On the Kobayashi pseudometric, complex automorphisms and hyperkähler manifolds

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Abstract
We define the Kobayashi quotient of a complex variety by identifying points with vanishing Kobayashi pseudodistance between them and show that if a compact complex manifold has an automorphism whose order is infinite, then the fibers of this quotient map are nontrivial. We prove that the Kobayashi quotients associated to ergodic complex structures on a compact manifold are isomorphic. We also give a proof of Kobayashi's conjecture on the vanishing of the pseudodistance for hyperkähler manifolds having Lagrangian fibrations without multiple fibers in codimension one. For a hyperbolic automorphism of a hyperkähler manifold, we prove that its cohomology eigenvalues are determined by its Hodge numbers, compute its dynamical degree and show that its cohomological trace grows exponentially, giving estimates on the number of its periodic points.

Contents

1 Introduction

Kobayashi conjectured that a compact Kähler manifold with semipositive Ricci curvature has vanishing Kobayashi pseudometric. In a previous paper ([KLV]) Kamenova-Lu-Verbitsky have proved the conjecture for all K3 surfaces and for certain hyperkähler manifolds that are deformation equivalent to Lagrangian fibrations. Here we give an alternative proof of this conjecture for hyperkähler Lagrangian fibrations without multiple fibers in codimension one, see Section 3.

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Theorem 1.1: Let $f\colon M \longrightarrow B = \mathbb{C}\mathbb{P}^n$ be a hyperkähler Lagrangian fibration without multiple fibers in codimension one over B. Then the Kobayashi pseudometric d_M vanishes identically on M and the Royden-Kobayashi pseudonorm $|\ |_M$ vanishes identically on a Zariski open subset of M.

In Section 4, we explore compact complex manifolds M having an automorphism of infinite order. For such manifolds the Kobayashi pseudometric is everywhere degenerate. For each point $x \in M$ we define the subset $M_x \subset M$ of points in M whose pseudo-distance to x is zero. Define the relation $x \sim y$ on M given by $d_M(x, y) = 0$. There is a well defined set-theoretic quotient map $\Psi : M \longrightarrow S = M/\sim$, called **the Kobayashi quotient map**. We say that $|\ |_M$ is **Voisin-degenerate** at a point $x \in M$ if there is a sequence of holomorphic maps $\varphi_n : D_{r_n} \to M$ such that $\varphi_n(0) \to x$, $|\varphi_n'(0)|_h = 1$ and $r_n \to \infty$.

Theorem 1.2: Let M be a compact complex manifold with an automorphism f of infinite order. Then the Kobayashi pseudo-metric d_M is everywhere degenerate in the sense that $M_x \neq \{x\}$ for all $x \in M$. The Royden-Kobayashi pseudo-norm $|\ |_M$ is everywhere Voisin-degenerate. Moreover, every fiber of the map $\Psi : M \longrightarrow S$ constructed above contains a Brody curve and is connected.

In Section 5, we show that the Kobayashi quotients for ergodic complex structures are isometric, equipped with the natural quotient pseudometric. This generalizes the key technical result of [KLV] for the identical vanishing of d_M for ergodic complex structures on hyperkähler manifolds.

Theorem 1.3: Let (M, I) be a compact complex manifold, and (M, J) its deformation. Assume that the complex structures I and J are both ergodic. Then the corresponding Kobayashi quotients are isometric.

Finally in Section 6, we prove that the cohomology eigenvalues of a hyperbolic automorphism of a hyperkähler manifold are determined by its Hodge numbers. We compute its dynamical degree in the even cases and give an upper bound in the odd cases.

Theorem 1.4: Let (M, I) be a hyperkähler manifold, and T a hyperbolic automorphism acting on cohomology as γ. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Replacing γ by γ^2 if necessary, we may assume that $\alpha > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d, and finally, on odd cohomology $H^{2d+1}(M)$ the maximal eigenvalue of γ is strictly less than $\alpha^{\frac{2d+1}{2}}$.

As a corollary we obtain that the trace $\operatorname{Tr}(\gamma^N)$ grows asymptotically as α^{nN}. We also show that the number of k-periodic points grows as α^{nk}.

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2 Preliminaries

Definition 2.1: A *hyperkähler* (or *irreducible holomorphic symplectic*) manifold M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$ where σ is everywhere non-degenerate.

Recall that a fibration is a connected surjective holomorphic map. On a hyperkähler manifold the structure of a fibration, if one exists, is limited by Matsushita's theorem.

Theorem 2.2: (Matsushita, [Mat1]) Let M be a hyperkähler manifold and $f\colon M \longrightarrow B$ a fibration with $0 < \dim B < \dim M$. Then $\dim B = \frac{1}{2}\dim M$ and the general fiber of f is a Lagrangian abelian variety. The base B has at worst $\mathbb{Q}$-factorial log-terminal singularities, has Picard number $\rho(B) = 1$ and $-K_B$ is ample.

Remark 2.3: B is smooth in all of the known examples. It is conjectured that B is always smooth.

Theorem 2.4: (Hwang [Hw]) In the settings above, if B is smooth then B is isomorphic to $\mathbb{CP}^n$, where $\dim_{\mathbb{C}} M = 2n$.

Definition 2.5: Given a hyperkähler manifold M, there is a non-degenerate integral quadratic form q on $H^2(M,\mathbb{Z})$, called the *Beauville-Bogomolov-Fujiki form* (BBK form for short), of signature $(3, b_2 - 3)$ and satisfying the *Fujiki relation*

$$\int_M \alpha^{2n} = c \cdot q(\alpha)^n \qquad \text{for } \alpha \in H^2(M,\mathbb{Z}),$$

with $c > 0$ a constant depending on the topological type of M. This form generalizes the intersection pairing on K3 surfaces. For a detailed description of the form we refer the reader to [F], [Bea] and [Bo].

Remark 2.6: Given $f\colon M \longrightarrow \mathbb{CP}^n$, h the hyperplane class on $\mathbb{CP}^n$, and $\alpha = f^*h$, then α is nef and $q(\alpha) = 0$.

Conjecture 2.7: [SYZ] If L is a nontrivial nef line bundle on M with $q(L) = 0$, then L induces a Lagrangian fibration, given as above.

Remark 2.8: This conjecture is known for deformations of Hilbert schemes of points on K3 surfaces (Bayer–Macrì [BM]; Markman [Mar]), and for deformations of the generalized Kummer varieties $K_n(A)$ (Yoshioka [Y]).

Definition 2.9: The *Kobayashi pseudometric* on M is the maximal pseudometric d_M such that all holomorphic maps $f\colon (D,\rho) \longrightarrow (M, d_M)$ are distance decreasing, where (D,ρ) is the unit disk with the Poincaré metric.

Definition 2.10: A manifold M is *Kobayashi hyperbolic* if d_M is a metric, otherwise it is called *Kobayashi non-hyperbolic.*

Remark 2.11: In [Ko1], it is asked whether a compact Kähler manifold M of semipositive Ricci curvature has identically vanishing pseudometric, which we denote by $d_M \equiv 0$. The question applies to hyperkähler manifolds but was unknown even for the case of surfaces outside the projective case. But Kamenova-Lu-Verbitsky (in [KLV]) have recently resolved completely the case of surfaces with the following affirmative results.

Theorem 2.12: [KLV] Let S be a K3 surface. Then $d_S \equiv 0$.

Theorem 2.13: [KLV] Let M be a hyperkähler manifold of non-maximal Picard rank and deformation equivalent to a Lagrangian fibration. Then $d_M \equiv 0$.

Theorem 2.14: [KLV] Let M be a hyperkähler manifold with $b_2(M) \geq 7$ (expected to always hold) and with maximal Picard rank $\rho = b_2 - 2$. Assume the SYZ conjecture for deformations of M. Then $d_M \equiv 0$.

Remark 2.15: Except for the proof of Theorem 2.14, we indicate briefly a proof of these theorems below. Theorem 2.14 is proved in [KLV] using the existence of double Lagrangian fibrations on certain deformations of M. Here we give a different proof of vanishing of the Kobayashi pseudometric for certain hyperkähler Lagrangian fibrations without using double fibrations.

Definition 2.16: Let M be a compact complex manifold and $\operatorname{Diff}^0(M)$ the connected component to identity of its diffeomorphism group. Denote by Comp the space of complex structures on M, equipped with a structure of Fréchet manifold. The *Teichmüller space* of M is the quotient $\text{Teich} := \text{Comp} / \operatorname{Diff}^0(M)$. The Teichmüller space is finite-dimensional for M Calabi-Yau ([Cat]). Let $\operatorname{Diff}^+(M)$ be the group of orientable diffeomorphisms of a complex manifold M. The *mapping class group* $\Gamma := \operatorname{Diff}^+(M)/\operatorname{Diff}^0(M)$ acts on Teich. An element $I \in \text{Teich}$ is called *ergodic* if the orbit $\Gamma \cdot I$ is dense in Teich, where

$$\Gamma \cdot I = \{I' \in \text{Teich} : (M, I) \sim (M, I')\}.$$

Theorem 2.17: (Verbitsky, [V4]) If M is hyperkähler and $I \in$ Teich, then I is ergodic if and only if $\rho(M,I) < b_2 - 2$.

Remark 2.18: For a K3 surface (M,I) not satisfying the above condition on the Picard rank ρ, it is easily seen to admit Lagrangian (elliptic) fibrations over $\mathbb{CP}^1$ without multiple fibers, and it is projective. Then $d_{(M,J)} \equiv 0$ by Theorem 3.2 below, for example.

Proposition 2.19: Let (M,J) be a compact complex manifold with $d_{(M,J)} \equiv 0$. Let $I \in$ Teich be an ergodic complex structure deformation equivalent to J. Then $d_{(M,I)} \equiv 0$.

Proof: Here we shall reproduce the proof from [KLV]. Consider the diameter function $\mathsf{diam}: \text{Teich} \longrightarrow \mathbb{R}_{\geqslant 0}$, the maximal distance between two points. It is upper semi-continuous (Corollary 1.23 in [KLV]). Since the complex structure J is in the limit set of the orbit of the ergodic structure I, by upper semi-continuity $0 \leqslant \mathsf{diam}(I) \leqslant \mathsf{diam}(J) = 0$. ■

3 (Royden-)Kobayashi pseudometric on Abelian fibrations

The following lemma is a generalization of Lemma 3.8 in [BL] to the case of abelian fibrations. The generalization is given for example in the Appendix of [KLV]. Recall that an abelian fibration is a connected locally projective surjective Kähler morphism with abelian varieties as fibers.

Lemma 3.1: Let $\pi\colon T \longrightarrow C$ be an abelian fibration over a non-compact complex curve C which locally has sections and such that not all components of the fibers are multiple. Then T has an analytic section over C. This is the case if π has no multiple fibers.

Proof: There is a Neron model N for T and a short exact sequence

$$F \longrightarrow \mathcal{O}(L) \longrightarrow \mathcal{O}(N)$$

where L is a vector bundle, F is a sheaf of groups $\mathbb{Z}^{2n}$ with degenerations and $\mathcal{O}(N)$ is the sheaf of local sections of N (whose general fibers are abelian varieties). Thus T corresponds to an element θ in $H^1(C,\mathcal{O}(N))$. There is an induced exact sequence of cohomologies: $H^1(C,\mathcal{O}(L)) \longrightarrow H^1(C,\mathcal{O}(N)) \longrightarrow H^2(C,F)$. Note that $H^1(C,\mathcal{O}(L)) = 0$ since C is Stein, and $H^2(C,F) = 0$ since it is topologically one-dimensional. Thus $\theta = 0$ and hence there is an analytic section. The last part of the lemma is given by Proposition 4.1 of [KLV]. ■

Theorem 3.2: Let $f\colon M \longrightarrow B = \mathbb{CP}^n$ be a hyperkähler Lagrangian fibration without multiple fibers in codimension one over B. Then $d_M \equiv 0$ and $|\ |_M$ vanishes on a nonempty Zariski open subset of M.

Proof: The fibers of f are projective, and furthermore, there is a canonical polarization on them (see [Og1] and [Og2], respectively). This also follows from [V5], Theorem 1.10, which implies that the given fibration is diffeomorphic to another fibration $f\colon M' \longrightarrow B$ with holomorphically the same fibers and the same base, but with projective total space M'. Standard argument (via the integral lattice in the "local" Neron-Severi group) now shows that f is locally projective.

By assumption, there are no multiple fibers outside a codimension 2 subset $S \subset B$ whose complement U contains at most the smooth codimension-one part D_0 of the discriminant locus of f where multiplicity of fibers are defined locally generically. Since the pseudometric is unchanged after removing codimension 2 subsets ([Ko2]), it is enough to restrict the fibration to that over U.

Let $C = \mathbb{P}^1$ be a line in $B = \mathbb{P}^n$ contained in U (and intersecting D_0 transversely). Then f restricts to an abelian fibration $X = f^{-1}(C)$ over C without multiple fibers and so Lemma 3.1 applies to give a section.

As S is codimension two or higher, we can connect any two points in U by a chain of such C's in U. One can thus connect two general points x and y on M by a chain consisting of fibers and sections over the above C's. Since the Kobayashi pseudometric vanishes on each fiber and each such section, the triangle inequality implies $d_M(x, y) = 0$. Therefore d_M vanishes on a dense open subset of M and hence $d_M \equiv 0$ by the continuity of d_M.

The same argument gives the vanishing statement of $|\ |_M$ via Theorem A.2 of [KLV]. ■

Remark 3.3: In the theorem above, it is sufficient to assume that B is nonsingular and that $d_B \equiv 0$, true if B is rationally connected. In fact, if one assumes further the vanishing of $|\ |_B$ on a nonempty Zariski open, then the same is true for $|\ |_M$, generalizing the corresponding theorems in [KLV]. The reader should have no difficulty to see these by the obvious modifications of the above proof.

4 Automorphisms of infinite order

We first sketch the proof of Kobayashi's theorem that Kobayashi hyperbolic manifolds have only finite order automorphisms (Theorem 9.5 in [Ko1]).

Theorem 4.1: Let M be a compact complex manifold with an automorprhism f of an infinite order. Then M is Kobayashi non-hyperbolic.

Proof: Assume M is Kobayashi hyperbolic. Observe that the automorphisms of a hyperbolic manifold are isometries of the Kobayashi metric. Also the group of isometries of a compact metric space is compact with respect to the compact open topology by a theorem of Dantzig and Van der Waerden, see for example [Ko2, Theorem 5.4.1]. On the other hand, compact Kobayashi hyperbolic manifolds have no holomorphic vector fields, because each such vector field gives an orbit which is an entire curve. This means that the automorphism group of M is discrete. Since the automorphism group $\operatorname{Aut}(M)$ is discrete and compact, this means it is finite, therefore f is of finite order. ■

Consider the pseudo-distance function $d_M : M \times M \longrightarrow \mathbb{R}$, defined by the Kobayashi pseudo-distance $d_M(x, y)$ on pairs (x, y). It is a symmetric continuous function which is bounded for compact M. Since it is symmetric, we can consider d_M as a function on the symmetric product $\operatorname{Sym}^2 M$ with $d_M = 0$ on the diagonal.

Lemma 4.2: There is a compact space S with a continuous map $\Psi : M \longrightarrow S$ and there is a distance function d_S on S making S into a compact metric space such that $d_M = d_S \circ \psi$, where $\psi : \operatorname{Sym}^2 M \longrightarrow \operatorname{Sym}^2 S$ is the map induced by Ψ.

Proof: The subset $M_x \subset M$ of points $y \in M$ with $d_M(x, y) = 0$ is compact and connected. The relation $x \sim y$ on M given by $d_M(x, y) = 0$ is symmetric and transitive so that $M_x = M_y$ if and only if $x \sim y$. So there is a well defined set-theoretic quotient map $\Psi : M \longrightarrow S = M/\sim$. Note that the set S is equipped with a natural metric induced from d_M. Indeed, $d_M(x', y')$ is the same for any points $x' \in M_x, y' \in M_y$, and hence d_M induces a metric d_S on S. This metric provides a topology on S, and since the set $U_{x,\varepsilon} = \{y \in M \mid d_M(x, y) < \varepsilon\}$ is open, the map $\Psi : M \longrightarrow S$ is continuous. Thus the metric space S is also compact. This completes the proof of the lemma. ■

Remark 4.3: The natural quotient considered above was already proposed in [Ko1] albeit little seems to be known about its possible structure. In particular, it is known that even when M is compact, S may not have the structure of a complex variety. But the same problem for a projective or a compact Kähler variety is still wide open and is the subject of strong conjectures.

Remark 4.4: If there is a holomorphic family of varieties X_t over a parameter space T of any dimension, then the relative construction also works by considering the problem via that of the total space over small disks in T. In particular, there is a monodromy action on the resulting family of compact metric spaces S_t by isometries over T.

Let M be a complex manifold and h a hermitian metric on M with its associated norm $|\ |_h$.

Recall that a theorem of Royden says that the Kobayashi pseudo-metric d_M can be obtained by taking the infimum of path-integrals of the infinitesimal pseudonorm $|\ |_M$, where

$$|v|_M = \inf \left\{ \frac{1}{R} \ \mid\ f : D_R \to M \text{ holomorphic}, R > 0, f'(0) = v \right\}.$$

Here D_R is the disk of radius R centred at the origin. Recall also that $|\ |_M$ is upper-semicontinuous [Siu].

Definition 4.5: We say that $|\ |_M$ is *Voisin-degenerate* at a point $x \in M$ if there is a sequence of holomorphic maps $\varphi_n : D_{r_n} \to M$ such that

$$\varphi_n(0) \to x,\ |\varphi_n'(0)|_h = 1 \text{ and } r_n \to \infty.$$

Observe that the locus Z_M of M consisting of points where $|\ |_M$ is Voisin-degenerate is a closed set. The following theorem and the arguments given are an elaboration of the original source of Voisin [Vo].

Theorem 4.6: Consider the equivalence relationship $x \sim y$ on M given by $d_M(x,y) = 0$ where d_M is the Kobayashi pseudo-metric on M. Then any non-trivial orbit (that is, equivalence class) of this relation consists of Voisin-degenerate points, and such orbits form a closed set. If, further, M is compact, then each orbit of $\sim$ contains the image of a nontrivial holomorphic map $\mathbb{C} \to M$ (with bounded derivative), called **a Brody curve.**

Proof: Let M_x be the orbit passing through x. Suppose that $|\ |_M$ is Voisin-degenerate at x. We have sequence of maps $\varphi_n : D_{r_n} \to M$ as given above. For sufficiently small ε, Schwarz' lemma tells us that $\varphi_n(D_{2\varepsilon})$ intersects the boundary of a polydisk Δ of radius ε (w.r.t. h) centred at x for n sufficiently large since $\varphi_n'(0) = 1$ and $\varphi_n(0) \to 0$. Hence $d_M(x, \partial\Delta) = 0$ and there is a point $x_0 \in \partial\Delta$ with $d_M(x, x_0) = 0$ by the compactness of the boundary and the continuity of d_M. So M_x is not a point. Conversely, suppose that M_x is not a point. Then the upper semicontinuity of $|\ |_M$ and Royden's definition of d_M implies the existence of a sequence $x_n \in M$ converging to x and unit tangent vectors v_n (w.r.t. h) at x_n for which $|v_n|_M \to 0$. It follows that $|\ |_M$ is Voisin-degenerate at x and thus at every point of M_x. Therefore, the nonsingleton orbits form precisely the set Z_M, which is closed in M.

For the last statement, we apply Brody's lemma to the sequence of maps φ_n that gives the Voisin-degeneration for x. This gives a sequence of suitably reparametrized holomorphic maps $\psi_n : D_{r_n} \to M$ such that

$$|\psi_n'(0)|_h = 1 = \sup\nolimits_{z \in D_{r_n}} |\psi_n'(z)|_h / \big(r_n \big|\tfrac{\partial}{\partial z}\big|_{D_{r_n}}(z)\big),$$

where for each n, $\psi_n = \varphi_n \circ m_{t_n} \circ g_n$, $m_{t_n}(z) = t_n z$, $t_n \in (0,1]$ and g_n is an automorphism of D_{r_n}. Note that for each compact in $\mathbb{C}$ the denominator in the supremum above converges to 1 and thus (ψ_n) form an equicontinuous family there. Applying the compactness of M then give us a Brody curve C in M by extracting a convergent subsequence of (ψ_n). Taking a further subsequence if necessary, we may assume that (t_n) converges, necessarily to a point t_0 in $[0,1]$. If $t_0 = 0$, then M_x contains the Brody curve C as $y_n = g_n(0) \in D_{r_n}$ and $d_M(x, C)$ is dominated by the limit of

$$d_M(\varphi_n(0), \psi_n(0)) = d_M(\varphi_n(0), \varphi_n(t_n y_n))) \leqslant d_{D_{r_n}}(0, t_n y_n) = d_{D_1}(0, t_n \frac{y_n}{r_n}).$$

If $t_0 \neq 0$, then the family $(\varphi_n \circ m_{t_n})$ is equicontinuous and so is convergent to a holomorphic $\varphi : \mathbb{C} \to M$ upon replacing by a subsequence. Then its image is a Brody curve in M_x since $|\varphi'(0)|_h$ is the limit of $|(\varphi_n \circ m_{t_n})'(0)|_h = t_n$ and it contains $x = \varphi(0)$ by construction. ■

Theorem 4.7: Assume M is compact. Then each orbit of the the equivalence relation given above is connected.

Proof: Let M_x be the orbit passing through x as before and

$$M_x(n) = \left\{ y \in X \,\middle|\, d_X(x, y) \leqslant \frac{1}{n} \right\}.$$

Then each $M_x(n)$ is compact and connected and $M_x = \cap_n M_x(n)$. If M_x is not connected, then there are disjoint open sets U, V in M separating M_x leading to the contradiction

$$\emptyset = (U \cup V)^c \cap M_x = \cap_n [(U \cup V)^c \cap M_x(n)] \neq \emptyset,$$

each $(U \cup V)^c \cap M_x(n)$ being nonempty compact as $M_x(n)$ is connected. ■

Remark 4.8: Without M being compact, our arguments only shows that each orbit M_x is locally connected.

5 Metric geometry of Kobayashi quotients

Definition 5.1: Let M be a complex manifold, and d_M its Kobayashi pseudometric. Define **the Kobayashi quotient** M_K of M as the space of all equivalence classes $\{x \sim y \;\mid\; d_M(x, y) = 0\}$ equipped with the metric induced from d_M.

The main result of this section is the following theorem.

Theorem 5.2: Let (M, I) be a compact complex manifold, and (M, J) its deformation. Assume that the complex structures I and J are both ergodic. Then the corresponding Kobayashi quotients are isometric.

Proof: Consider the limit $\lim \nu_i(I) = J$, where ν_i is a sequence of diffeomorphisms of M. For each point $x \in (M, I)$, choose a limiting point $\nu(x) \in (M, J)$ of the sequence $\nu_i(x)$. By the upper-semicontinuity of the Kobayashi pseudometric, we have

$$d_{(M,J)}(\nu(x), \nu(y)) \geqslant d_{(M,I)}(x, y). \tag{5.1}$$

Let C be the union of all $\nu(x)$ for all $x \in (M, I)$. Define a map $\psi :\ C \longrightarrow (M, I)$ mapping $z = \nu(x)$ to x (if there are several choices of such x, choose one in arbitrary way). By (5.1), the map ψ is 1-Lipschitz with respect to the Kobayashi pseudometric. For any $x \in (M, J)$, the Kobayashi distance between x and

$\psi(\nu(x))$ is equal zero, also by (5.1). Therefore, ψ defines a surjective map on Kobayashi quotients: $\Psi : C_K \longrightarrow (M, I)_K$. Exchanging I and J, we obtain a 1-Lipshitz surjective map $\Phi : C'_K \longrightarrow (M, J)_K$, where C'_K is a subset of $(M, I)_K$. Taking a composition of Ψ and Φ, we obtain a 1-Lipschitz, surjective map from a subset of $(M, I)_K$ to $(M, I)_K$. The following proposition shows that such a map is always an isometry, finishing the proof of Theorem 5.2. ■

Proposition 5.3: Let M be a compact metric space, $C \subset M$ a subset, and $f : C \longrightarrow M$ a surjective 1-Lipschitz map. Then $C = M$ and f is an isometry.

Proposition 5.3 is implied by the following three lemmas, some which are exercises found in [BBI].

Lemma 5.4: Let M be a compact metric space, $C \subset M$ a subset, and $f : C \longrightarrow M$ a surjective 1-Lipschitz map. Then M is the closure of C.

Proof: Suppose that M is not the closure $\overline{C}$ of C. Take $q \in M \backslash \overline{C}$, and let $\varepsilon = d(q, \overline{C})$. Define p_i inductively, $p_0 = q$, $f(p_{i+1}) = p_i$. Let $p \in \overline{C}$ be any limit point of the sequence $\{p_i\}$, with $\lim_i p_{n_i} = p$. Since $f^m(p_n) \in C$ for any $m < n$, one has $f^m(p) \in \overline{C}$.

Clearly, $f^{n_i}(p_{n_i}) = q$. Take n_i such that $d(p, p_{n_i}) < \varepsilon$. Then $d(f^{n_i}(p), q) < \varepsilon$. This is a contradiction, because $f^n(p) \in \overline{C}$ and $\varepsilon = d(q, \overline{C})$. ■

Lemma 5.5: Let M be a compact metric space, and $f : M \longrightarrow M$ an isometric embedding. Then f is bijective.

Proof: Follows from Lemma 5.4 directly. ■

Lemma 5.6: Let M be a compact metric space, and $f : M \longrightarrow M$ a 1-Lipschitz, surjective map. Then f is an isometry.

Proof: Let d be the diameter of M, and let K be the space of all 1-Lipschitz functions $\mu : M \longrightarrow [0, d]$ with the sup-metric. By the Arzela-Ascoli theorem, K is compact. Now, f^* defines an isometry from K to itself, $\mu \longrightarrow \mu \circ f$. For any $z \in M$, the function $d_z(x) = d(x, z)$ belongs to K. However, $d_{f(z)}$ does not belong to the image of f^* unless $d(z, x) = d(f(z), f(x))$ for all x, because if $d(z, x) < d(f(z), f(x))$, one has $(f^*)^{-1}(d_{f(z)})(f(x)) = d(z, x) > d(f(z), f(x))$, hence $(f^*)^{-1}(d_{f(z)})$ cannot be Lipschitz. This is impossible by Lemma 5.5, because an isometry from K to itself must be bijective. Therefore, the map $f : M \longrightarrow M$ is an isometry. ■

The proof of Proposition 5.3 easily follows from Lemma 5.6 and Lemma 5.4. Indeed, by Lemma 5.4, f is a surjective, 1-Lipschitz map from M to itself, and by Lemma 5.6 it is an isometry. ■

6 Eigenvalues and periodic points of hyperbolic automorphisms

The following proposition follows from a simple linear-algebraic observation.

Proposition 6.1: Let T be a holomorphic automorphism of a hyperkähler manifold (M, I), and $\gamma : \ H^2(M) \longrightarrow H^2(M)$ the corresponding isometry of $H^2(M)$. Then γ has at most 1 eigenvalue α with $|\alpha| > 1$, and such α is real.

Proof: Since T is holomorphic, γ preserves the Hodge decomposition

$$H^2(M, \mathbb{R}) = H^{(2,0)+(0,2)}(M, \mathbb{R}) \oplus H^{1,1}(M, \mathbb{R}).$$

Since the BBF form is positive definite on $H^{(2,0)+(0,2)}(M, \mathbb{R})$, the eigenvalues of γ are $|\alpha_i| = 1$ on this space. On $H^{1,1}(M, \mathbb{R})$, the BBF form has signature $(+, -, -, ..., -)$, hence γ can be considered as an element of $O(1, n)$. However, it is well known that any element of $SO(1, n)$ has at most 1 eigenvalue α with $|\alpha| > 1$, and such α is real. ■

Definition 6.2: An automorphism of a hyperkähler manifold (M, I) or an automorphism of its cohomology algebra preserving the Hodge type is called **hyperbolic** if it acts with an eigenvalue α, $|\alpha| > 1$ on $H^2(M, \mathbb{R})$.

In holomorphic dynamics, there are many uses for the **d-th dynamical degree of an automorphism**, which is defined as follows. Given an automorphism T of a manifold M, we consider the corresponding action on $H^d(M, \mathbb{R})$, and d-th dynamical degree is logarithm of the maximal absolute value of its eigenvalues. In [Og3], K. Oguiso has shown that the dynamical degree of a hyperbolic automorphism is positive for all even d, and computed it explicitly for automorphisms of Hilbert schemes of K3 which come from automorphisms of K3.

We compute the dynamical degree for all even d and give an upper bound for odd ones. We also compute asymptotical growth of the trace of the action of T^N in cohomology, which could allow one to prove that the number of quasi-periodic points grows polynomially as the period grows. One needs to be careful here, because there could be periodic and fixed subvarieties, and their contribution to the Lefschetz fixed point formula should be calculated separately.

Theorem 6.3: Let (M, I) be a hyperkähler manifold, and T a hyperbolic automorphism acting on cohomology as γ. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Replacing γ by γ^2 if necessary, we may assume that $\alpha > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d, and finally, on odd cohomology $H^{2d+1}(M)$ the maximal eigenvalue of γ is strictly less than $\alpha^{\frac{2d+1}{2}}$.

Remark 6.4: From Theorem 6.3, it follows immediately that $\mathrm{Tr}(\gamma^N)$ grows asymptotically as α^{nN}.

We prove Theorem 6.3 at the end of this section.

Recall that the Hodge decomposition defines multiplicative action of $U(1)$ on cohomology $H^*(M)$, with $t \in U(1) \subset \mathbb{C}$ acting on $H^{p,q}(M)$ as t^{p-q}. In [V1], the group generated by $U(1)$ for all complex structures on a hyperkähler manifold was computed explicitly, and it was found that it is isomorphic $G = \mathrm{Spin}^+(H^2(M, \mathbb{R}), q)$ (with center acting trivially on even-dimensional forms and as -1 on odd-dimensional forms; see [V2]). Here Spin^+ denotes the connected component.

In [V3], it was shown that the connected component of the group of automorphisms of $H^*(M)$ is mapped to G surjectively and with compact kernel ([V3, Theorem 3.5]). Therefore, to study the eigenvalues of automorphisms of $H^*(M)$, we may always assume that they belong to G.

Now, the eigenvalues of $g \in G$ on its irreducible representations can always be computed using the Weyl character formula. The computation is time-consuming, and instead of using Weyl character formula, we use the following simple observation.

Claim 6.5: Let G be a group, and V its representation. Then the eigenvalues of g and xgx^{-1} are equal for all $x, g \in G$. ■

To prove Theorem 6.3, we replace one-parametric group containing the hyperbolic automorphism by another one-parametric group adjoint to it in G, and describe this second one-parametric group in terms of the Hodge decomposition.

Proposition 6.6: Let (M, I) be a hyperkähler manifold, and f an automorphism of M. Assume that f acts on $H^2(M)$ with an eigenvalue $\alpha > 0$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the maximal of these eigenvalues on even cohomology $H^{2d}(M)$ is equal to α^d (with eigenspace of dimension 1), and on odd cohomology $H^{2d+1}(M)$ it is strictly less than $\alpha^{\frac{2d+1}{2}}$.

Proof: Write the polar decomposition $\gamma = \gamma_1 \circ \beta$, where $\gamma_1 \in G$ has eigenvalues $\alpha, \alpha^{-1}, 1, 1, ..., 1$, β belongs to the maximal compact subgroup, and they commute. Clearly, the eigenvalues of β on V are of absolute value 1, and absolute values of eigenvalues of γ and γ_1 are equal. Therefore, we can without restricting generality assume that $\gamma = \gamma_1$ has eigenvalues $\alpha, \alpha^{-1}, 1, 1, ..., 1$.

Consider now the following one-parametric subgroup of the complexification $G_{\mathbb{C}} \subset \mathrm{Aut}(H^*(M, \mathbb{C}))$: $\rho(t)$ acts on $H^{p,q}$ as t^{p-q}, $t \in \mathbb{R}$. The corresponding element of the Lie algebra has only two non-zero real eigenvalues in adjoint action. Clearly, all one-parametric subgroups of $G_{\mathbb{C}} = \mathrm{Spin}(H^2(M, \mathbb{C}))$ with this property are conjugate. This implies that γ is conjugate to an element

$\rho(\alpha)$.

By Claim 6.5, γ and $\rho(\alpha)$ have the same eigenvalues, and $\rho(\alpha)$ clearly has eigenvalues $\alpha^{\frac{d-i}{2}}, \alpha^{\frac{d-i-1}{2}}, ...\alpha^{\frac{i-d}{2}}$ on V. ■

Corollary 6.7:

$$\lim_{n \longrightarrow \infty} \frac{\log \operatorname{Tr}(f^n)\Big|_{H^*(M)}}{n} = \log \alpha.$$

In particular, the number of k-periodic points grows as α^{nk}. ■

The same argument as in Proposition 6.6 also proves the following theorem.

Theorem 6.8: Let M be a hyperkähler manifold, and $\gamma \in \operatorname{Aut}(H^*(M))$ an automorphism preserving the Hodge decomposition and acting on $H^{1,1}(M)$ hyperbolically. Denote by α the eigenvalue of γ on $H^2(M, \mathbb{R})$ with $|\alpha| > 1$. Replacing γ by γ^2 if necessary, we may assume that $\alpha > 1$. Then all eigenvalues of γ have absolute value which is a power of $\alpha^{1/2}$. Moreover, the eigenspace of eigenvalue $\alpha^{k/2}$ on $H^d(M)$ is isomorphic to $H^{\frac{(d+k)}{2}, \frac{(d-k)}{2}}(M)$. ■

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