Maxwell fields are divergence free (Theorem 2.1). $\square$

We will now show that the strict area/charge inequality is also a sufficient condition for exhaustion. This will require some preparation. In [1], it was assumed without loss of generality that the initial data possessed harmonic asymptotics. Similarly, for the results of this section, we may assume that the initial data (M, g, E, B) possess the so called charged harmonic asymptotics, developed by Corvino in [6]. This means that in the asymptotic end $g = U_0^4 \delta$ (where δ is the Euclidean metric) for some function U_0 satisfying $R_g = -8U_0^{-5} \Delta_\delta U_0 = 2|E|_g^2$, with $E^i = U_0^{-6} E_\delta^i$ and $E_\delta = -q_e \nabla r^{-1}$. Observe that the magnetic field is excluded here, since when the asymptotics are imposed B has the same form as E with q_e replaced by q_b. However, as mentioned in the introduction, nothing is lost by assuming $q_b = 0$ ($q_e = q$), so that $B = 0$ in the end with such asymptotics. It should also be noted that the asymptotics used here for the electric field differ slightly from those in [6], where $E = U_0^{-6} \nabla \chi$ for some function $\chi = -qr^{-1} + O(r^{-2})$ which is harmonic in the end; the choice of χ ensures that E is divergence free on M. Thus, in our version of the asymptotics, E is no longer divergence free everywhere, a property which is of no use for the results in the current section.

Write $U_t = u_t U_0$ and $V_t = v_t u_t U_0$. Then in the asymptotic end

$$(4.5) \qquad L_\delta U_t = U_t^5 L_{g_t} 1 = -\frac{1}{8} U_t^5 R_{g_t}, \qquad L_\delta V_t = U_t^5 L_{g_t} v_t = U_t^5 \left(|E_t|_{g_t}^2 v_t - \frac{1}{8} R_{g_t} v_t \right).$$

According to the charged harmonic asymptotics and (2.10) we have that $R_{g_t} = 2|E_t|_{g_t}^2 = 2U_t^{-8}|E_\delta|_\delta^2$, therefore

$$(4.6) \qquad \Delta_\delta U_t = -\frac{1}{4}|E_\delta|_\delta^2 U_t^{-3}, \qquad \Delta_\delta V_t = \frac{3}{4} U_t^{-4}|E_\delta|_\delta^2 V_t.$$

Let $S_{r(t)}$ be a large coordinate sphere in the asymptotic end, and define $\tilde{V}_t$ to be the unique solution of the boundary value problem

$$(4.7) \qquad \Delta_\delta \tilde{V}_t = \frac{3}{4}\tilde{U}_t^{-4}|E_\delta|_\delta^2 \tilde{V}_t, \quad \tilde{V}_t = 0 \quad \text{on} \quad S_{r(t)}, \quad \tilde{V}_t \to -e^{-t} \quad \text{as} \quad |x| \to \infty,$$

where $\tilde{U}_t$ is the function U_t in the conformal flow of the Reissner-Nordström initial data (see Appendix B). Note that $\tilde{V}_t$ is the velocity function V_t in the conformal flow of the Reissner-Nordström initial data, and in particular

$$(4.8) \qquad \tilde{V}_t = \frac{-e^{-2t} + e^{2t}\frac{\tilde{m}^2 - q^2}{4|x|^2}}{\sqrt{e^{-2t} + \frac{\tilde{m}}{|x|} + e^{2t}\frac{\tilde{m}^2 - q^2}{4|x|^2}}}$$

for some constant $\tilde{m}$. We choose $\tilde{m}$ so that the boundary condition of (4.7) is satisfied, namely

$$(4.9) \qquad \tilde{m} = \sqrt{4e^{-4t}r(t)^2 + q^2}.$$

It follows that

$$(4.10) \qquad \tilde{U}_t = \left(e^{-2t} + \frac{\sqrt{4e^{-4t}r(t)^2 + q^2}}{|x|} + e^{-2t}\frac{r(t)^2}{|x|^2} \right)^{1/2}.$$

For reasons that will become clear in the proof of Proposition 4.4 below, we would like to compare the solution of the conformal flow U_t, or more precisely a radial approximation $\hat{U}_t$, with the model solution from the Reissner-Nordström example $\tilde{U}_t$. The desired radial approximation is given as the unique (radial) solution of
$$(4.11)$$

$$\Delta_\delta \hat{U}_t = -\frac{1}{4}|E_\delta|_\delta^2 \hat{U}_t^{-3}, \qquad \hat{U}_t = \left(\frac{1}{4\pi r(t)^2} \int_{S_{r(t)}} U_t^4 dA_\delta \right)^{1/4} \quad \text{on} \quad S_{r(t)}, \qquad \hat{U}_t \to e^{-t} \quad \text{as} \quad |x| \to \infty.$$

The corresponding radial velocity function $\hat{V}_t = \frac{d}{dt}\hat{U}_t$ is the unique solution of the boundary value problem

$$(4.12) \qquad \Delta_\delta \hat{V}_t = \frac{3}{4}\hat{U}_t^{-3}|E_\delta|_\delta^2 \hat{V}_t, \qquad \hat{V}_t = \frac{d}{dt}[\hat{U}_t(r(t))] \quad \text{on} \quad S_{r(t)}, \quad \hat{V}_t \to -e^{-t} \quad \text{as} \quad |x| \to \infty.$$

It turns out that $\hat{U}_t$ has a relatively simple explicit form.

Lemma 4.2. *Let $r(t) = \varepsilon\sqrt{A_0}e^{2t}$ with $A_0 = |\partial M|_g$. If ε is sufficiently small, then there exists a constant $\alpha > -\frac{1}{2}q^2$, depending on $U_t|_{S_{r(t)}}$, such that*

$$(4.13) \qquad \hat{U}_t^4(x) = e^{-4t} + \frac{e^{-2t}\sqrt{\frac{8}{3}\left(\alpha + \frac{1}{2}q^2\right)}}{|x|} + \frac{\alpha}{|x|^2} + \frac{e^{2t}\sqrt{\frac{8}{3}\left(\alpha + \frac{1}{2}q^2\right)}(\alpha - q^2)}{6|x|^3} + \frac{e^{4t}(\alpha - q^2)^2}{36|x|^4}.$$

Proof. Consider the equation satisfied by $\hat{U}_t^4$:

$$(4.14) \qquad \Delta_\delta \hat{U}_t^4 = -|E_\delta|_\delta^2 + \frac{3}{4}\hat{U}_t^{-4}|\nabla\hat{U}_t^4|_\delta^2.$$

Since the equation and all coefficients are analytic in their arguments, we may assume that the solution is given by an expansion

$$(4.15) \qquad \hat{U}_t^4 = e^{-4t} + \frac{c_1}{|x|} + \frac{c_2}{|x|^2} + \frac{c_3}{|x|^3} + \cdots .$$

We then proceed to calculate each term in (4.14). For instance

$$(4.16) \qquad \Delta_\delta \hat{U}_t^4 = \frac{2c_2}{|x|^4} + \frac{6c_3}{|x|^5} + \frac{12c_4}{|x|^6} + \frac{20c_5}{|x|^7} + \frac{30c_6}{|x|^8} + \cdots ,$$

so that

$$(4.17) \qquad \begin{aligned} \hat{U}_t^4 \Delta_\delta \hat{U}_t^4 = &\frac{2e^{-4t}c_2}{|x|^4} + \frac{2c_1c_2 + 6e^{-4t}c_3}{|x|^5} + \frac{2c_2^2 + 6c_1c_3 + 12e^{-4t}c_4}{|x|^6} \\ &+ \frac{8c_2c_3 + 12c_1c_4 + 20e^{-4t}c_5}{|x|^7} + \frac{2c_2c_4 + 6c_3^2 + 12c_2c_4 + 20c_1c_5 + 30e^{-4t}c_6}{|x|^8} + \cdots . \end{aligned}$$

Next observe that

$$(4.18) \qquad \partial_r \hat{U}_t^4 = -\frac{c_1}{|x|^2} - \frac{2c_2}{|x|^3} - \frac{3c_3}{|x|^4} - \frac{4c_4}{|x|^5} - \frac{5c_5}{|x|^6} - \cdots ,$$

which yields

$$(4.19) \qquad |\nabla\hat{U}_t^4|^2 = |\partial_r\hat{U}_t^4|^2 = \frac{c_1^2}{|x|^4} + \frac{4c_1c_2}{|x|^5} + \frac{4c_2^2 + 6c_1c_3}{|x|^6} + \frac{8c_1c_4 + 12c_2c_3}{|x|^7} + \frac{9c_3^2 + 16c_2c_4 + 10c_1c_5}{|x|^8} + \cdots .$$

Finally

$$(4.20) \qquad \hat{U}_t^4|E_\delta|^2 = \frac{e^{-4t}q^2}{|x|^4} + \frac{c_1q^2}{|x|^5} + \frac{c_2q^2}{|x|^6} + \frac{c_3q^2}{|x|^7} + \frac{c_4q^2}{|x|^8} + \cdots .$$

By combining these expansions and using equation (4.14), we find the following relations

$$(4.21) \qquad \begin{aligned} 2e^{-4t}c_2 &= -e^{-4t}q^2 + \frac{3}{4}c_1^2, \\ 6e^{-4t}c_3 + 2c_1c_2 &= -c_1q^2 + 3c_1c_2, \\ 12e^{-4t}c_4 + 6c_1c_3 + 2c_2^2 &= -q^2c_2 + 3c_2^2 + \frac{9}{2}c_1c_3, \\ 20e^{-4t}c_5 + 12c_1c_4 + 6c_2c_3 + 2c_2c_3 &= -q^2c_3 + 6c_1c_4 + 9c_2c_3. \end{aligned}$$

From this we can solve for the constants c_i:

$$(4.22) \qquad \begin{aligned} c_2 &= \frac{3}{8}e^{4t}c_1^2 - \frac{1}{2}q^2, \\ c_3 &= \frac{1}{16}e^{8t}c_1^3 - \frac{1}{4}e^{4t}q^2c_1, \\ c_4 &= \frac{3}{48}e^{4t}q^4 - \frac{1}{32}e^{8t}c_1^2q^2 + \frac{1}{256}e^{12t}c_1^4, \\ c_i &= 0, \quad i \geq 5. \end{aligned}$$

Although the higher order terms for $i > 5$ have not been computed here, one may deduce that they all vanish by simply checking that (4.15), with these coefficients, solves (4.14). The constant c_1 may be chosen in order to realize the correct boundary condition.

Let us now obtain the form (4.13). The first task is to show that $c_1 > 0$. To see this, first recall the result of Bray and Iga [2], which states that

$$(4.23) \qquad U_t^4 \geq \frac{cA_0}{|x|^2} \quad \text{outside of} \quad S_{r(t)}$$

for some positive constant c. Since the average value of U_t^4 agrees with that of $\hat{U}_t^4$ on $S_{r(t)}$, we find that

$$(4.24) \qquad \hat{U}_t^4 \geq c\varepsilon^{-2}e^{-4t} > e^{-4t} \quad \text{on} \quad S_{r(t)},$$

if ε is small enough. Moreover $\hat{U}_t^4 \to e^{-4t}$ as $|x| \to \infty$. From equation (4.14), it is clear that $\hat{U}_t^4$ cannot obtain an interior minimum, thus

$$(4.25) \qquad \hat{U}_t^4 > e^{-4t} \quad \text{outside of} \quad S_{r(t)}.$$

It follows that $c_1 > 0$. Now solve for c_i in terms of c_2 to produce

$$(4.26) \qquad c_1 = e^{-2t}\sqrt{\frac{8}{3}\left(c_2 + \frac{1}{2}q^2\right)}, \qquad c_3 = \frac{e^{2t}}{6}\sqrt{\frac{8}{3}\left(c_2 + \frac{1}{2}q^2\right)}(c_2 - q^2), \qquad c_4 = \frac{1}{36}e^{4t}(c_2 - q^2)^2.$$

The desired result is obtained by setting $\alpha = c_2$ and noting that $\alpha > -\frac{1}{2}q^2$ by (4.22). $\qquad \square$

Notice that the Reissner-Nordström conformal factors have a similar expansion to that of $\hat{U}_t$, namely

$$(4.27) \qquad \begin{aligned} \tilde{U}_t^4 &= \left(e^{-2t} + \frac{\sqrt{4e^{-4t}r(t)^2 + q^2}}{|x|} + \frac{e^{-2t}r(t)^2}{|x|^2}\right)^2 \\ &= e^{-4t} + \frac{2e^{-2t}\sqrt{4e^{-4t}r(t)^2 + q^2}}{|x|} + \frac{6e^{-4t}r(t)^2 + q^2}{|x|^2} \\ &\quad + \frac{2e^{-2t}r(t)^2\sqrt{4e^{-4t}r(t)^2 + q^2}}{|x|^3} + \frac{e^{-4t}r(t)^4}{|x|^4}. \end{aligned}$$

This is not too surprising, since $\tilde{U}_t$ satisfies the same equation (4.14) as $\hat{U}_t^4$, and has the same asymptotic behavior as $|x| \to \infty$. Observe also that

$$(4.28) \qquad \tilde{V}_t = \frac{d}{dt}\tilde{U}_t = -\frac{e^{-2t}\left(1 - \frac{r(t)^2}{|x|^2}\right)}{\left(e^{-2t} - \frac{\sqrt{4e^{-4t}r(t)^2 + q^2}}{|x|} + e^{2t}\frac{r(t)^2}{|x|^2}\right)^{1/2}},$$

satisfies

$$(4.29) \qquad \Delta_\delta \tilde{V}_t = \frac{3}{4}\tilde{U}_t^{-4}|E_\delta|_\delta^2 \tilde{V}_t, \quad \tilde{V}_t = 0 \quad \text{on} \quad S_{r(t)}, \quad \tilde{V}_t \to -e^{-t} \quad \text{as} \quad |x| \to \infty.$$

The next lemma gives the foundational estimate on which the exhaustion proof is based. It is also the primary place where the area/charge inequality is required.

Lemma 4.3. *Let $r(t) = \varepsilon\sqrt{A_0}e^{2t}$ with $A_0 = |\partial M|_g$. If $A_0 > 4\pi q^2$ and ε is sufficiently small, then*

$$(4.30) \qquad \hat{U}_t(x) \geq \tilde{U}_t(x) \quad \text{for} \quad |x| \geq r(t).$$

Proof. Note that by setting $\alpha = q^2 + 6e^{-4t}r(t)^2$, $\hat{U}_t$ becomes $\tilde{U}_t$. In fact, by directly comparing the coefficients in the expressions for these two functions, it is apparent that the desired result follows if

$$(4.31) \qquad \alpha - q^2 \geq 6e^{-4t}r(t)^2 = 6\varepsilon^2 A_0 \quad \text{and} \quad (\alpha - q^2)^2 \geq 36e^{-8t}r(t)^4 = 36\varepsilon^4 A_0^2.$$

Observe that since $|S_{r(t)}|_{g_t} \geq |\partial M_t|_{g_t} = |\partial M|_g$, we have

$$
\begin{aligned}
\frac{A_0}{4\pi} &\leq \frac{1}{4\pi}\int_{S_{r(t)}} U_t^4 dA_\delta = \frac{1}{4\pi}\int_{S_{r(t)}} \hat{U}_t^4 dA_\delta \\
(4.32) \\
&= \varepsilon^2 A_0 + \varepsilon\sqrt{A_0}\sqrt{\frac{8}{3}\left(\alpha + \frac{1}{2}q^2\right)} + \alpha + \frac{1}{6\varepsilon\sqrt{A_0}}\sqrt{\frac{8}{3}\left(\alpha + \frac{1}{2}q^2\right)}(\alpha - q^2) + \frac{(\alpha - q^2)^2}{36\varepsilon^2 A_0}.
\end{aligned}
$$

Suppose that both inequalities in (4.31) are violated, then

$$(4.33) \qquad \frac{A_0}{4\pi} < 2\varepsilon^2 A_0 + \varepsilon\sqrt{A_0}\sqrt{\frac{8}{3}\left(\frac{3}{2}q^2 + 6\varepsilon^2 A_0\right)} + q^2 + 6\varepsilon^2 A_0 + \varepsilon\sqrt{\frac{8A_0}{3}\left(\frac{3}{2}q^2 + 6\varepsilon^2 A_0\right)}.$$

However this is impossible for small ε, since $A_0 > 4\pi q^2$ independent of ε. Therefore, at least one of the inequalities in (4.31) must be satisfied. If the first inequality is satisfied, then so is the second. So assume now that the second inequality is satisfied but not the first. The only way that this can happen is if $\alpha - q^2 < -6\varepsilon^2 A_0$. We claim however, that $\alpha \geq q^2$ as a result of the positive mass theorem with charge [12], and hence (4.31) holds.

To verify the claim, consider the initial data $(\mathbb{R}^3 \setminus \{0\}, \hat{U}_t^4\delta, \hat{U}_t^{-6}E_\delta)$, which satisfies the charged dominant energy condition, the Maxwell constraint, and has mass $\hat{m} = \frac{1}{2}\sqrt{\frac{8}{3}\left(\alpha + \frac{1}{2}q^2\right)}$. Note that although $\hat{U}_t$ was initially defined in (4.11) only on $\mathbb{R}^3 \setminus B_{r(t)}$, the explicit expression for $\hat{U}_t$ in Lemma 4.2 is valid on $\mathbb{R}^3 \setminus \{0\}$. If $\alpha = q^2$ then we are done, so assume that $\alpha \neq q^2$. Then according to the expansion of $\hat{U}_t$, this initial data set has an asymptotically flat end corresponding to $\{0\}$, and therefore a minimal surface exists which separates the two ends. We may now apply the positive mass theorem with charge to conclude that $\hat{m} \geq |q|$, and the claim follows. $\qquad \square$

We have now finished the preparation, and are ready to establish the first step in the proof of exhaustion.

Proposition 4.4. *Let $r(t) = \varepsilon\sqrt{A_0}e^{2t}$ with $A_0 = |\partial M|_g$. If $A_0 > 4\pi q^2$ and ε is sufficiently small, then ∂M_t cannot be entirely enclosed by the coordinate sphere $S_{r(t)}$ for all t.*

Proof. The proof is by contradiction. Thus assume that ∂M_t is entirely enclosed by $S_{r(t)}$ for all $t \geq \bar{t}$. It will then be shown that if $\bar{t}$ is sufficiently large, then ∂M_t is not the outermost minimal area enclosure of ∂M with respect to the metric g_t, yielding a contradiction.

Consider the equation satisfied by the difference $W_t = \tilde{V}_t - \hat{V}_t$

$$(4.34) \qquad \Delta_\delta W_t = \frac{3}{4}\hat{U}_t^{-4}|E_\delta|_\delta^2 W_t + \frac{3}{4}(\tilde{U}_t^{-4} - \hat{U}_t^{-4})\tilde{V}_t|E_\delta|_\delta^2.$$

Moreover $W_t \to 0$ as $|x| \to \infty$, and

$$(4.35) \qquad \hat{V}_t(r(t)) = \frac{d}{dt}[\hat{U}_t(r(t))] < 0 = \tilde{V}_t(r(t))$$

so that $W_t > 0$ on $S_{r(t)}$. In (4.35) we used the formula (4.13) to show that $\frac{d}{dt}[\hat{U}_t(r(t))] < 0$. Since $\tilde{U}_t^{-4} - \hat{U}_t^{-4} \geq 0$ by Lemma 4.3, we may apply the maximum principle to conclude that $W_t \geq 0$ outside $S_{r(t)}$.

The remaining arguments proceed similar to those in the proof of Theorem 12 in [1]. From the above, we have that $\hat{V}_t \leq \tilde{V}_t$ outside of $S_{r(t)}$.[2] This allows an estimate of $\hat{U}_t$ from above, since $\hat{V}_t = \frac{d}{dt}\hat{U}_t$. In this direction, first notice that

(4.36)
$$\tilde{V}_t = -\frac{e^{-2t}\left(1 - \frac{r(t)^2}{|x|^2}\right)}{\left(e^{-2t} - \frac{\sqrt{4e^{-4t}r(t)^2 + q^2}}{|x|} + e^{2t}\frac{r(t)^2}{|x|^2}\right)^{1/2}} \leq -\frac{e^{-2t}\left(1 - \frac{r(t)^2}{|x|^2}\right)}{\left(e^{-2t} + \frac{2e^{-2t}r(t)}{|x|} + e^{2t}\frac{r(t)^2}{|x|^2}\right)^{1/2}} = -e^{-t}\left(1 - \frac{r(t)}{|x|}\right).$$

Now choose a constant $c > 0$ such that $\hat{U}_{\bar{t}}(x) \leq e^{-\bar{t}} + \frac{c}{|x|}$ for all x outside of $S_{r(t)}$. Then for all x outside of $S_{r(t)}$,

(4.37)
$$\hat{U}_t(x) = \hat{U}_{\bar{t}}(x) + \int_{\bar{t}}^t \hat{V}_s ds \leq \hat{U}_{\bar{t}}(x) + \int_{\bar{t}}^t \tilde{V}_s ds \leq e^{-t} + \frac{1}{|x|}\left[c + \varepsilon\sqrt{A_0}(e^t - e^{\bar{t}})\right].$$

It follows that

(4.38)
$$|S_{r(t)}|_{g_t} = \int_{S_{r(t)}} U_t^4 dA_\delta = \int_{S_{r(t)}} \hat{U}_t^4 dA_\delta \leq 4\pi r(t)^2 \hat{U}_t^4(r(t)) \leq 4\pi\varepsilon^2 A_0[2 + O(\varepsilon^{-1}e^{-t})]^4.$$

Therefore, for ε sufficiently small and $t \geq \bar{t}$ sufficiently large $|S_{r(t)}|_{g_t} < A_0$. $\square$

We are now ready to state the main result of this section.

Theorem 4.5. *If $|\partial M|_g > 4\pi q^2$, then the collection of subdomains $\{M_t\}$ exhausts the manifold M. In particular, the flowing surfaces ∂M_t eventually become connected (topological 2-spheres) for all sufficiently large times.*

Given Proposition 4.4, the proof of this statement is identical to that which appears in Section 10 of [1], after noting that U_t is superharmonic by (4.6).

5. Monotonicity of the Mass

Monotonicity of the mass is proven with a doubling argument similar to that in [1]. However here, the doubling procedure is based on the proof of uniqueness for the Reissner-Nördstrom black hole given by Masood-ul-Alam [23]. Let $(M_t^- \cup M_t^+, g_t^\pm)$ be the doubled manifold with $M_t^\pm$ representing two copies of M_t glued along their boundaries and $g_t^\pm = (w_t^\pm)^4 g_t$, where

(5.1)
$$w_t^\pm = \frac{1}{2}\sqrt{(1 \pm \overline{v}_t)^2 - \phi_t^2}.$$

The function $\overline{v}_t$ approximates v_t, and ϕ_t imitates the role of an electromagnetic potential in the static case; analogues of both functions are used in [23]. Ultimately though, these functions are chosen to impart positivity to the scalar curvature of $g_t^\pm$.

In order to define $\overline{v}_t$, let τ_0 be sufficiently small, and set $\tau(x) = \text{dist}_{g_t}(x, \partial M_t)$. Denote surfaces of constant distance to the boundary and the domain consisting of points whose distance to the boundary is larger than τ, by S_τ and $M(\tau)$ respectively. Then $\overline{v}_t$ is the unique solution of the boundary value problem

(5.2)
$$\Delta_{g_t}\overline{v}_t - f_t\overline{v}_t = 0 \quad \text{on} \quad M_t, \quad \overline{v}_t = 0 \quad \text{on} \quad \partial M_t,$$

---

[2]It may also be possible to prove this inequality directly from the explicit formulas for $\hat{V}_t$ and $\tilde{V}_t$, with help from the area/charge inequality.

$$(5.3) \qquad \overline{v}_t = -1 + \frac{\overline{\gamma}_t}{r} + O\left(\frac{1}{r^2}\right) \qquad \text{as} \quad r \to \infty,$$

where

$$(5.4) \qquad f_t = \left\{ \begin{array}{ll} \lambda^2 \eta(\tau) & \tau < \frac{5}{4}\tau_0 \\ |E_t|^2_{g_t} + |B_t|^2_{g_t} & \text{on } M(2\tau_0) \end{array} \right\}.$$

Here λ is a small parameter to be determined, η is a cut-off function such that $\eta(\tau) = 1 - 2\tau_0^{-1}(\tau - \tau_0)$ for $\frac{3}{4}\tau_0 < \tau < \frac{5}{4}\tau_0$, $\eta(\tau) = 0$ for $\tau < \frac{1}{2}\tau_0$, $|\eta'(\tau)| \le c\tau_0^{-1}$, and $|\eta''(\tau)| \le c\tau_0^{-2}$. On the transition region $\frac{5}{4}\tau_0 < \tau < 2\tau_0$, f_t is defined so that

$$(5.5) \qquad f_t \le \lambda^2 \eta\left(\frac{5}{4}\tau_0\right) + |E_t|^2_{g_t} + |B_t|^2_{g_t} = \frac{1}{2}\lambda^2 + |E_t|^2_{g_t} + |B_t|^2_{g_t},$$

and so as to make a smooth positive function on $M(\tau_0)$.

The function ϕ_t is also defined piecewise. Namely it will be shown in the next section that if $\lambda, \tau_0 > 0$ and τ_0 is sufficiently small, then there is a positive solution of the following Dirichlet problem

$$(5.6) \qquad \Delta_{g_t}\phi_t - \frac{\nabla \overline{v}_t \cdot \nabla \phi_t}{\overline{v}_t} = \frac{\Lambda}{\phi_t}\left|f_t - \frac{|\nabla \phi_t|^2}{\overline{v}_t^2}\right| \qquad \text{on} \quad M(\tau_0),$$

$$(5.7) \qquad \phi_t = \lambda \tau_0^4 \quad \text{on} \quad S_{\tau_0}, \qquad \phi_t \to 0 \quad \text{as} \quad r \to \infty,$$

where Λ is a positive constants to be specified. On the interior region define

$$(5.8) \qquad \phi_t = \lambda \tau_0^4 + \partial_\tau \phi_t|_{S_{\tau_0}}(\tau - \tau_0)\eta(\tau) \quad \text{when} \quad 0 \le \tau \le \tau_0,$$

where there is a slight abuse of notation in that $\partial_\tau \phi_t|_{S_{\tau_0}}$ is defined for all $0 \le \tau \le \tau_0$ by the fact that it is constant along the geodesic flow. Observe that ϕ_t is $C^{1,1}$ across S_{τ_0}, and thus ϕ_t is $C^{1,1}(M_t)$.

Notice that if $|E|_g = |B|_g = 0$ and $\lambda = 0$ then $f_t = 0$, which implies that $\phi_t = 0$ and $\overline{v}_t = v_t$. It follows that in the absence of the electromagnetic field, the conformal factors (5.1) reduce, modulo the choice of λ, to the same expressions used in [1]. Let us now establish positivity of the conformal factors.

Lemma 5.1. *If λ, τ_0 are appropriately small, and $\Lambda > 1$, then $(1 \pm \overline{v}_t)^2 - \phi_t^2 > 0$ on M_t.*

Proof. Observe that $(1 \pm \overline{v}_t)^2 - \phi_t^2 = (1 \pm \overline{v}_t + \phi_t)(1 \pm \overline{v}_t - \phi_t)$. Thus since $\overline{v}_t < 0$ and $\phi_t > 0$, it is enough to show that $1 + \overline{v}_t - \phi_t > 0$. First we show this on $M(2\tau_0)$. Equations (5.2) and (5.6) imply that

$$(5.9)$$

$$\begin{aligned} \Delta_{g_t}(1 + \overline{v}_t - \phi_t) + \frac{\nabla \phi_t \cdot \nabla(1 + \overline{v}_t - \phi_t)}{\overline{v}_t} &= \left(|E_t|^2_{g_t} + |B_t|^2_{g_t} - \frac{|\nabla \phi_t|^2}{\overline{v}_t^2}\right)\overline{v}_t - \frac{\Lambda}{\phi_t}\left|f_t - \frac{|\nabla \phi_t|^2}{\overline{v}_t^2}\right| \\ &\le -\left(\frac{\Lambda}{\phi_t} + \overline{v}_t\right)\left|f_t - \frac{|\nabla \phi_t|^2}{\overline{v}_t^2}\right| + (|E_t|^2_{g_t} + |B_t|^2_{g_t} - f_t)\overline{v}_t \\ &\le -\left(\frac{\Lambda}{\phi_t} + \overline{v}_t\right)\left|f_t - \frac{|\nabla \phi_t|^2}{\overline{v}_t^2}\right|. \end{aligned}$$

Also

$$(5.10) \qquad (1 + \overline{v}_t - \phi_t)|_{S_{2\tau_0}} > 0, \qquad (1 + \overline{v}_t - \phi_t) \to 0 \quad \text{as} \quad r \to \infty,$$

if λ, τ_0 are small enough. Clearly the right-hand side of (5.9) is nonpositive if $\Lambda > 1$. This is due to the fact that $\phi_t \leq \lambda\tau_0^4$ and $|\overline{v}_t| < 1$ by the maximum principle. Thus, by the minimum principle $1 + \overline{v}_t - \phi_t > 0$ on $M(2\tau_0)$.

Let us now consider the remaining region. Again use the fact that $\phi_t \leq \lambda\tau_0^4$ on the region between S_{τ_0} and $S_{2\tau_0}$. Moreover, in Appendix A it is shown that $\phi_t \leq c(\tau_0)\lambda$ for $\tau < \tau_0$. Hence, since $|\overline{v}_t| \leq c\tau_0$ on $M_t \setminus M(2\tau_0)$ (by the mean value theorem), if λ, τ_0 are chosen appropriately (small), then $1 + \overline{v}_t - \phi_t > 0$ for $\tau < 2\tau_0$. $\square$

In order to justify the use of $\overline{v}_t$ in place of v_t, in connection with monotonicity of the mass, it must be established that the monopoles of these two functions at spatial infinity remain arbitrarily close.

Theorem 5.2. *If an upper bound for λ is fixed, then $|\gamma_t - \overline{\gamma}_t| \leq c\tau_0^{1/4}$ where c is independent of λ and τ_0.*

Proof. We have

$$\Delta_{g_t}(v_t - \overline{v}_t) - f_t(v_t - \overline{v}_t) = (|E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t)v_t \quad \text{on} \quad M_t, \tag{5.11}$$

$$(v_t - \overline{v}_t)|_{\partial M_t} = 0, \quad (v_t - \overline{v}_t) = \frac{\gamma_t - \overline{\gamma}_t}{r} + O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty. \tag{5.12}$$

Thus

$$\begin{aligned}
-4\pi(\gamma_t - \overline{\gamma}_t) &= \int_{M_t} \Delta_{g_t}(v_t - \overline{v}_t) + \int_{\partial M_t} \partial_\tau(v_t - \overline{v}_t) \\
&= \int_{M_t} [f_t(v_t - \overline{v}_t) + (|E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t)v_t] + \int_{\partial M_t} \partial_\tau(v_t - \overline{v}_t),
\end{aligned} \tag{5.13}$$

and hence

$$|\gamma_t - \overline{\gamma}_t| \leq C\left(|v_t - \overline{v}_t|_{C^1(\partial M_t)} + \| f_t(v_t - \overline{v}_t) \|_{L^1(M_t)} + \| |E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t \|_{L^1(M_t)}\right). \tag{5.14}$$

In order to estimate $|v_t - \overline{v}_t|_{C^1(\partial M_t)}$, use that

$$|v_t - \overline{v}_t|_{C^1(\partial M_t)} \leq |v_t - \overline{v}_t|_{C^1(M_t)} \leq C \| v_t - \overline{v}_t \|_{W^{2,p}(M_t)} \tag{5.15}$$

for $p > 3$. By the L^p estimates for (5.11)

$$\| v_t - \overline{v}_t \|_{W^{2,p}(M_t)} \leq C(\| |E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t \|_{L^p(M_t)} + \| v_t - \overline{v}_t \|_{L^p(M_t)}). \tag{5.16}$$

We choose p large enough and even, and estimate $\| v_t - \overline{v}_t \|_{L^p(M_t)}$. To do this, multiply the equation (5.11) by $(v_t - \overline{v}_t)^{\frac{p}{3}-1}$ and integrate by parts. It follows that

$$\int_{M_t} \left(\frac{p}{3} - 1\right)\left(\frac{6}{p}\right)^2 |\nabla(v_t - \overline{v}_t)^{\frac{p}{6}}|^2 + f_t|v_t - \overline{v}_t|^{\frac{p}{3}} = -\int_{M_t} (v_t - \overline{v}_t)^{\frac{p}{3}-1}h_t, \tag{5.17}$$

where $h_t = v_t(|E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t)$. Since $(v_t - \overline{v}_t)^{\frac{p}{6}} = 0$ on ∂M_t and vanishes sufficiently fast as $r \to \infty$, we may apply the Gagliardo-Nirenberg-Sobolev inequality to obtain

$$
\int_{M_t} |v_t - \overline{v}_t|^p = \int_{M_t} \left(|v_t - \overline{v}_t|^{\frac{p}{6}} \right)^6
$$

$$
\leq C \left(\int_{M_t} |\nabla (v_t - \overline{v}_t)^{\frac{p}{6}}|^2 \right)^3
$$

(5.18)

$$
\leq C \left(\int_{M_t} |v_t - \overline{v}_t|^{\frac{p}{3}-1} |h_t| \right)^3
$$

$$
\leq C \left(\int_{M_t} \left(|v_t - \overline{v}_t|^{\frac{p}{3}-1} \right)^{\overline{q}} \right)^{\frac{3}{\overline{q}}} \left(\int_{M_t} |h_t|^{\overline{p}} \right)^{\frac{3}{\overline{p}}},
$$

where $\overline{p}^{-1} + \overline{q}^{-1} = 1$. We want $\overline{q}(\frac{p}{3} - 1) = p$, which implies $\overline{q} = \frac{3p}{p-3}$ and $\overline{p} = \frac{3p}{2p+3}$. Thus

(5.19) $\| v_t - \overline{v}_t \|_{L^p(M_t)} \leq C \| h_t \|_{L^{\frac{3p}{2p+3}}(M_t)} \leq C \| |E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t \|_{L^{\frac{3p}{2p+3}}(M_t)}.$

It follows that

(5.20) $\| v_t - \overline{v}_t \|_{W^{2,p}(M_t)} \leq C \left(\| |E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t \|_{L^{\frac{3p}{2p+3}}(M_t)} + \| |E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t \|_{L^p(M_t)} \right).$

Notice also that if $B(r)$ denotes the domain contained within the coordinate sphere S_r in the asymptotic end, then for r_0 sufficiently large

(5.21) $\| f_t(v_t - \overline{v}_t) \|_{L^1(M_t)} \leq C \left(\int_{B(r_0)} |v_t - \overline{v}_t| + \int_{M_t \setminus B(r_0)} r^{-4} |v_t - \overline{v}_t| \right),$

since $f_t \leq cr^{-4}$ on $M_t \setminus B(r_0)$. Thus
(5.22)
$\| f_t(v_t - \overline{v}_t) \|_{L^1(M_t)} \leq C \left(\mathrm{Vol}(B(r_0))^{\frac{1}{q}} \| v_t - \overline{v}_t \|_{L^p(B(r_0))} + \| r^{-4} \|_{L^q(M_t \setminus B(r_0))} \| v_t - \overline{v}_t \|_{L^p(M_t \setminus B(r_0))} \right),$

so that

(5.23) $\| f_t(v_t - \overline{v}_t) \|_{L^1(M_t)} \leq C \| v_t - \overline{v}_t \|_{L^p(M_t)}.$

Therefore

(5.24) $|\gamma_t - \overline{\gamma}_t| \leq C \left(\| |E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t \|_{L^{\frac{3p}{2p+3}}(M_t)} + \| |E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t \|_{L^p(M_t)} \right),$

for $p > 3$.

Recall the definition of f_t in (5.4). We see that $|E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t \equiv 0$ except on a set of small measure depending on τ_0. In particular by choosing $p = 4$ we obtain the desired result.   □

As in Bray's doubling argument [1], we have that the time derivative of the mass of g_t is given by

(5.25) $m_t' = -2(m_t - e^{-2t}\gamma_t) = -2\widetilde{m}_t + 2e^{-2t}(\gamma_t - \overline{\gamma}_t),$

where $\widetilde{m}_t = m_t - e^{-2t}\overline{\gamma}_t$ is the mass of the doubled manifold. Note that ϕ_t can be ignored in this computation, since $\phi_t = O(r^{-1})$ as $r \to \infty$. Moreover, since $\partial_\tau \phi_t = 0$ at ∂M_t, the mean curvatures across the glued boundaries agree. Therefore as $\phi_t \in C^{1,1}(M_t)$ and $\overline{v}_t \in C^\infty(M_t)$, we may apply the positive mass theorem with corners [25], [29] to conclude that $\widetilde{m}_t \geq 0$, provided the scalar curvature of the doubled manifold is nonnegative. Thus, it remains to show that the scalar curvature is nonnegative.

Lemma 5.3. *The scalar curvature of the doubled manifold is given by*
(5.26)

$$
R_{g_t^\pm} = \frac{1}{2}(\overline{v}_t)^{-2}(w_t^\pm)^{-8} \left| \phi_t \overline{v}_t \nabla \overline{v}_t - \frac{1}{2}(\overline{v}_t^2 + \phi_t^2 - 1)\nabla \phi_t \right|^2
$$
$$
+ (w_t^\pm)^{-4}(R_{g_t} - 2|E_t|^2_{g_t} - 2|B_t|^2_{g_t}) + 2(w_t^\pm)^{-4}(|E_t|^2_{g_t} + |B_t|^2_{g_t} - f_t)
$$
$$
+ \frac{1}{2}(w_t^\pm)^{-6} \left[((1 \pm \overline{v}_t)^2 - \phi_t^2 \mp 4\overline{v}_t(1 \pm \overline{v}_t)) \left(f_t - \frac{|\nabla \phi_t|^2}{\overline{v}_t^2} \right) + \phi_t \left(\Delta_{g_t} \phi_t - \frac{\nabla \overline{v}_t \cdot \nabla \phi_t}{\overline{v}_t} \right) \right].
$$

Proof. A standard formula yields

(5.27)
$$
R_{g_t^-} = -8(w_t^-)^{-5} \left(\Delta_{g_t} w_t^- - \frac{1}{8}R_{g_t} w_t^- \right).
$$

Next, observe that

(5.28)
$$
\nabla w_t^- = -\frac{1}{2}\frac{(1-\overline{v}_t)\nabla \overline{v}_t + \phi_t \nabla \phi_t}{\sqrt{(1-\overline{v}_t)^2 - \phi_t^2}}
$$

so

(5.29)
$$
\Delta_{g_t} w_t^- = -\frac{1}{2}\frac{(1-\overline{v}_t)\Delta_{g_t}\overline{v}_t + \phi_t \Delta_{g_t}\phi_t}{\sqrt{(1-\overline{v}_t)^2 - \phi_t^2}} + \frac{1}{2}\frac{|\nabla \overline{v}_t|^2 - |\nabla \phi_t|^2}{\sqrt{(1-\overline{v}_t)^2 - \phi_t^2}} - \frac{1}{2}\frac{|(1-\overline{v}_t)\nabla \overline{v}_t + \phi_t \nabla \phi_t|^2}{((1-\overline{v}_t)^2 - \phi_t^2)^{3/2}}.
$$

It follows that
(5.30)

$$
R_{g_t^-} = \frac{1}{2}(\overline{v}_t)^{-2}(w_t^-)^{-8} \left| \phi_t \overline{v}_t \nabla \overline{v}_t - \frac{1}{2}(\overline{v}_t^2 + \phi_t^2 - 1)\nabla \phi_t \right|^2 + (w_t^-)^{-4}(R_{g_t} - 2|E_t|^2_{g_t} - 2|B_t|^2_{g_t})
$$
$$
+ 2(w_t^-)^{-4}(|E_t|^2_{g_t} + |B_t|^2_{g_t} - f_t) + 2(w_t^-)^{-6}(1-\overline{v}_t)(\Delta_{g_t}\overline{v}_t - f_t\overline{v}_t)
$$
$$
+ \frac{1}{2}(w_t^-)^{-6} \left[((1-\overline{v}_t)^2 - \phi_t^2 + 4\overline{v}_t(1-\overline{v}_t)) \left(f_t - \frac{|\nabla \phi_t|^2}{\overline{v}_t^2} \right) + \phi_t \left(\Delta_{g_t}\phi_t - \frac{\nabla \overline{v}_t \cdot \nabla \phi_t}{\overline{v}_t} \right) \right].
$$

Since $\overline{v}_t$ satisfies (5.2), we obtain the desired result for $R_{g_t^-}$. A similar calculation yields the formula for $R_{g_t^+}$. $\square$

We are now ready to establish monotonicity of the mass.

Theorem 5.4. *The mass m_t is nonincreasing.*

Proof. First, we note that there exist small perturbations $g_t \to g_t^\varepsilon$, $E_t \to E_t^\varepsilon$, $B_t \to B_t^\varepsilon$ for which a strict charged dominant energy condition holds on M_t, and in particular

(5.31)
$$
R_{g_t^\varepsilon} - 2\left(|E_t^\varepsilon|^2_{g_t^\varepsilon} + |B_t^\varepsilon|^2_{g_t^\varepsilon} \right) \geq C_t^\varepsilon \quad \text{on} \quad M_t \setminus M(\tau_0),
$$

for some constant $C_t^\varepsilon > 0$. Here ε is the perturbation parameter, and $C_t^\varepsilon \to 0$ as $\varepsilon \to 0$. Moreover, this perturbation may be constructed to preserve the property of minimality for the boundary, as well as the divergence free property of the Maxwell fields. After the perturbation, although ∂M_t is minimal it may not be outermost, however, for the present purpose the outermost condition is not necessary. That is, the doubling argument only requires that the boundary be minimal, not outermost. Let us now construct the desired deformation. Fix a smooth positive function ϱ which vanishes sufficiently fast at spatial infinity, and solve the semi-linear boundary value problem

(5.32)
$$
\Delta_{g_t} z_{t,\varepsilon} - \frac{1}{8}R_{g_t} z_{t,\varepsilon} + \frac{1}{8}(R_{g_t} + \varepsilon\varrho) z_{t,\varepsilon}^{-3} = 0 \quad \text{on} \quad M_t,
$$

$$(5.33) \qquad\qquad \partial_\tau z_{t,\varepsilon} = 0 \quad \text{on} \quad \partial M_t, \qquad z_{t,\varepsilon} \to 1 \quad \text{as} \quad r \to \infty.$$

It is easily seen that a smooth positive solution exists for small ε, and is pointwise close to 1, by the implicit function theorem. Furthermore, equation (5.32) implies that the conformal metric $g_t^\varepsilon = z_{t,\varepsilon}^4 g_t$ has scalar curvature $R_{g_t^\varepsilon} = (R_{g_t} + \varepsilon\varrho) z_{t,\varepsilon}^{-8}$, and the Neumann boundary condition guarantees that the boundary is still a minimal surface with respect to the new metric. Now define $(E_t^\varepsilon)^i = z_{t,\varepsilon}^{-6} E_t^i$ and $(B_t^\varepsilon)^i = z_{t,\varepsilon}^{-6} B_t^i$, so that the perturbed Maxwell fields remain divergence free. Lastly, a strict charged dominant energy condition holds

$$(5.34) \qquad R_{g_t^\varepsilon} = (R_{g_t} + \varepsilon\varrho) z_{t,\varepsilon}^{-8} \geq 2 z_{t,\varepsilon}^{-8} \left(|E_t|_{g_t}^2 + |B_t|_{g_t}^2 \right) + \frac{\varepsilon}{2}\varrho > 2 \left(|E_t^\varepsilon|_{g_t^\varepsilon}^2 + |B_t^\varepsilon|_{g_t^\varepsilon}^2 \right).$$

The constant C_t^ε may then be taken to be $\min_{M_t \setminus M(\tau_0)} \frac{\varepsilon}{2}\varrho$.

We will now apply the doubling argument to $(M_t, g_t^\varepsilon, E_t^\varepsilon, B_t^\varepsilon)$. Note that since $\lambda, \tau_0 > 0$ are small enough, Theorem 6.9 guarantees existence of the conformal factors (5.1). According to (5.25)

$$(5.35) \qquad\qquad m_t' = -2\widetilde{m}_t^\varepsilon + 2e^{-2t}(\gamma_t - \overline{\gamma}_t) + \theta_t^\varepsilon,$$

where $\theta_t^\varepsilon \to 0$ as $\varepsilon \to 0$. Since $|\gamma_t - \overline{\gamma}_t|$ and θ_t^ε may be made arbitrarily small, it will follow that $m_t' \leq 0$ if the mass of the doubled manifold is nonnegative $\widetilde{m}_t^\varepsilon \geq 0$. In light of the discussion preceding Lemma 5.3, it suffices to show that the scalar curvature of the doubled manifold is nonnegative. This will be accomplished in two cases associated with different regions. For convenience, in what follows, the superscript ε will be omitted from most of the notation.

Case 1: $\tau \geq \tau_0$.

In this region, with the help of (5.34), we find that

$$(5.36) \qquad \begin{aligned} R_{g_t^\pm} \geq &\frac{1}{2}(w_t^\pm)^{-6} \left[\left((1 \pm \overline{v}_t)^2 - \phi_t^2 \mp 4\overline{v}_t(1 \pm \overline{v}_t) \right) \left(f_t - \frac{|\nabla\phi_t|^2}{\overline{v}_t^2} \right) + \Lambda \left| f_t - \frac{|\nabla\phi_t|^2}{\overline{v}_t^2} \right| \right] \\ &+ (w_t^\pm)^{-4} \left[\frac{\varepsilon}{2}\varrho + 2 \left(|E_t|_{g_t}^2 + |B_t|_{g_t}^2 - f_t \right) \right]. \end{aligned}$$

The first line on the right-hand side is clearly nonnegative if $\Lambda \geq 12$, since $|\overline{v}_t| < 1$. Moreover, the second line is nonnegative for $\tau \geq 2\tau_0$ in light of (5.4), and is nonnegative for $\tau_0 \leq \tau \leq 2\tau_0$ by (5.5) if λ is sufficiently small, depending on ε.

Case 2: $0 \leq \tau < \tau_0$.

In this region (5.31) holds, and $|\overline{v}_t|, \phi_t, f_t \sim 0$, $w_t^\pm \sim 1$. Therefore (5.26) implies

$$(5.37) \qquad\qquad R_{g_t^\pm} \geq \frac{1}{2}(w_t^\pm)^{-6} \left[C_t^\varepsilon - 4f_t - 2\frac{|\nabla\phi_t|^2}{\overline{v}_t^2} + \phi_t \left(\Delta_{g_t}\phi_t - \frac{\nabla\overline{v}_t \cdot \nabla\phi_t}{\overline{v}_t} \right) \right].$$

According to (5.8) we may write

$$(5.38) \qquad\qquad \phi_t = \lambda\tau_0^4 + \beta(\overline{x}, \tau)(\tau - \tau_0)\eta(\tau) \quad \text{when} \quad 0 \leq \tau < \tau_0,$$

where $\overline{x}$ are coordinates on S_τ and $\beta(\overline{x}, \tau) = \partial_\tau\phi_t(\overline{x}, \tau_0)$. Then

$$(5.39) \qquad |\partial_\tau\phi_t| \leq |\beta|, \quad |\partial_\tau^2\phi_t| \leq c\tau_0^{-1}|\beta|, \quad |\overline{\nabla}\phi_t| \leq c\tau_0|\overline{\nabla}\beta|, \quad |\overline{\nabla}^2\phi_t| \leq c\tau_0|\overline{\nabla}^2\beta|,$$

where $\overline{\nabla}$ represents the induced connection on S_τ. Estimates for β are established in Appendix A (Theorem A.1), namely

$$(5.40) \qquad\qquad |\beta| + |\overline{\nabla}\beta| + |\overline{\nabla}^2\beta| \leq c(\tau_0, \varepsilon)\lambda.$$

Here, unlike in the appendix, the constant $c(\tau_0, \varepsilon)$ depends on ε since ϕ_t depends on the perturbed initial data. It follows that

$$
\phi_t \left| \Delta_{g_t} \phi_t - \frac{\nabla \overline{v}_t \cdot \nabla \phi_t}{\overline{v}_t} \right| = \phi_t \left| \overline{\Delta} \phi_t + \partial_\tau^2 \phi_t + H \partial_\tau \phi_t - \frac{\nabla \overline{v}_t \cdot \nabla \phi_t}{\overline{v}_t} \right|
$$

(5.41)
$$
\leq c \phi_t \left(\tau_0^{-1} |\beta| + \tau_0 (|\overline{\nabla} \beta| + |\overline{\nabla}^2 \beta|) \right)
$$

$$
\leq c(\tau_0, \varepsilon) \lambda^2.
$$

Similarly

(5.42)
$$
\frac{|\nabla \phi_t|^2}{\overline{v}_t^2} \leq c \left(\frac{|\partial_\tau \phi_t|^2 + |\overline{\nabla} \phi_t|^2}{\tau_0^2} \right) \leq c \left(\tau_0^{-2} |\beta|^2 + |\overline{\nabla} \beta|^2 \right) \leq c(\tau_0, \varepsilon) \lambda^2.
$$

Therefore by choosing λ sufficiently small, dependent on ε and τ_0, we find that the scalar curvature is nonnegative. $\qquad\square$

6. Existence of the Conformal Factor

In this section we will show that a positive solution of the Dirichlet problem (5.6), (5.7) exists, by constructing solutions to an auxiliary problem on a finite domain and then taking a limit as the finite domains exhaust $M(\tau_0)$.

Let $M(\tau_0, r_0)$ denote the complement in M_t of the region of distance less than τ_0 from ∂M_t and the region outside S_{r_0} in the asymptotically flat end. Define

(6.1)
$$
f_{t,r_0} = \left\{ \begin{array}{ll} \lambda^2 \eta(\tau) & \tau < \frac{5}{4} \tau_0 \\ |E_t|_{g_t}^2 + |B_t|_{g_t}^2 & \text{on } M(2\tau_0, \frac{1}{4} r_0) \\ \frac{\delta^2}{r_0^4} \chi(r) & \frac{1}{2} r_0 < r \end{array} \right\},
$$

which agrees with f_t on $M(0, \frac{1}{4} r_0)$. Here $\delta > 0$ is a small parameter to be determined and χ is a smooth cut-off function with $\chi \equiv 1$ on $\frac{1}{2} r_0 < r < r_0 + 1$, $\chi \equiv 0$ on $r > 2r_0$, $|\nabla \chi| \leq c r_0^{-1}$, and $|\nabla^2 \chi| \leq c r_0^{-2}$. On the transition region $\frac{1}{4} r_0 < r < \frac{1}{2} r_0$, f_{t,r_0} is chosen so as to make a smooth positive function. Next, solve (5.2), (5.3) with f_t replaced by f_{t,r_0}

(6.2)
$$
\Delta_{g_t} \overline{v}_{t,r_0} - f_{t,r_0} \overline{v}_{t,r_0} = 0 \quad \text{on} \quad M_t, \qquad \overline{v}_{t,r_0} = 0 \quad \text{on} \quad \partial M_t,
$$

(6.3)
$$
\overline{v}_{t,r_0} = -1 + \frac{\overline{\gamma}_{t,r_0}}{r} + O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty.
$$

The first main task is to establish existence of a positive solution to the auxiliary Dirichlet problem

(6.4)
$$
\Delta_{g_t} \phi_{t,r_0} - \frac{\nabla \overline{v}_{t,r_0} \cdot \nabla \phi_{t,r_0}}{\overline{v}_{t,r_0}} = \frac{\Lambda}{\phi_{t,r_0}} \left| f_{t,r_0} - \frac{|\nabla \phi_{t,r_0}|_{g_t}^2}{\overline{v}_{t,r_0}^2} \right| \quad \text{on} \quad M(\tau_0, r_0),
$$

(6.5)
$$
\phi_t = \lambda \tau_0^4 \quad \text{on} \quad S_{\tau_0}, \qquad \phi_t = \frac{\delta}{4r_0} \quad \text{on} \quad S_{r_0}.
$$

A priori estimates will be shown to hold independent of r_0, so that the desired solution of (5.6), (5.7) will arise as the limit $\phi_{t,r_0} \to \phi_t$ as $r_0 \to \infty$.

The following version of the Leray-Schauder fixed point theorem will be applied to (6.4), (6.5). In what follows, we will temporarily drop all references to the subindices t and r_0 associated with functions, as well as the subscript g_t associated with operators and norms.

Theorem 6.1. *Suppose that $\mathcal{B}$ is a Banach space with norm $\| \cdot \|$, $\mathcal{C} \subset \mathcal{B}$ is a closed convex subset, ϕ_0 is a point of $\mathcal{C}$, $T \colon \mathcal{C} \times [0,1] \to \mathcal{C}$ is continuous and compact with $T(\phi, 0) = \phi_0$, for all $\phi \in \mathcal{C}$, and suppose that there is a fixed constant $\Gamma > 0$ such that*

$$\tag{6.6} \| \phi \| < \Gamma$$

is satisfied whenever $\phi \in \mathcal{C}$ satisfies $T(\phi, s) = \phi$ for some $s \in [0,1]$. Then there exists $\phi \in \mathcal{C}$ such that $T(\phi, 1) = \phi$.

Note that this version of the theorem is slightly more general than that given in [13] (Theorem 11.6). The only difference is that T is defined on $\mathcal{C}$ instead of $\mathcal{B}$. This generalization is easily obtained using Dugundjis extension theorem [8] (Theorem 7.2).

In order to set up the fixed point theorem, fix a positive function $\varrho \in C^\infty(M)$ with $\varrho \sim r^{-3}$ as $r \to \infty$, and consider the regularized equation

$$\tag{6.7} \Delta\phi - \frac{\nabla\overline{v} \cdot \nabla\phi}{\overline{v}} = \frac{s\Lambda\phi}{(s\phi + \varepsilon)^2}\left| f - \frac{|\nabla\phi|^2}{\overline{v}^2} \right| + (1-s)\varrho\phi \quad \text{on} \quad M(\tau_0, r_0),$$

$$\tag{6.8} \phi = \lambda\tau_0^4 \quad \text{on} \quad S_{\tau_0}, \qquad \phi = \frac{\delta}{4r_0} \quad \text{on} \quad S_{r_0}.$$

In equation (6.7), there are actually two regularizations at play. One is the ε-regularization which avoids problems when ϕ vanishes, and the other is a capillarity regularization associated with the extra term $(1-s)\varrho\phi$, which aids in establishing C^1 estimates when s is sufficiently far away from the important value 1. Later we will let $\varepsilon \to 0$ in order to obtain a solution of (6.4).

Let $\mathcal{C}$ be the cone of nonnegative $C^2(M(\tau_0, r_0))$ functions; note that since $M(\tau_0, r_0)$ is closed, this is the space of functions which are C^2 up to the boundary. It is clear that $\mathcal{C}$ is closed and convex. Define the map $T : \mathcal{C} \times [0,1] \to \mathcal{C}$ by $T(\phi, s) = \psi$, where ψ solves

$$\tag{6.9} \Delta\psi - \frac{\nabla\overline{v} \cdot \nabla\psi}{\overline{v}} - \left(\frac{s\Lambda}{(s\phi + \varepsilon)^2}\left| f - \frac{|\nabla\phi|^2}{\overline{v}^2} \right| + (1-s)\varrho \right)\psi = 0 \quad \text{on} \quad M(\tau_0, r_0),$$

$$\tag{6.10} \psi = \lambda\tau_0^4 \quad \text{on} \quad S_{\tau_0}, \qquad \psi = \frac{\delta}{4r_0} \quad \text{on} \quad S_{r_0}.$$

Then given $\phi \in \mathcal{C}$ there exists a unique solution $\psi \in C^{2,\alpha}(M(\tau_0, r_0))$ by elliptic theory, for any $0 < \alpha < 1$. Moreover $\psi > 0$ by the maximum principle, so $\psi \in \mathcal{C}$. The Schauder estimates imply that

$$\tag{6.11} |\psi|_{C^{2,\alpha}(M(\tau_0, r_0))} \leq C(|\phi|_{C^{1,\alpha}(M(\tau_0, r_0))}, \varepsilon),$$

and that T is continuous. Since $C^{2,\alpha} \hookrightarrow C^2$ is compact, we also find that T is compact.

Next observe that if $s = 0$ then ψ does not depend on ϕ. Thus, in order to apply the Leray-Schauder fixed point theorem, it remains only to prove the a priori estimate

$$\tag{6.12} |\phi|_{C^{2,\alpha}(M(\tau_0, r_0))} \leq C$$

for a fixed point $T(\phi, s) = \phi$, where C is independent of s. Note that a fixed point satisfies the boundary value problem (6.7), (6.8). The estimate (6.12) will be established in several steps. First, maximum principle techniques produce C^0 bounds and also reduce C^1 bounds to boundary gradient estimates, which are then obtained with a local barrier argument. A positive subsolution is then constructed, which allows a boot-strap procedure to yield higher order bounds.

Proposition 6.2. *For any s and ε, $\sup_{M(\tau_0, r_0)} \phi \leq \max\{\lambda\tau_0^4, \frac{\delta}{4r_0}\}$.*

Proof. This follows directly from the maximum principle applied to (6.7), (6.8). □

Proposition 6.3. *If* $\Lambda > 8$ *then there exists a constant C independent of s, ε, and r_0 such that*

$$(6.13) \qquad \sup_{M(\tau_0, r_0)} |\nabla\phi| \leq C(1 + |f|_{C^1}) + \sup_{\partial M(\tau_0, r_0)} |\nabla\phi|.$$

Proof. We will apply a maximum principle argument to the equation satisfied by $|\nabla\phi|$. Observe that

$$
\begin{aligned}
\Delta|\nabla\phi| &= \nabla^j\left(\frac{\phi^i \nabla_{ij}\phi}{|\nabla\phi|}\right) \\
(6.14) \qquad &= \frac{|\nabla^2\phi|^2}{|\nabla\phi|} + \frac{\phi^i \nabla_j \nabla_i \nabla_j \phi}{|\nabla\phi|} - \frac{\phi^i(\nabla_{ij}\phi)\phi^l \nabla_l^j \phi}{|\nabla\phi|^3} \\
&= \frac{|\nabla^2\phi|^2}{|\nabla\phi|} + \frac{\phi^i \partial_i \Delta\phi}{|\nabla\phi|} + \frac{R_{ij}\phi^i \phi^j}{|\nabla\phi|} - \frac{|\nabla|\nabla\phi||^2}{|\nabla\phi|},
\end{aligned}
$$

where R_{ij} denotes components of the Ricci tensor. Suppose that a global interior maximum exists for $|\nabla\phi|$. Then at this point we may assume that

$$(6.15) \qquad \frac{|\nabla\phi|^2}{\overline{v}^2} > \frac{C^2(1 + |f|_{C^1})^2}{\overline{v}^2} \geq 2f$$

for some constant $C > 0$, otherwise the desired result holds immediately. Thus by (6.7) and (6.14), and setting $h(\phi) = s\Lambda\phi(s\phi + \varepsilon)^{-2}$, it follows that

$$(6.16)$$
$$
\begin{aligned}
\Delta|\nabla\phi| = {} & \frac{|\nabla^2\phi|^2}{|\nabla\phi|} + \frac{1}{|\nabla\phi|}\left[\frac{\phi^i \nabla_{il}\overline{v}\nabla^l\phi}{\overline{v}} + \frac{\phi^i \nabla_l \overline{v}\nabla_i^l \phi}{\overline{v}} - \frac{(\nabla\overline{v}\cdot\nabla\phi)^2}{\overline{v}^2} + h'|\nabla\phi|^2\left(\frac{|\nabla\phi|^2}{\overline{v}^2} - f\right)\right] \\
& + \frac{h}{|\nabla\phi|}\left(\frac{\phi^i \partial_i |\nabla\phi|^2}{\overline{v}^2} - 2\frac{(\nabla\phi\cdot\nabla\overline{v})}{\overline{v}^3}|\nabla\phi|^2 - \nabla\phi\cdot\nabla f\right) + \frac{R_{ij}\phi^i\phi^j}{|\nabla\phi|} - \frac{|\nabla|\nabla\phi||^2}{|\nabla\phi|} + (1-s)\frac{\phi^i \partial_i(\varrho\phi)}{|\nabla\phi|}.
\end{aligned}
$$

At the maximum

$$(6.17) \qquad 0 = \partial_i |\nabla\phi|^2 = 2\phi_j \nabla_i^j \phi, \qquad \Delta|\nabla\phi| \leq 0.$$

Hence

$$(6.18) \quad 0 \geq \frac{|\nabla^2\phi|^2}{|\nabla\phi|} + \frac{h'}{\overline{v}^2}|\nabla\phi|^3 - \frac{2h|\nabla\overline{v}|}{|\overline{v}|^3}|\nabla\phi|^2 + \left((1-s)\varrho - |h'|f - c(x)\right)|\nabla\phi| - h|\nabla f| - (1-s)|\nabla\varrho|\phi,$$

for some positive function $c(x)$ independent of s, ε, and r_0, and falling-off at least on the order of r^{-3} in the asymptotic end.

Let $\{e_1, e_2, e_3 = \frac{\nabla\phi}{|\nabla\phi|}\}$ be an orthonormal basis of tangent vectors at the maximum point, then with the help of (6.17)

$$(6.19) \qquad |\nabla^2\phi|^2 = \sum_{i,j=1,2}\left[\nabla^2\phi(e_i, e_j)\right]^2 \geq \frac{1}{2}\left[\sum_{i=1,2}\nabla^2\phi(e_i, e_i)\right]^2 = \frac{1}{2}(\Delta\phi)^2.$$

Moreover, using (6.7) and (6.15) produces

$$(6.20) \qquad (\Delta\phi)^2 \geq \frac{h^2}{4\overline{v}^4}|\nabla\phi|^4 - \frac{2h|\nabla\overline{v}|}{|\overline{v}|^3}|\nabla\phi|^3 - \frac{2(1-s)\varrho\phi|\nabla\overline{v}|}{|\overline{v}|}|\nabla\phi| - \frac{2hf|\nabla\overline{v}|}{|\overline{v}|}|\nabla\phi|.$$

Combining this with (6.18) then yields

$$
(6.21) \quad
\begin{aligned}
0 \geq & \left(\frac{h^2}{8\overline{v}^4} + \frac{h'}{\overline{v}^2} \right) |\nabla\phi|^3 - \frac{3h|\nabla\overline{v}|}{|\overline{v}|^3} |\nabla\phi|^2 + \left((1-s)\varrho - |h'|f - c(x) \right) |\nabla\phi| \\
& - h|\nabla f| - (1-s)|\nabla\varrho|\phi - \frac{(1-s)\varrho\phi|\nabla\overline{v}|}{|\overline{v}|} - \frac{hf|\nabla\overline{v}|}{|\overline{v}|}.
\end{aligned}
$$

Let us now calculate

$$
(6.22) \quad
\begin{aligned}
h' + \frac{1}{8}h^2 &= \frac{s\Lambda(-s\phi + \varepsilon)}{(s\phi + \varepsilon)^3} + \frac{s^2\Lambda^2\phi^2}{8(s\phi + \varepsilon)^4} \\
&= \frac{\Lambda}{8} \left[\frac{(\Lambda - 8s)(s\phi + \varepsilon)^2 + \varepsilon(16s - 2\Lambda)(s\phi + \varepsilon) + \Lambda\varepsilon^2}{(s\phi + \varepsilon)^4} \right].
\end{aligned}
$$

By Young's inequality

$$
(6.23) \quad \varepsilon(16s - 2\Lambda)(s\phi + \varepsilon) \leq \Lambda\varepsilon^2 + \frac{(8s - \Lambda)^2(s\phi + \varepsilon)^2}{\Lambda},
$$

and hence

$$
(6.24) \quad h' + \frac{1}{8}h^2 \geq \frac{s(\Lambda - 8s)}{(s\phi + \varepsilon)^2}.
$$

Then since $|\overline{v}| \leq 1$,

$$
(6.25) \quad \frac{h^2}{8\overline{v}^4} + \frac{h'}{\overline{v}^2} \geq \frac{s(\Lambda - 8s)}{(s\phi + \varepsilon)^2\overline{v}^2}.
$$

We are now in a position to obtain a contradiction to the assumption (6.15), if C is chosen sufficiently large and independent of s, ε, and r_0. If $s \leq \frac{1}{2}$, then apply (6.15) and (6.25) to dominate all terms in (6.18) involving h. That is,

$$
(6.26) \quad \left(\frac{h^2}{8\overline{v}^4} + \frac{h'}{\overline{v}^2} \right) |\nabla\phi|^3 - \frac{3h|\nabla\overline{v}|}{|\overline{v}|^3} |\nabla\phi|^2 - |h'|f|\nabla\phi| - h|\nabla f| - \frac{hf|\nabla\overline{v}|}{|\overline{v}|} > 0
$$

if C is large enough. Similarly, if $s \geq \frac{1}{2}$, then (6.15) and (6.25) may be used to dominate all terms in (6.18) whether or not they involve h. Furthermore, for $s \leq \frac{1}{2}$, $(1-s)\varrho|\nabla\phi|$ may be used to dominate all terms not involving h. That is,

$$
(6.27) \quad \left((1-s)\varrho - c(x) \right) |\nabla\phi| - (1-s)|\nabla\varrho|\phi - \frac{(1-s)\varrho\phi|\nabla\overline{v}|}{|\overline{v}|} > 0
$$

if C is large enough. Notice that inequalities (6.26) and (6.27) are not consistent with (6.18). We conclude that there must exist a finite C, independent of s, ε, and r_0, such that $|\nabla\phi| \leq C(1 + |f|_{C^1})$ at a global interior max, if this point exists. If the global maximum is not attained on the interior, then it must be obtained on the boundary, and the desired result (6.13) follows.   □

We will now establish boundary gradient estimates by constructing appropriate local barriers.

Lemma 6.4. *If $\lambda\tau_0^4 \geq \frac{\delta}{4r_0}$, and $\tau_0 > 0$ is sufficiently small, then there exists a constant C independent of s, ε, λ, and τ_0 such that*

$$
(6.28) \quad |\nabla\phi|_{S_{\tau_0}} \leq C\lambda\tau_0.
$$

Proof. Since ϕ is constant on S_{τ_0} it suffices to estimate the normal derivative. As $\lambda \tau_0^4 \geq \frac{\delta}{4r_0}$, an upper barrier is trivial to construct. Namely, by the maximum principle $\phi \leq \lambda \tau_0^4$ globally. Hence we have

$$(6.29) \qquad\qquad\qquad\qquad \partial_\tau \phi|_{S_{\tau_0}} \leq 0.$$

A lower barrier will now be constructed as the solution to an Eikonal equation near the boundary

$$(6.30) \qquad\qquad |\nabla \underline{\phi}|^2 = f \overline{v}^2 \quad \text{on} \quad D(\tau_0, \tau_1), \qquad \underline{\phi} = \lambda \tau_0^4 \quad \text{on} \quad S_{\tau_0},$$

where $\tau_1 = \tau_0 + \tau_0^{5/2}$ and $D(\tau_0, \tau_1)$ denotes the domain enclosed by S_{τ_0} and S_{τ_1}. Note that the surface S_{τ_0} is noncharacteristic for this initial value problem, since it is possible to solve for $\partial_\tau \underline{\phi}|_{S_{\tau_0}}$. We claim that there is a solution with $\underline{\phi} < 0$ on S_{τ_1}. This will follow from an implicit function theorem argument. First construct an approximate solution ϕ_0. Expand

$$(6.31) \qquad\qquad\qquad \overline{v} = \overline{v}_0 + \overline{v}_1(\tau - \tau_0) + O(|\tau - \tau_0|^2),$$

and observe that since $\partial_\tau \overline{v}|_{\partial M} < 0$ we have $\partial_\tau \overline{v}|_{S_{\tau_0}} = \overline{v}_1 < 0$, and also $-c\tau_0 \leq \overline{v}_0 \leq -c^{-1}\tau_0$. Then plugging (6.31) into equation (6.30), and using $f = \lambda^2[1 - 2\tau_0^{-1}(\tau - \tau_0)]$ on $D(\tau_0, \tau_1)$, yields

$$(6.32) \qquad \phi_0 = \lambda \tau_0^4 + \lambda \overline{v}_0(\tau - \tau_0) + \frac{\lambda}{2}\left(\overline{v}_1 - \tau_0^{-1}\overline{v}_0\right)(\tau - \tau_0)^2 + \cdots + O\left(\frac{\lambda|\tau - \tau_0|^{N+1}}{\tau_0^{N-1}}\right),$$

with

$$(6.33) \qquad\qquad\qquad |\nabla \phi_0|^2 = f \overline{v}^2 + O\left(\frac{\lambda^2 |\tau - \tau_0|^N}{\tau_0^{N-2}}\right)$$

for any large N, depending on how many terms are given in ϕ_0. Below, $N \geq 3$ is sufficient.

Consider now the linearized equation

$$(6.34) \qquad\qquad \mathcal{L}\underline{\varphi} := \nabla \phi_0 \cdot \nabla \underline{\varphi} = \psi \quad \text{on} \quad D(\tau_0, \tau_1), \qquad \underline{\varphi} = 0 \quad \text{on} \quad S_{\tau_0}.$$

Since $\partial_\tau \phi_0|_{S_{\tau_0}} = \lambda \overline{v}_0 \neq 0$, the method of characteristics shows that we can always solve this initial value problem. Moreover $\mathcal{L} : \widetilde{C}^1 \to C^0$ is an isomorphism if $\widetilde{C}^1$ consists of all C^1 functions vanishing on S_{τ_0}. We may now apply the implicit function theorem to obtain a local smooth solution $\underline{\phi}$ of (6.30) for $|\tau_0 - \tau_1| = \tau_0^{5/2}$ sufficiently small.

To show that $\underline{\phi}|_{S_{\tau_1}} < 0$, choose τ_0 sufficiently small so that $\phi_0|_{S_{\tau_1}} < 0$. This is satisfied if

$$(6.35) \qquad\qquad\qquad \lambda \tau_0^4 < \lambda |\overline{v}_0||\tau_1 - \tau_0| \sim \lambda \tau_0^{7/2}.$$

It follows that

$$(6.36) \qquad \underline{\phi} = \phi_0 + (\underline{\phi} - \phi_0) = \phi_0 + O\left(\frac{\lambda|\tau - \tau_0|^N}{\tau_0^{N-2}}\right) < 0 \quad \text{on} \quad S_{\tau_1},$$

if τ_0 is sufficiently small.

Continuing with the proof of the boundary gradient estimate, we will use $\underline{\phi}$ as a lower barrier. To see that $\underline{\phi}$ is a subsolution of (6.7) on $D(\tau_0, \tau_1)$, calculate

$$\Delta\underline{\phi} = \partial_\tau^2\underline{\phi} + H\partial_\tau\underline{\phi} + \Delta_{S_\tau}\underline{\phi}$$

(6.37)
$$= \partial_\tau^2\phi_0 + H\partial_\tau\phi_0 + \Delta_{S_\tau}\phi_0 + O\left(\frac{\lambda|\tau_1 - \tau_0|^{N-2}}{\tau_0^{N-2}}\right)$$

$$= \lambda(\overline{v}_1 - \tau_0^{-1}\overline{v}_0) + O\left(\frac{\lambda|\tau_1 - \tau_0|}{\tau_0}\right),$$

and

(6.38)
$$\frac{\nabla\overline{v}\cdot\nabla\underline{\phi}}{\overline{v}} = \frac{\partial_\tau\overline{v}\partial_\tau\underline{\phi}}{\overline{v}} + \frac{\overline{\nabla}\overline{v}\cdot\overline{\nabla}\underline{\phi}}{\overline{v}}$$

$$= \frac{\partial_\tau\overline{v}\partial_\tau\phi_0}{\overline{v}} + \frac{\overline{\nabla}\overline{v}\cdot\overline{\nabla}\phi_0}{\overline{v}} + O\left(\frac{\lambda|\tau_1 - \tau_0|^{N-1}}{\tau_0^{N-1}}\right)$$

$$= \left(\frac{1}{\tau} + O(1)\right)(\lambda\overline{v}_0 + O(\lambda|\tau_1 - \tau_0|)) + O\left(\frac{\lambda|\tau_1 - \tau_0|}{\tau_0}\right)$$

$$= \frac{\lambda\overline{v}_0}{\tau} + O\left(\lambda|\overline{v}_0| + \frac{\lambda|\tau_1 - \tau_0|}{\tau_0}\right),$$

where we have used $\overline{v} = \widetilde{v}\tau + O(\tau^2)$ with $\widetilde{v} < 0$; $\overline{\nabla}$ represents the induced connection on S_τ. On the other side of (6.7), equation (6.30) eliminates one term and the other satisfies

(6.39)
$$(1 - s)\varrho\underline{\phi} \le c\lambda\tau_0^3.$$

Now observe that

(6.40)
$$\overline{v}_1 = \partial_\tau\overline{v}|_{S_{\tau_0}} = \partial_\tau\overline{v}|_{\partial M} + O(\tau_0), \qquad \overline{v}_0 = \overline{v}|_{S_{\tau_0}} = \partial_\tau\overline{v}|_{\partial M}\tau_0 + O(\tau_0^2),$$

which implies

(6.41)
$$\overline{v}_1 - \left(\frac{1}{\tau_0} + \frac{1}{\tau}\right)\overline{v}_0 = -\frac{\tau_0}{\tau}\partial_\tau\overline{v}|_{\partial M} + O(\tau_0) \ge 2c + O\left(\frac{|\tau_1 - \tau_0|}{\tau_0}\right)$$

for some constant $c > 0$. Hence, if τ_0 is sufficiently small

(6.42)
$$\Delta\underline{\phi} - \frac{\nabla\overline{v}\cdot\nabla\underline{\phi}}{\overline{v}} = \lambda\left[\overline{v}_1 - \left(\frac{1}{\tau_0} + \frac{1}{\tau}\right)\overline{v}_0\right] + O\left(\lambda|\overline{v}_0| + \frac{\lambda|\tau_1 - \tau_0|}{\tau_0}\right) \ge 2c\lambda + O(\lambda\tau_0^{3/2}) > 0.$$

Note that this positive lower bound is a result of the choice $f = \lambda^2[1 - 2\tau_0^{-1}(\tau - \tau_0)]$ on $D(\tau_0, \tau_1)$, and is the reason for defining f in this way. In light of (6.30) and (6.39), we find that $\underline{\phi}$ is a subsolution of (6.7) in the following sense

(6.43)
$$\Delta\underline{\phi} - \frac{\nabla\overline{v}\cdot\nabla\underline{\phi}}{\overline{v}} > \frac{s\Lambda\phi}{(s\phi + \varepsilon)^2}\left|f - \frac{|\nabla\phi|^2}{\overline{v}^2}\right| + (1 - s)\varrho\underline{\phi}.$$

It is important that the coefficient $\frac{s\Lambda\phi}{(s\phi+\varepsilon)^2}$ involves ϕ instead of $\underline{\phi}$, for the comparison argument below. This motivates using the Eikonal equation (6.30) to construct the subsolution.

Having shown that $\underline{\phi}$ is a subsolution, we now compare it with ϕ. Note that

(6.44)
$$\Delta(\phi - \underline{\phi}) - \frac{\nabla\overline{v}\cdot\nabla(\phi - \underline{\phi})}{\overline{v}} < \frac{s\Lambda\phi}{(s\phi + \varepsilon)^2}\left(\left|f - \frac{|\nabla\phi|^2}{\overline{v}^2}\right| - \left|f - \frac{|\nabla\underline{\phi}|^2}{\overline{v}^2}\right|\right) + (1 - s)\varrho(\phi - \underline{\phi}),$$

and

$$(6.45) \qquad (\phi - \underline{\phi})|_{S_{\tau_0}} = 0, \qquad (\phi - \underline{\phi})|_{S_{\tau_1}} > 0.$$

At an interior minimum of $\phi - \underline{\phi}$, $|\nabla \phi| = |\nabla \underline{\phi}|$ and $\Delta(\phi - \underline{\phi}) \geq 0$, which yields a contradiction. We conclude that $\phi - \underline{\phi} \geq 0$ on $D(\tau_0, \tau_1)$, and hence $\partial_\tau \phi|_{S_{\tau_0}} \geq \partial_\tau \underline{\phi}|_{S_{\tau_0}}$. Since

$$(6.46) \qquad \partial_\tau \underline{\phi} = \partial_\tau \phi_0 + O\left(\frac{\lambda |\tau_1 - \tau_0|^{N-1}}{\tau_0^{N-1}}\right) = \lambda \bar{v}_0 + O\left(\frac{\lambda |\tau_1 - \tau_0|}{\tau_0}\right) \geq -c\lambda\tau_0,$$

the desired result follows. □

Using similar methods a boundary gradient estimate may be established at S_{r_0}.

Lemma 6.5. *If r_0 is sufficiently large, then*

$$(6.47) \qquad |\nabla \phi|_{S_{r_0}} \leq \frac{2}{r_0}.$$

Proof. Let $r_1 = \frac{1}{2} r_0$. An upper barrier may be constructed in the form

$$(6.48) \qquad \bar{\phi} = \frac{\delta}{4r_0} + a(r_0 - r) \quad \text{for} \quad r \in [r_1, r_0].$$

A basic calculation shows that

$$(6.49) \qquad \Delta\bar{\phi} - \frac{\nabla\bar{v} \cdot \nabla\bar{\phi}}{\bar{v}} = -\frac{2a}{r} + O\left(\frac{a}{r^2}\right).$$

Hence, if $a > 0$ and r_0 is large enough, $\bar{\phi}$ is a supersolution

$$(6.50) \qquad \begin{aligned} \Delta\bar{\phi} - \frac{\nabla\bar{v} \cdot \nabla\bar{\phi}}{\bar{v}} &= -\frac{2a}{r} + O\left(\frac{a}{r^2}\right) \\ &< 0 \leq \frac{s\Lambda\phi}{(s\phi + \varepsilon)^2}\left|f - \frac{|\nabla\bar{\phi}|^2}{\bar{v}^2}\right| + (1-s)\varrho\bar{\phi}. \end{aligned}$$

Moreover if $a = 2r_0^{-1}$, then

$$(6.51) \qquad \bar{\phi}|_{S_{r_1}} = \frac{\delta}{4r_0} + 1 > \lambda\tau_0^4 = \sup_{M(\tau_0, r_0)} \phi.$$

A comparison argument then implies that $\bar{\phi} \geq \phi$ for $r_1 \leq r \leq r_0$. Therefore

$$(6.52) \qquad \partial_r \phi|_{S_{r_0}} \geq -\frac{2}{r_0}.$$

As in the proof of Lemma 6.4, a lower barrier will be constructed as a solution to the Eikonal equation

$$(6.53) \qquad |\nabla \underline{\phi}|^2 = f\bar{v}^2 \quad \text{on} \quad B(r_1, r_0), \qquad \underline{\phi} = \frac{\delta}{4r_0} \quad \text{on} \quad S_{r_0},$$

where $B(r_1, r_0)$ is the region between S_{r_1} and S_{r_0}. Note that S_{r_0} is noncharacteristic for this initial value problem, since it is possible to solve for $\partial_r \underline{\phi}|_{S_{r_0}}$. In order to apply the implicit function theorem, an approximate solution may be found in the form

$$(6.54) \qquad \phi_0 = \frac{\delta}{4r_0} - \frac{\delta}{r_0^2}(r_0 - r).$$

Using the fact that $f = \frac{\delta^2}{r_0^4}$, a direct calculation produces

$$(6.55) \qquad |\nabla\phi_0|^2 - f\overline{v}^2 = O\left(\frac{\delta^2}{rr_0^4}\right).$$

The implicit function theorem now yields a solution

$$(6.56) \qquad \underline{\phi} = \phi_0 + O\left(\frac{\delta}{r_0^2}\right).$$

Moreover

$$(6.57) \qquad \underline{\phi}|_{S_{r_1}} = -\frac{\delta}{4r_0} + O\left(\frac{\delta}{r_0^2}\right) < 0,$$

if r_0 is large enough.

In order to show that $\underline{\phi}$ is a subsolution on $B(r_1, r_0)$, observe that

$$(6.58) \qquad \Delta\underline{\phi} = \partial_r^2\underline{\phi} + \frac{2}{r}\partial_r\underline{\phi} + O(r^{-1}|\nabla^2\underline{\phi}| + r^{-2}|\nabla\underline{\phi}|) = \frac{2\delta}{rr_0^2} + O\left(\frac{\delta}{r_0^4}\right),$$

and

$$(6.59) \qquad \frac{\nabla\overline{v}\cdot\nabla\underline{\phi}}{\overline{v}} = O\left(\frac{|\nabla\underline{\phi}|}{r^2}\right) = O\left(\frac{\delta}{r_0^4}\right).$$

On the other side of equation (6.7), one term is eliminated with the aid of (6.53) and the other satisfies

$$(6.60) \qquad (1-s)\varrho|\underline{\phi}| = O\left(\frac{\delta}{r_0^4}\right),$$

since $\varrho \le cr^{-3}$. It follows that

$$(6.61) \qquad \begin{aligned} \Delta\underline{\phi} - \frac{\nabla\overline{v}\cdot\nabla\underline{\phi}}{\overline{v}} &= \frac{2\delta}{rr_0^2} + O\left(\frac{\delta}{r_0^4}\right) \\ &\ge \frac{2\delta}{r_0^3} + O\left(\frac{\delta}{r_0^4}\right) \\ &> O\left(\frac{\delta}{r_0^4}\right) = \frac{s\Lambda\underline{\phi}}{(s\underline{\phi}+\varepsilon)^2}\left|f - \frac{|\nabla\underline{\phi}|^2}{\overline{v}^2}\right| + (1-s)\varrho\underline{\phi}. \end{aligned}$$

A comparison argument now shows that $\phi \ge \underline{\phi}$ on $B(r_1, r_0)$, which yields

$$(6.62) \qquad \partial_r\phi|_{S_{r_0}} \le \frac{\delta}{2r_0^2}$$

if r_0 is sufficiently large. $\qquad\square$

By combining Proposition 6.3, Lemma 6.4, and Lemma 6.5 we obtain global C^1 bounds.

Corollary 6.6. *If $\tau_0 > 0$ is sufficiently small, r_0 is sufficiently large, and $\Lambda > 8$ then there exists a constant C independent of s, ε, and r_0 such that*

$$(6.63) \qquad \sup_{M(\tau_0, r_0)} |\nabla\phi| \le C(1 + |f|_{C^1}).$$

We are now in position to establish a basic existence results.

Theorem 6.7. *If $\lambda, \tau_0 > 0$, τ_0 is sufficiently small, r_0 is sufficiently large, and $\Lambda > 8$ then there exists a positive solution $\phi_{r_0,\varepsilon} \in C^{2,\alpha}(M(\tau_0, r_0))$, for any $\alpha \in [0, 1)$, of (6.7), (6.8) with $s = 1$.*

Proof. In order to apply the Leray-Schauder theorem, it suffices to establish a $C^{2,\alpha}$ estimate. Observe that with the aid of Proposition 6.2 and Corollary 6.6, and the fact that $s\phi + \varepsilon \geq \varepsilon > 0$, we have $\Delta\phi - \frac{\nabla\overline{v}\cdot\nabla\phi}{\overline{v}} \in L^\infty$ and the corresponding bound is independent of s. This implies that $\phi \in W^{2,p}$ for any p, and hence $\phi \in C^{1,\alpha}$ for any $\alpha < 1$, again independent of s. Thus $\Delta\phi - \frac{\nabla\overline{v}\cdot\nabla\phi}{\overline{v}} \in C^{0,\alpha}$, which implies that $\phi \in C^{2,\alpha}$. That is,

$$|\phi|_{C^{2,\alpha}(M(\tau_0, r_0))} \leq C \tag{6.64}$$

where C is independent of s. This guarantees the existence of a positive solution $\phi_{r_0,\varepsilon} \in C^{2,\alpha}(M(\tau_0, r_0))$ of (6.7), (6.8) with $s = 1$, which satisfies (6.64). $\qquad\square$

In order to proceed further towards the goal of establishing $C^{2,\alpha}$ estimates independent of ε, a uniform positive lower bound for $\phi_{r_0,\varepsilon}$ is needed. To this end let us rewrite (6.7), with $s = 1$, as an equation for $\zeta := (\phi_{r_0,\varepsilon} + \varepsilon)^3$. Observe that

$$\Delta\zeta = 3(\phi_{r_0,\varepsilon} + \varepsilon)^2 \Delta\phi_{r_0,\varepsilon} + 6(\phi + \varepsilon)|\nabla\phi_{r_0,\varepsilon}|^2 \tag{6.65}$$

and

$$\frac{\nabla\overline{v}\cdot\nabla\zeta}{\overline{v}} = \frac{3(\nabla\overline{v}\cdot\nabla\phi_{r_0,\varepsilon})}{\overline{v}}(\phi_{r_0,\varepsilon} + \varepsilon)^2. \tag{6.66}$$

It follows that

$$\Delta\zeta - \frac{\nabla\overline{v}\cdot\nabla\zeta}{\overline{v}} = 3\Lambda[(\phi_{r_0,\varepsilon} + \varepsilon) - \varepsilon]\left|f - \frac{|\nabla\phi_{r_0,\varepsilon}|^2}{\overline{v}^2}\right| + 6|\nabla\phi_{r_0,\varepsilon}|^2\zeta^{\frac{1}{3}}. \tag{6.67}$$

Hence

$$\Delta\zeta - \frac{\nabla\overline{v}\cdot\nabla\zeta}{\overline{v}} - \Phi\zeta^{\frac{1}{3}} = -3\varepsilon\Lambda\left|f - \frac{|\nabla\phi_{r_0,\varepsilon}|^2}{\overline{v}^2}\right| \leq 0, \tag{6.68}$$

where

$$\Phi = 3\Lambda\left|f - \frac{|\nabla\phi_{r_0,\varepsilon}|^2}{\overline{v}^2}\right| + 6|\nabla\phi_{r_0,\varepsilon}|^2, \tag{6.69}$$

and

$$\zeta|_{S_{\tau_0}} = (\lambda\tau_0^4 + \varepsilon)^3, \qquad \zeta|_{S_{r_0}} = \left(\frac{\delta}{4r_0} + \varepsilon\right)^3. \tag{6.70}$$

According to the C^0 and C^1 estimates $\Phi \leq \Phi_0$, where Φ_0 is a function independent of ε, r_0, and $\phi_{r_0,\varepsilon}$.

Suppose that there exists a solution of the Dirichlet problem

$$\Delta\underline{\zeta} - \frac{\nabla\overline{v}\cdot\nabla\underline{\zeta}}{\overline{v}} - \Phi_0\underline{\zeta}^{\frac{1}{3}} = 0 \quad \text{on} \quad M(\tau_0, r_0), \tag{6.71}$$

$$\underline{\zeta} = (\lambda\tau_0^4)^3 \quad \text{on} \quad S_{\tau_0}, \qquad \underline{\zeta} = \left(\frac{\delta}{4r_0}\right)^3 \quad \text{on} \quad S_{r_0}. \tag{6.72}$$

By the maximum principle the solution is positive. Moreover, it is independent of ε, and $\phi_{r_0,\varepsilon}$, and $\zeta \geq \underline{\zeta}$. To see this, use a comparison argument. Observe that

$$\Delta(\zeta - \underline{\zeta}) - \frac{\nabla\overline{v}\cdot\nabla(\zeta - \underline{\zeta})}{\overline{v}} \leq \Phi\zeta^{\frac{1}{3}} - \Phi_0\underline{\zeta}^{\frac{1}{3}}, \tag{6.73}$$

(6.74)
$$(\zeta - \underline{\zeta})|_{S_{\tau_0}} > 0, \qquad (\zeta - \underline{\zeta})|_{S_{r_0}} > 0.$$

Suppose that $\zeta - \underline{\zeta} < 0$ somewhere, and let x_0 be the point at which the global minimum is achieved. Then $\zeta^{\frac{1}{3}}(x_0) < \underline{\zeta}^{\frac{1}{3}}(x_0)$, $|\nabla(\zeta - \underline{\zeta})(x_0)| = 0$, and $\Delta(\zeta - \underline{\zeta})(x_0) \geq 0$, so that

(6.75)
$$\begin{aligned}
0 &\leq \Delta(\zeta - \underline{\zeta})(x_0) - \frac{\nabla \overline{v} \cdot \nabla(\zeta - \underline{\zeta})}{\overline{v}}(x_0) \\
&\leq \Phi \zeta^{\frac{1}{3}}(x_0) - \Phi_0 \underline{\zeta}^{\frac{1}{3}}(x_0) \\
&\leq \underline{\zeta}^{\frac{1}{3}}(\Phi - \Phi_0)(x_0) < 0.
\end{aligned}$$

This contradiction implies that $\zeta \geq \underline{\zeta}$ on all of $M(\tau_0, r_0)$, and yields the desired uniform positive lower bound. Hence $(\phi_{r_0,\varepsilon} + \varepsilon)^3 \geq \underline{\zeta}$ or rather $\phi_{r_0,\varepsilon} \geq \underline{\zeta}^{\frac{1}{3}} - \varepsilon$. Since $\underline{\zeta} > 0$ is independent of ε, we find that $\phi_{r_0,\varepsilon} \geq \frac{1}{2}\underline{\zeta}^{\frac{1}{3}}$ for all sufficiently small ε.

Proposition 6.8. *If $\lambda, \tau_0 > 0$ then there exists a smooth positive function $\underline{\zeta}$ independent of ε, and $\phi_{r_0,\varepsilon}$, such that $\phi_{r_0,\varepsilon} \geq \underline{\zeta}^{\frac{1}{3}} - \varepsilon$ on $M(\tau_0, r_0)$. In particular, $\phi_{r_0,\varepsilon} \geq \frac{1}{2}\underline{\zeta}^{\frac{1}{3}}$ for all sufficiently small ε.*

Proof. It only remains to establish the existence of $\underline{\zeta}$. Write the equation as

(6.76)
$$\mathrm{div}\left(\frac{\nabla \underline{\zeta}}{\overline{v}}\right) - \frac{\Phi_0}{\overline{v}} \underline{\zeta}^{\frac{1}{3}} = 0,$$

and consider the functional

(6.77)
$$I[\zeta] = \int_{M(\tau_0, r_0)} \frac{1}{2} \frac{|\nabla \zeta|^2}{|\overline{v}|} + \frac{3}{4} \frac{\Phi_0}{|\overline{v}|} \zeta^{\frac{4}{3}}.$$

We will minimize it over all $H^1(M(\tau_0, r_0))$-functions with the fixed boundary values as in (6.72). To show that this functional is bounded from below, apply Hölder's inequality and use $\mathrm{Vol}(M(\tau_0, r_0)) < \infty$ to find

(6.78)
$$\int_{M(\tau_0, r_0)} \frac{\Phi_0}{|\overline{v}|} \zeta^{\frac{4}{3}} \leq \sigma \parallel \zeta \parallel_{L^2}^2 + c_1,$$

where σ is a small parameter. Moreover, by Poincaré's inequality

(6.79)
$$\begin{aligned}
\int_{M(\tau_0, r_0)} |\nabla \zeta|^2 &= \int_{M(\tau_0, r_0)} |\nabla(\zeta - (\lambda \tau_0^4)^3)|^2 \\
&\geq \int_{M(\tau_0, r_0)} c_2^{-1} |\zeta - (\lambda \tau_0^4)^3|^2 \\
&= \int_{M(\tau_0, r_0)} c_2^{-1} (\zeta^2 - 2(\lambda \tau_0^4)^3 \zeta + (\lambda \tau_0^4)^6) \\
&\geq \int_{M(\tau_0, r_0)} 2c_2^{-1} \zeta^2 - c_3.
\end{aligned}$$

It follows that if σ is sufficiently small then

(6.80)
$$I[\zeta] \geq c_4^{-1} \parallel \zeta \parallel_{H^1}^2 - c_5,$$

and hence $I[\zeta]$ is bounded from below.

By direct methods in the calculus of variations, there is a minimizing sequence that has a weakly convergent (in H^1) subsequence $\zeta_i \rightharpoonup \underline{\zeta}$. Then since $H^1 \hookrightarrow L^2$ is compact, $\zeta_i \to \underline{\zeta}$ strongly in L^2. In particular $\zeta_i \to \underline{\zeta}$ in $L^{\frac{4}{3}}$, so that

$$\text{(6.81)} \qquad \int_{M(\tau_0,r_0)} \zeta_i^{\frac{4}{3}} \to \int_{M(\tau_0,r_0)} \underline{\zeta}^{\frac{4}{3}}.$$

Since the H^1 norm is weakly lower semicontinuous, it follows that $\underline{\zeta}$ realizes the infimum, and is hence a weak solution.

Now use elliptic regularity. Since $\underline{\zeta}^{\frac{1}{3}} \in L^6$, by (6.71) we have $\Delta\underline{\zeta} - \frac{\nabla\overline{v}\cdot\nabla\underline{\zeta}}{\overline{v}} \in L^6$, which implies that $\underline{\zeta} \in W^{2,6} \hookrightarrow C^{1,\frac{1}{2}}$. This in turn implies $\underline{\zeta}^{\frac{1}{3}} \in C^0$, so that $\underline{\zeta} \in W^{2,p}$ for any $p > 1$, and hence $\underline{\zeta} \in C^{1,\alpha}$ for any $\alpha < 1$. Since $\underline{\zeta} > 0$ it follows that $\underline{\zeta}^{\frac{1}{3}} \in C^{0,\alpha}$, and hence $\underline{\zeta} \in C^{2,\alpha}$. Boot-strapping then produces $\underline{\zeta} \in C^\infty$. $\qquad\square$

Having established a uniform lower bound, we may let $\varepsilon \to 0$ and $r_0 \to \infty$ to obtain the main existence result.

Theorem 6.9. *If $\lambda, \tau_0 > 0$, τ_0 is sufficiently small, and $\Lambda > 8$, then there exists a positive solution $\phi_t \in C^{2,\alpha}(M(\tau_0))$, for any $\alpha \in [0,1)$, of (5.6), (5.7).*

Proof. In light of the uniform lower bound in Proposition 6.8, the same arguments appearing in the proof of Theorem 6.7 yield a $C^{2,\alpha}$ estimate (6.64), with C independent of ε. Thus, after possibly passing to a subsequence $\phi_{r_0,\varepsilon} \to \phi_{t,r_0}$ as $\varepsilon \to 0$, yielding a positive solution of (6.4), (6.5) which also satisfies (6.64).

We will now let $r_0 \to \infty$. In order to control the decay at spatial infinity, we construct a global upper barrier by solving
(6.82)
$$\Delta\overline{\phi}_{t,r_0} - \frac{\nabla\overline{v}_{t,r_0}\cdot\overline{\phi}_{t,r_0}}{\overline{v}_t} = 0 \quad \text{on} \quad M(\tau_0), \quad \overline{\phi}_{t,r_0}|_{S_{\tau_0}} = 1, \quad \overline{\phi}_{t,r_0} = \frac{c_{t,r_0}}{r} + O\left(\frac{1}{r^2}\right) \quad \text{as} \quad r \to \infty,$$

where c_{t,r_0} is a positive constant. Since $\overline{v}_{t,r_0}$ smoothly converges to $\overline{v}_t$ as $r_0 \to \infty$, it holds that $c_{t,r_0} \to c_t > 0$ and $\overline{\phi}_{t,r_0} \to \overline{\phi}_t$, the solution of (6.82) with $\overline{v}_{t,r_0}$ replaced by $\overline{v}_t$. Thus if δ is chosen sufficiently small (independent of r_0), then $\overline{\phi}_{t,r_0}|_{S_{r_0}} > \phi_{t,r_0}|_{S_{r_0}}$. A standard comparison argument now applies to yield $\overline{\phi}_{t,r_0} > \phi_{t,r_0}$ on $M(\tau_0,r_0)$.

Let $\underline{\zeta}_{t,r_0}$ denote the lower bound which arises from the construction in Proposition 6.8. Since Φ_0 and $\sup\underline{\zeta}_{t,r_0}$ are controlled independent of r_0, higher order a priori estimates for $\underline{\zeta}_{t,r_0}$ are also independent of r_0, so that $\underline{\zeta}_{t,r_0} \to \underline{\zeta}_t$ on compact subsets as $r_0 \to \infty$, where $\underline{\zeta}_t > 0$ satisfies equation (6.71) with $\overline{v}_{t,r_0}$ replaced by $\overline{v}_t$. Therefore, since $\underline{\zeta}_{t,r_0}^{\frac{1}{3}} \le \phi_{t,r_0}$, the function ϕ_{t,r_0} is uniformly bounded below by a positive constant on any compact subset, independent of r_0. As the C^0 and C^1 bounds of Proposition 6.2 and Corollary 6.6 are also independent of r_0, $C^{2,\alpha}$ estimates on compact subsets may be established analogously to (6.64) and independently of r_0. This implies that, after possibly passing to a subsequence, $\phi_{t,r_0} \to \phi_t$ as $r_0 \to \infty$, where ϕ_t satisfies (5.6). Moreover since $\underline{\zeta}_t^{\frac{1}{3}} \le \phi_t \le \overline{\phi}_t$, (5.7) is also valid and ϕ_t is strictly positive. $\qquad\square$

Remark 6.10. It is unlikely that higher order regularity better than $C^{2,\alpha}$ is possible, due to the presence of an absolute value on the right-hand side of (5.6).

7. Proof of the Main Theorem

The purpose of this section is to prove Theorem 1.1. First observe that if $\rho = |q|$, or equivalently $|\partial M|_g = 4\pi q^2$, then the Penrose inequality with charge (1.2) is equivalent to the positive mass theorem with charge $m \geq |q|$, which has already been established in [12]. The case of equality for this theorem, asserts that the initial data must be isometric to the canonical slice of the Majumdar-Papapetrou spacetime [5]. However, such initial data does not possess a minimal surface boundary, and hence does not fall under the hypotheses of Theorem 1.1. Hence, when the area/charge inequality is saturated we must have $m > |q|$.

Now assume that $|\partial M|_g > 4\pi q^2$, so that in particular Theorem 4.5 applies. Note that the results of Section 4 require a perturbation of the initial data to achieve charged harmonic asymptotics, thus we assume here that such a perturbation has been made. Theorems 2.1 and 5.4 imply that the areas $|\partial M_t|_{g_t}$ and charges q_t remain constant throughout the flow, while the mass $m(t)$ is nonincreasing. Moreover, Theorem 4.5 guarantees that for some finite time $\bar{t}$ the minimal boundary $\partial M_{\bar{t}}$ is connected. Consider now the initial data set $(M_{\bar{t}}, g_{\bar{t}}, E_{\bar{t}}, B_{\bar{t}})$. It satisfies all the hypotheses of the Penrose inequality with charge for a single black hole, namely the boundary is an outermost minimal surface, the charged dominant energy condition is valid, and the Maxwell constraints holds without charged matter. As the inequality has been proven for a single black hole [15], [16], it follows that

$$(7.1) \qquad m(0) \geq m(\bar{t}) \geq \sqrt{\frac{|\partial M_{\bar{t}}|_{g_t}}{16\pi}} + \sqrt{\frac{\pi}{|\partial M_{\bar{t}}|_{g_t}}} q_{\bar{t}}^2 = \sqrt{\frac{|\partial M_0|_{g_0}}{16\pi}} + \sqrt{\frac{\pi}{|\partial M_0|_{g_0}}} q_0^2.$$

Since this holds for an arbitrarily small perturbation of the data, (1.2) holds for the original initial data. Notice that we avoid the convergence issue concerning the flow, by employing the inverse mean curvature flow once the minimal surface becomes connected. Whether the charged conformal flow converges to the canonical Reissner-Nordström data is an interesting question, which we strongly believe has an affirmative answer. Ultimately though, it is not necessary for the current result, and so it will be left for future investigation.

It remains to establish the rigidity statement. Here we will not need to perturb the initial data to achieve charged harmonic asymptotics. Suppose that equality holds in (1.2). We conclude that the mass remains constant throughout the flow $m'(t) = 0$. To see this, suppose not, then

$$(7.2) \qquad m(\widetilde{t}) < \sqrt{\frac{|\partial M_0|_{g_0}}{16\pi}} + \sqrt{\frac{\pi}{|\partial M_0|_{g_0}}} q_0^2$$

for some $\widetilde{t}$. A contradiction is obtained by applying the results of the previous paragraph to the initial data set $(M_{\widetilde{t}}, g_{\widetilde{t}}, E_{\widetilde{t}}, B_{\widetilde{t}})$. Equation (5.35) now implies that

$$(7.3) \qquad e^{-2t}(\gamma_t - \overline{\gamma}_t) + \frac{1}{2}\theta_t^\varepsilon = \widetilde{m}^\varepsilon(t) \quad \text{for all} \quad t \geq 0,$$

where $\widetilde{m}^\varepsilon(t)$ is the mass of the doubled manifold $(M_t^+ \cup M_t^-, (g_t^\varepsilon)^+ \cup (g_t^\varepsilon)^-)$ and g_t^ε is defined in the proof of Theorem 5.4.

We claim that the doubled manifold must be diffeomorphic to $\mathbb{R}^3$. To see this, suppose that it is not true. Then ∂M_t must have two or more components. According to the result of Meeks-Simon-Yau [24], there is then an outermost minimal surface $\mathcal{S}_t(\varepsilon, \tau_0)$ in the doubled manifold which encloses the nontrivial topology. Since the scalar curvature of the doubled manifold is nonnegative, the Penrose

inequality and (7.3) then yield

$$(7.4) \qquad e^{-2t}(\gamma_t - \overline{\gamma}_t) + \frac{1}{2}\theta_t^\varepsilon \geq \sqrt{\frac{|\mathcal{S}_t(\varepsilon, \tau_0)|_{(g_t^\varepsilon)^+ \cup (g_t^\varepsilon)^-}}{16\pi}} \qquad \text{for all} \quad t \geq 0.$$

For each fixed t, Theorem 5.2 shows that $|\gamma_t - \overline{\gamma}_t| \to 0$ as $\tau_0 \to 0$, and from the proof of Theorem 5.4 we have $\theta_t^\varepsilon \to 0$ as $\varepsilon \to 0$. On the other hand $\phi_t \to 0$ and $\overline{v}_t \to v_t$ both in C^0 as $\tau_0 \to 0$ and $\varepsilon \to 0$ (note that higher order convergence of ϕ_t is not generally possible), so that the conformal factors defining the doubled metric converge as $w_t^\pm \to \frac{1 \pm v_t}{2}$ in C^0. Let $\widetilde{g}_t^\pm = \left(\frac{1 \pm v_t}{2}\right)^4 g_t$. It follows that

$$(7.5) \qquad |\mathcal{S}_t(\varepsilon, \tau_0)|_{(g_t^\varepsilon)^+ \cup (g_t^\varepsilon)^-} \geq \frac{1}{2}|\mathcal{S}_t(\varepsilon, \tau_0)|_{\widetilde{g}_t^+ \cup \widetilde{g}_t^-},$$

for τ_0, ε sufficiently small. There is, however, a positive lower bound for the area of any surface enclosing the nontrivial topology in $(M_t^+ \cup M_t^-, \widetilde{g}_t^+ \cup \widetilde{g}_t^-)$. Hence there is a positive lower bound independent of τ_0 and ε for the right-hand side of (7.5). This leads to a contradiction with (7.4) for sufficiently small τ_0 and ε. Therefore the doubled manifold must have trivial topology, or equivalently ∂M_t consists of one component for all $t \geq 0$.

Rigidity for the Penrose inequality with charge in the case of one black hole was established in [9]. This result relies on monotonicity of the so called charged Hawking mass

$$(7.6) \qquad M_{CH}(S) = \sqrt{\frac{|S|}{16\pi}} \left(1 + \frac{4\pi q^2}{|S|} - \frac{1}{16\pi} \int_S H^2 dA\right),$$

under inverse mean curvature flow. In [9] only the electric field was present, and q in (7.6) represented the total electric charge. However, the same proof applies when both the electric and magnetic fields are present, if $q^2 = q_e^2 + q_b^2$. It follows that (M, g) is isometric to the canonical slice of the Reissner-Nordström spacetime, and $E = -q_e \nabla r^{-1}$, $B = -q_b \nabla r^{-1}$ in the usual anisotropic coordinates. This concludes the proof.

Appendix A. Estimates at S_{τ_0}

In this section we will establish the estimates for ϕ_t at S_{τ_0} appearing in (5.40). For simplicity, the subindex t will be suppressed. Note that by the Hopf lemma $\partial_\tau \phi|_{S_{\tau_0}} < 0$, so the level sets of ϕ foliate a neighborhood of S_{τ_0}. Consider the domain

$$(A.1) \qquad \Omega_\rho = \{x \in M \mid \sigma \lambda \tau_0^4 < \phi < \lambda \tau_0^4\},$$

where σ is sufficiently small to guarantee that $\Omega_\sigma \subset D(\tau_0, \frac{5}{4}\tau_0)$, the domain enclosed by S_{τ_0} and $S_{\frac{5}{4}\tau_0}$. Then $f = \lambda^2[1 - 2\tau_0^{-1}(\tau - \tau_0)]$ on Ω_σ. We may then apply the method of proof for Proposition 6.3 to obtain a C^1 estimate on Ω_σ. The key inequality which leads to the desired estimate is the analogue of (6.18). Namely using (6.25), $c^{-1}\tau_0 \leq |\overline{v}| \leq c\tau_0$, and $s = 1$, $\varepsilon = 0$ we find that at a global maximum for $|\nabla \phi|$,

$$(A.2) \qquad 0 \geq \frac{c^{-1}(\Lambda - 8)}{\tau_0^2 \phi^2}|\nabla \phi|^3 - \frac{c\Lambda}{\tau_0^3 \phi}|\nabla \phi|^2 - c\left(\frac{\Lambda f}{\phi^2} + 1\right)|\nabla \phi| - \frac{\Lambda}{\phi}|\nabla f| - \frac{c\Lambda f}{\tau_0 \phi}$$

for some constant $c > 0$ independent of λ, τ_0, and σ. It is clear that if $|\nabla \phi| > c_0 \lambda \tau_0$ for c_0 sufficiently large, then a contradiction is obtained from (A.2). We conclude that at a global interior maximum, there exists a finite constant c such that $|\nabla \phi| \leq c\lambda \tau_0$. It follows that

$$(A.3) \qquad \sup_{\Omega_\sigma} |\nabla \phi| \leq c\lambda \tau_0 + \sup_{\partial \Omega_\sigma} |\nabla \phi|.$$

The boundary of Ω_σ consists of two types of components. One is S_{τ_0}, where a gradient estimate has already been established in Lemma 6.4. The other type of component is also a level set of ϕ, and possesses a neighborhood in which f has the desired expression, so Lemma 6.4 applies here as well to yield $|\nabla\phi|_{\partial\Omega_\sigma} \le c\lambda\tau_0$. Together with (A.3), this implies that

$$(A.4) \qquad\qquad \sup_{\Omega_\sigma} |\nabla\phi| \le c\lambda\tau_0.$$

We now claim that $S_{\overline{\tau}_0} \subset \Omega_\sigma$ for some $\overline{\tau}_0 > \tau_0$ independent of λ. To see this, let $\overline{x}$ be coordinates on S_τ, and observe that (A.4) implies

$$(A.5) \qquad\qquad \phi(\overline{x}, \tau) - \phi(\overline{x}, \tau_0) = \int_{\tau_0}^{\tau} \partial_\tau \phi(\overline{x}, \varsigma) d\varsigma \ge -c\lambda\tau_0(\tau - \tau_0),$$

so that

$$(A.6) \qquad\qquad \phi(\overline{x}, \tau) \ge \lambda\tau_0^4 - c\lambda\tau_0(\tau - \tau_0).$$

Thus, if $\overline{\tau}_0 = \tau_0 + \frac{1}{2c}\tau_0^3$ then

$$(A.7) \qquad\qquad \phi|_{S_{\overline{\tau}_0}} \ge \frac{1}{2}\lambda\tau_0^4 > \sigma\lambda\tau_0^4,$$

assuming that $\sigma < 1/2$. This verifies the claim. It follows that $\mathrm{Vol}(\Omega_\sigma) \ge C_0 > 0$ where C_0 is independent of λ. Therefore the constants appearing in the L^p and Schauder estimates, as well as the Sobolev embeddings, are independent of λ.

By the L^p estimates and the equation (5.6) satisfied by ϕ,

$$(A.8) \quad \| \phi \|_{W^{2,p}(\Omega_\sigma)} \le c \left(\Lambda \| \phi^{-1}f \|_{L^p(\Omega_\sigma)} + \Lambda \| \phi^{-1}|\nabla\phi|^2 \|_{L^p(\Omega_\sigma)} + \| \phi \|_{L^p(\Omega_\sigma)} + \| \phi \|_{W^{2,p}(\partial\Omega_\sigma)} \right)$$

where the constant c depends on τ_0 but not on λ. With the aid of (A.4), $\sigma\lambda\tau_0^4 \le \phi \le \lambda\tau_0^4$, and $f \le \lambda^2$, this implies that

$$(A.9) \qquad \| \phi \|_{W^{2,p}(\Omega_\sigma)} \le c(\tau_0)\lambda, \qquad \text{which yields} \qquad |\phi|_{C^{1,\alpha}(\Omega_\sigma)} \le c(\tau_0)\lambda.$$

By the Schauder estimates

$$(A.10) \quad |\phi|_{C^{2,\alpha}(\Omega_\sigma)} \le c \left(\Lambda|\phi^{-1}f|_{C^{0,\alpha}(\Omega_\sigma)} + \Lambda|\phi^{-1}|\nabla\phi|^2|_{C^{0,\alpha}(\Omega_\sigma)} + |\phi|_{C^0(\Omega_\sigma)} + |\phi|_{C^{2,\alpha}(\partial\Omega_\sigma)} \right) \le c(\tau_0)\lambda.$$

Due to the absolute value on the right-hand side of (5.6), $C^{2,\alpha}$-estimates are generally the highest order estimates possible. However, since one-sided derivatives of the absolute value of a smooth function are Lipschitz, taking the one-sided derivative ∂_τ at S_{τ_0} yields an equation for $\partial_\tau\phi$ whose right-hand side is Lipschitz near S_{τ_0}. Then $C^{2,\alpha}$-estimates follow for $\partial_\tau\phi$. We have thus proven the following theorem.

Theorem A.1. *If τ_0 is sufficiently small and $\Lambda > 8$, then the solution of (5.6), (5.7) constructed in Theorem 6.9 satisfies*

$$(A.11) \qquad\qquad |\nabla^2\partial_\tau\phi_t|_{S_{\tau_0}} + |\nabla^2\phi_t|_{S_{\tau_0}} + |\nabla\phi_t|_{S_{\tau_0}} + \phi_t|_{S_{\tau_0}} \le c(\tau_0)\lambda,$$

where $c(\tau_0)$ is independent of λ.

Appendix B. The Reissner-Nordström Flow

In this section we construct the charged conformal flow in the canonical slice of the Reissner-Nordström spacetime. Consider the exterior Reissner-Nordström metric with mass m and squared total charge $q^2 = q_e^2 + q_b^2$, in non-isotropic coordinates,

$$(\text{B.1}) \qquad -\left(1 - \frac{2m}{\bar{r}} + \frac{q^2}{\bar{r}^2}\right) d\bar{t}^2 + \left(1 - \frac{2m}{\bar{r}} + \frac{q^2}{\bar{r}^2}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\sigma^2, \qquad \bar{r} \geq m + \sqrt{m^2 - q^2},$$

where $d\sigma^2$ is the round metric on the 2-sphere. The electric and magnetic fields are given by

$$(\text{B.2}) \qquad E_i = -\left(1 - \frac{2m}{\bar{r}} + \frac{q^2}{\bar{r}^2}\right)^{-1} \partial_i \left(\frac{q_e}{\bar{r}}\right), \qquad B_i = -\left(1 - \frac{2m}{\bar{r}} + \frac{q^2}{\bar{r}^2}\right)^{-1} \partial_i \left(\frac{q_b}{\bar{r}}\right).$$

Change coordinates by

$$(\text{B.3}) \qquad \bar{r} = r + m + \frac{m^2 - q^2}{4r}$$

to obtain the expression of the metric in isotropic coordinates

$$(\text{B.4}) \qquad -\left(\frac{1 - \frac{m^2 - q^2}{4r^2}}{1 + \frac{m}{r} + \frac{m^2 - q^2}{4r^2}}\right)^2 d\bar{t}^2 + \left(1 + \frac{m}{r} + \frac{m^2 - q^2}{4r^2}\right)^2 \delta, \qquad r \geq \frac{\sqrt{m^2 - q^2}}{2},$$

with corresponding electric and magnetic fields fields

$$(\text{B.5}) \qquad E_i = -\left(1 + \frac{m}{r} + \frac{m^2 - q^2}{4r^2}\right)^{-1} \partial_i \left(\frac{q_e}{r}\right), \qquad B_i = -\left(1 + \frac{m}{r} + \frac{m^2 - q^2}{4r^2}\right)^{-1} \partial_i \left(\frac{q_b}{r}\right),$$

where δ is the Euclidean metric.

We may write the Reissner-Nordström spacetime metric as $-V^2 d\bar{t}^2 + g$. In isotropic coordinates $g = U^4 \delta$, where

$$(\text{B.6}) \qquad U(x) = \sqrt{1 + \frac{m}{r} + \frac{m^2 - q^2}{4r^2}}.$$

The electric field may now be expressed as
(B.7)
$$E_i = -V^{-1} \partial_i \left(\frac{q_e}{\rho}\right) \quad \text{in non-isotropic coordinates}, \quad E_i = -U^{-2} \partial_i \left(\frac{q_e}{r}\right) \quad \text{in isotropic coordinates}.$$

Notice that this makes sense from previous formulas, since (in isotropic coordinates) we know that $E^i = U^{-6} E_\delta^i$ is divergence free whenever E_δ is divergence free with respect to δ; this is of course the case, as $E_\delta = q_e \nabla r^{-1}$. We can also check that the electric fields agree in the two different coordinates:

$$(\text{B.8}) \quad \begin{aligned} E_{\bar{r}} d\bar{r} &= \frac{1}{V} \frac{q_e}{\bar{r}^2} d\bar{r} = \frac{1}{V} \frac{q_e}{\bar{r}^2} \frac{d\bar{r}}{dr} dr = \frac{1}{V} \frac{q_e}{\bar{r}^2} \left(1 - \frac{m^2 - q^2}{4r^2}\right) dr \\ &= \frac{1}{V} \frac{q_e}{r^2 U^4} \left(1 - \frac{m^2 - q^2}{4r^2}\right) dr = -U^{-2} \partial_r \left(\frac{q_e}{r}\right) dr = E_r dr. \end{aligned}$$

Similar considerations hold for the magnetic field.

The conformal flow $g_t = u_t^4 g$ is given by rescaling coordinates by $x \mapsto e^{-2t} x$, thus

$$(\text{B.9}) \qquad u_t(x) = \frac{\sqrt{e^{-2t} + \frac{m}{r} + \frac{e^{2t}(m^2 - q^2)}{4r^2}}}{\sqrt{1 + \frac{m}{r} + \frac{m^2 - q^2}{4r^2}}}.$$

In order to calculate the flow velocity v_t, observe that

$$(B.10) \qquad v_t = \frac{d}{dt} \log u_t = \frac{-e^{-2t} + e^{2t} \frac{m^2 - q^2}{4r^2}}{e^{-2t} + \frac{m}{r} + e^{2t} \frac{m^2 - q^2}{4r^2}}.$$

As usual $v_t = 0$ on the minimal surface $\partial M_t = \{r = \frac{\sqrt{m^2 - q^2}}{2} e^{2t}\}$, and $v_t \to -1$ as $r \to \infty$. Set $U_t = u_t U$, then $g_t = U_t^4 \delta$. In order to calculate the equation satisfied by v_t, recall the identities

$$(B.11) \qquad L_{g_t} v_t = U_t^{-5} L_\delta(U_t v_t), \qquad R_{g_t} = -8 U_t^{-5} L_\delta U_t = -8 U_t^{-5} \Delta_\delta U_t.$$

It follows that

$$(B.12) \qquad \begin{aligned} \Delta_{g_t} v_t &= \frac{1}{8} R_{g_t} v_t + U_t^{-5} \Delta_\delta(U_t v_t) \\ &= -U_t^{-5} v_t \Delta_\delta U_t + U_t^{-5}(U_t \Delta_\delta v_t + 2\nabla U_t \cdot \nabla v_t + v_t \Delta_\delta U_t) \\ &= U_t^{-4}(\Delta_\delta v_t + 2\nabla \log U_t \cdot \nabla v_t). \end{aligned}$$

A computation shows that

$$(B.13) \qquad \begin{aligned} \Delta_\delta v_t &= \partial_r^2 v_t + \frac{2}{r} \partial_r v_t \\ &= \frac{\frac{m^2 - q^2}{r^4} + e^{2t} \frac{m(m^2 - q^2)}{2r^5}}{\left(e^{-2t} + \frac{m}{r} + e^{2t} \frac{m^2 - q^2}{4r^2}\right)^2} - \frac{2\left(e^{-2t} \frac{m}{r^2} + \frac{m^2 - q^2}{r^3} + e^{2t} \frac{m(m^2 - q^2)}{4r^4}\right)\left(\frac{m}{r^2} + e^{2t} \frac{m^2 - q^2}{2r^3}\right)}{\left(e^{-2t} + \frac{m}{r} + e^{2t} \frac{m^2 - q^2}{4r^2}\right)^3}, \end{aligned}$$

and

$$(B.14) \qquad \partial_r v_t = -\frac{e^{-2t} \frac{m}{r^2} + \frac{m^2 - q^2}{r^3} + e^{2t} \frac{m(m^2 - q^2)}{4r^2}}{\left(e^{-2t} + \frac{m}{r} + e^{2t} \frac{m^2 - q^2}{4r^2}\right)^2},$$

$$(B.15) \qquad \partial_r \log U_t = -\frac{\frac{m}{r^2} + e^{2t} \frac{m^2 - q^2}{2r^3}}{2\left(e^{-2t} + \frac{m}{r} + e^{2t} \frac{m^2 - q^2}{4r^2}\right)}.$$

Therefore

$$(B.16) \qquad \Delta_{g_t} v_t = \frac{q^2 r^{-4}}{\left(e^{-2t} + \frac{m}{r} + e^{2t} \frac{m^2 - q^2}{4r^2}\right)^4} v_t.$$

However since $E_t^i = U_t^{-6} E_\delta^i$ we have

$$(B.17) \qquad |E_t|_{g_t}^2 = U_t^{-8} |E_\delta|_\delta^2 = \frac{q_e^2 r^{-4}}{\left(e^{-2t} + \frac{m}{r} + e^{2t} \frac{m^2 - q^2}{4r^2}\right)^4},$$

with an identical formula for the squared norm of the magnetic field when q_e is replaced by q_b. It is also true that $R_{g_t} = 2\left(|E_t|_{g_t}^2 + |B_t|_{g_t}^2\right)$, which may be verified from the identity

$$(B.18) \qquad R_{g_t} = -8 U_t^{-5} L_\delta U_t = -8 U_t^{-5} \Delta_\delta U_t.$$

Thus, the equation satisfied by the velocity function is

$$(B.19) \qquad \Delta_{g_t} v_t - \left(|E_t|_{g_t}^2 + |B_t|_{g_t}^2\right) v_t = 0 \quad \text{or} \quad \Delta_{g_t} v_t - \frac{1}{2} R_{g_t} v_t = 0.$$

We remark that this equation for v_t is precisely the one satisfied by the warping factor of the Reissner-Nordström spacetime. In fact the Reissner-Nordström spacetime metric may be written as $-v_t^2 d\bar{t}^2 + u_t^4 g$. This structure is also valid for the Schwarzschild spacetime if u_t and v_t arise from Bray's original conformal flow. Given an arbitrary metric g, this shows how to associate a static spacetime with (M, g), namely use the conformal factor and velocity functions from the conformal flow, to generate a static spacetime as above.

References

[1] H. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom., **59** (2001), 177-267. arXiv:math/9911173

[2] H. Bray, and K. Iga, *Superharmonic functions in $\mathbb{R}^n$ and the Penrose inequality in general relativity*, Comm. Anal. Geom., **10** (2002), no. 5, 999-1016.

[3] H. Bray, and M. Khuri, *A Jang equation approach to the Penrose inequality*, Discrete Contin. Dyn. Syst., **27** (2010), no. 2, 741-766. arXiv:0910.4785

[4] H. Bray, and M. Khuri, *P.D.E.'s which imply the Penrose conjecture*. Asian J. Math., **15** (2011), no. 4, 557-610. arXiv:0905.2622

[5] P. Chruściel, H. Reall, and P. Tod, *On Israel-Wilson-Perjés black holes*, Classical Quantum Gravity, **23** (2006), 2519-2540. arXiv:gr-qc/0512116

[6] J. Corvino, *On the asymptotics for the Einstein-Maxwell constraint equations*, In preparation (2014).

[7] S. Dain, M. Khuri, G. Weinstein, and S. Yamada, *Lower bounds for the area of black holes in terms of mass, charge, and angular momentum*, Phys. Rev. D, **88** (2013), 024048. arXiv:1306.4739

[8] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

[9] M. Disconzi, and M. Khuri, *On the Penrose inequality for charged black holes*, Classical Quantum Gravity, **29** (2012), 245019. arXiv:1207.5484

[10] R. Geroch, *Energy Extraction*, Annals of the New York Academy of Sciences **224** (1973), no. 1, 108-117.

[11] G. Gibbons, *Some comments on gravitational entropy and the inverse mean curvature flow*, Classical Quantum Gravity **16** (1999), no. 6, 1677-1687. arXiv:hep-th/9809167

[12] G. Gibbons, S. Hawking, G. Horowitz, and M. Perry, *Positive mass theorem for black holes*, Commun. Math. Phys., **88** (1983), 295-308.

[13] D. Gilbarg, and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[14] S. Hawking, and G. Ellis, *The Large Structure of Space-Time*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1973.

[15] G. Huisken, and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom., **59** (2001), 353-437.

[16] P.-S. Jang, *Note on cosmic censorship*, Phys. Rev. D, **20** (1979), no. 4, 834-838.

[17] P.-S. Jang, and R. Wald, *The positive energy conjecture and the cosmic censor hypothesis*, J. Math. Phys., **18** (1977), no. 1, 41-44.

[18] M. Khuri, *A Penrose-like inequality with charge*, Gen. Relativity Gravitation, **45** (2013), 2341-2361. arXiv:1308.3591

[19] M. Khuri, and G. Weinstein, *Rigidity in the positive mass theorem with charge*, J. Math. Phys., **54** (2013), 092501. arXiv:1307.5499

[20] M. Khuri, G. Weinstein, and S. Yamada, *On the Riemannian Penrose inequality with charge and the cosmic censorhip conjecture*, Res. Inst. Math. Sci. Kokyuroku, Kyoto Univ., #1862 (2012), 63-66. arXiv:1306.0206

[21] M. Khuri, G. Weinstein, and S. Yamada, *The Riemannian Penrose inequality with charge for multiple black holes*, Complex Analysis and Dynamical Systems VI, conference proceedings to appear. arXiv:1308.3771

[22] M. Khuri, G. Weinstein, and S. Yamada, *Extensions of the charged Riemannian Penrose inequality*, Class. Quantum Grav., **32** (2015), 035019. arXiv:1410.5027

[23] A. Masood-ul-Alam, *Uniqueness proof of static charged black holes revisited*, Classical Quantum Gravity **9** (1992), no. 5, L53-L55.

[24] W. Meeks III, L. Simon, and S.-T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. (2), **116** (1982), no. 3, 621-659.

[25] P. Miao, *Positive mass theorem on manifolds admitting corners along a hypersurface*, Adv. Theor. Math. Phys., **6** (2002), 1163-1182. arXiv:math-ph/0212025

[26] R. Penrose, *Naked singularities*, Ann. New York Acad. Sci., **224** (1973), 125-134.

[27] R. Penrose, *Some unsolved problems in classical general relativity*, Seminar on Differential Geometry, Ann. Math. Study, **102** (1982), 631-668.

[28] R. Schoen, and S.-T. Yau, *On the proof of the positive mass conjecture in General Relativity*, Comm. Math. Phys., **65** (1979), no. 1, 45-76.

[29] Y. Shi, and L.-F. Tam, *Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature*, J. Differential Geometry, **62** (2002), 79-125. arXiv:math/0301047

[30] G. Weinstein, and S. Yamada, *On a Penrose inequality with charge*, Comm. Math. Phys., **257** (2005), no. 3, 703-723. arXiv:math/0405602

[31] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys., **80** (1981) 381-402.

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA
E-mail address: khuri@math.sunysb.edu

Physics Department and Department of Computer Science and Mathematics, Ariel University of Samaria, Ariel, 40700, Israel
E-mail address: gilbert.weinstein@gmail.com

Department of Mathematics, Gakushuin University, Tokyo, 171-8588, Japan
E-mail address: yamada@math.gakushuin.ac.jp