Similarity. Draft

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1. Ratio of commeasurable segments

If a segment AB can be obtained by summing up n copies of a segment CD, then we say that $\frac{CD}{AB} = n$ and $\frac{AB}{CD} = \frac{1}{n}$.

If for segments AB and CD there exists a segment EF and natural numbers p and q such that $\frac{AB}{EF} = p$ and $\frac{CD}{EF} = q$, then AB and CD are said to be *commeasurable*, $\frac{AB}{CD}$ is defined as $\frac{p}{q}$ and the segment EF is called a *common measure* of AB and CD. The ratio $\frac{AB}{CD}$ does not depend on the common measure EF. This follows from the following two statements.

For any two commeasurable segments there exists the greatest common measure. It can be found by the *Euclidean algorithm*, see Sections 145, 146 of the textbook.

If EF is the greatest common measure of segments AB and CD and GH is a common measure of AB and CD, then there exists a natural number n such that $\frac{EF}{GH} = n$.

If a segment AB is longer than a segment CD and these segments are commeasurable with a segment EF, then $\frac{AB}{EF} > \frac{CD}{EF}$.

2. Incommeasurable segments

There exist segments that are not commeasurable. For example, a side and diagonal of a square are not commeasurable, see Section 148 of the textbook. Segments that are not commeasurable are called *incommeasurable*.

For incommeasurable segments AB and CD the ratio $\frac{AB}{CD}$ is defined as the unique real number r such that

- $r < \frac{EF}{CD}$ for any segment EF, which is longer than AB and commeasurable with CD;

- $\dfrac{EF}{CD} < r$ for any segment EF, which is shorter than AB and commeasurable with CD.

3. Lemma

Lemma A. *Let ABC be a triangle, D be a point on AB and E be a point on BC. If $DE \parallel AC$, then*

$$\frac{BD}{BA} = \frac{BE}{BC} = \frac{DE}{AC}$$

Proof. The proof is contained in the proof of statement (2) of Lemma 159 in the textbook. □

4. Distances and metric spaces

If we choose a segment AB and call it the unit, then we can assign to any other segment CD the number $\dfrac{CD}{AB}$, call it the *length* of CD and denote it by $|CD|$.

Further, the length $|CD|$ of segment CD is called then the *distance* between points C and D and denote by $\operatorname{dist}(C,D)$. Of course, $\operatorname{dist}(C,D)$ depends on the choice of AB. Define $|CD|$ and $\operatorname{dist}(C,D)$ to be 0 if $C = D$.

The distance between points has the following properties:

- it is symmetric, $\operatorname{dist}(C,D) = \operatorname{dist}(D,C)$ for any points C, D;
- $\operatorname{dist}(C,D) = 0$ if and only if $C = D$;
- triangle inequality, $\operatorname{dist}(C,D) \le \operatorname{dist}(C,E) + \operatorname{dist}(E,D)$.

The first two of these properties are obvious, the last one was proven, see Section 48 of the textbook.

In mathematics there many functions which have these 3 properties. Therefore it was productive to create the following notion of metric space. A *metric space* is an arbitrary set X equipped with a function $d : X \times X \to \mathbb{R}_+$ such that

- $d(a,b) = d(b,a)$ for any $a, b \in X$;
- $d(a,b) = 0$ if and only if $a = b$;
- $d(a,b) \le d(a,c) + d(c,b)$ for any $a, b, c \in X$.

Thus the plane with a selected unit segment and $d(C,D) = d(C,D)$ is a metric space.

Definition of similarity transformations

Let X and Y be metric spaces with distances d_X and d_Y, respectively. A map $T : X \to Y$ is said to be a *similarity transformation* with *ratio* $k \in \mathbb{R}$, $k \geq 0$, if $d_Y(T(a), T(b)) = kd_X(a, b)$ for any $a, b \in X$.

Other terms used in the same situation: a similarity transformation may call a *dilation*, or *dilatation*, the ratio may call also the *coefficient*.

Most often the notion of similarity transformation is applied when $X = Y$ and $d_X = d_Y$. We will consider it when $X = Y$ is the plane or the 3-space and the distance is defined via the length of the corresponding segment, and the length is defined by a choice of unit segment, as above.

General properties of similarity transformations.
1. Any isometry is a similarity transformation with ratio 1.
2. Composition $S \circ T$ of similarity transformations T and S with ratios k and l, respectively, is a similarity transformation with ratio kl.

Homothety

A profound example of dilation with ratio different from 1 is a homothety.

Definition. Let k be a positive real number, O be a point on the plane. The map which maps O to itself and any point $A \neq O$ to a point B such that $\dfrac{OB}{OA} = k$ and the rays OA and OB coincide is called the *homothety* centered at O with ratio k.

Composition $T \circ S$ of homotheties T and S with the same center and ratios k and l, respectively, is the homothety with the same center and the ratio kl. In particular, any homothety is invertible and the inverse transformation is the homothety with the same center and the inverse ratio.

Theorem B. *A homothety T with ratio k is a similarity transformation with ratio k.*

Proof. We need to prove that $\frac{T(A)T(B)}{AB} = k$ for any segment AB. Consider, first, the case when O does not belong to the line AB. Then OAB is a triangle, and $OT(A)T(B)$ is also a triangle.

Assume that $k < 1$. Then $T(A)$ belongs to OA. Draw a segment $T(A)C$ parallel to AB with the end point C belonging to OB. Then by Lemma A, $\frac{OC}{OB} = \frac{OT(A)}{OA} = k$. Therefore $C = T(B)$. Again, by Lemma A, $\frac{T(A)T(B)}{AB} = \frac{OT(A)}{OA} = k$.

If $k > 1$, then A belongs to $OT(A)$. Draw the segment AC parallel to $T(A)T(B)$ and having the end point C on the segment $OT(B)$. By Lemma

A, $\frac{OT(B)}{OC} = \frac{OT(A)}{OA} = k$. Therefore $C = T(B)$. Again, by Lemma A, $\frac{T(A)T(B)}{AB} = \frac{OT(A)}{OA} = k$.

The easy case, when points A, B, O are collinear, consider as an exercise. □

Corollary C. *Any similarity transformation T with ratio k of the plane is a composition of an isometry and a homothety with ratio k.*

Proof. Consider a composition $T \circ H$ of T with a homothety with ratio k^{-1}. This composition is a similarity transformation with ratio $k^{-1}k = 1$, that is an isometry. Denote this isometry by I. Thus $I = T \circ H$. Multiply both sides of this equality by H^{-1} from the right hand side: $I \circ H^{-1} = T \circ H \circ H^{-1} = T$. □

Theorem D. *A similarity transformation of a plane is invertible.*

Proof. By C, any similarity transformation T is a composition of an isometry and a homothety. A homothety is invertible, as was noticed above. An isometry of the plane is a composition of at most three reflections. Each reflection is invertible, because its composition with itself is the identity. A composition of invertible maps is invertible. □

Corollary E. *The transformation inverse to a similarity transformation T with ratio k is a similarity transformation with ratio k^{-1}.*

5. Similar figures

Plane figures F_1 and F_2 are said to be *similar* if there exists a similarity transformation T such that $T(F_1) = F_2$.

Any two congruent figures are similar. In particular, any two lines are congruent and hence similar, any two rays are congruent and hence similar.

Segments are not necessarily congruent, but nonetheless any two segments are similar. Indeed, first, by a congruence transformation one can make any segment parallel to another segment, and then find a homothety mapping one of the segments to the other one.

Any two circles (or any two disks) are similar. Indeed, if the circles have the same radius, then one can find a translation mapping one them onto the other one, otherwise one can find a homothety mapping one of them onto the other one.

Theorem F. *A figure similar to a segment is a segment.*

Lemma G (Characterization of points belonging to segment). *A point C belongs to a segment AB if and only if $|AC| + |CB| = |AB|$.*

Lemma was proven in Section 48 of the textbook. □

PROOF OF THEOREM F.. The relation $|AC|+|CB| = |AB|$ characterizing the set of points of segment AB is invariant under a similarity transformation. Indeed, if T is a similarity transformation with ratio k, then $|T(A)T(C)| = k|AC|$, $|T(C)T(B)| = k|CB|$, and $|T(A)T(B)| = k|AB|$. Therefore, if C belongs to AB, then $|AC| + |CB| = |AB|$,

$$\begin{aligned}|T(A)T(C)| + |T(C)T(B)| &= k|AC| + k|CB| \\ &= k(|AC| + |CB|) = k|AB| = |T(A)T(B)|\end{aligned}$$

and hence $T(C)$ belongs to the segment $T(A)T(B)$. Thus the image of segment $[AB]$ under T is contained in the segment $[T(A)T(B)]$:

$$T([AB]) \subset [T(A)T(B)].$$

Similarly, $T{-}1[T(A)T(B)] \subset [AB]$. Therefore,

$$[T(A)T(B)] = TT^{-1}[T(A)T(B)] \subset T[AB].$$

Hence $T([AB]) = [T(A)T(B)]$. □

Exercises.
1. Prove that a figure similar to a line is a line.
2. Prove that a figure similar to a ray is a ray.
3. Prove that a figure similar to a circle is a circle.

Theorem H. *A figure similar to an angle is an angle. Two angles are similar if and only they are congruent.*

Proof. Recall that an angle is a figure consisting of two rays starting from the same point. Since a figure similar to a ray is a ray and a similarity transformation of an angle onto another figure should map the common point of the rays to a common point of their images, the image of an angle under a similarity transformation is an angle.

Consider now two similar angles and prove that they are congruent. Let T be a similarity mapping an angle $\angle A$ to an angle $\angle B$. Let the ratio of T be k. Let H be the homothety centered at the vertex of B with ratio k^{-1}. The image of $\angle B$ under H is $\angle B$. Therefore the composition $H \circ T$ maps $\angle A$ onto $\angle B$. This composition is a similarity mapping with ratio $k \times k^{-1} = 1$. Hence, $H \circ T$ is an isometry mapping $\angle A$ onto $\angle B$. □

6. Similarity tests for triangles

Theorem I (AA-test). *If in triangles ABC and $A'B'C'$ the angles $\angle A$, $\angle A'$ are congruent and angles $\angle B$, $\angle B'$ are congruent, then $\triangle ABC$ is similar to $\triangle A'B'C'$.*

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By ASA test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. By Lemma A, $\frac{DB}{AB} = \frac{BE}{BC}$. Hence, the homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □

Theorem J (SAS-test). *If in triangles ABC and $A'B'C'$ the angles $\angle A$, $\angle A'$ are congruent and*

$$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$

then $\triangle ABC$ is similar to $\triangle A'B'C'$.

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By Lemma A, $\frac{DB}{AB} = \frac{BE}{BC}$, and therefore $|BE| = |B'C'|$. By SAS test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. The homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □

Theorem K (SSS-test). *If in triangles ABC and $A'B'C'$*

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA}$$

then $\triangle ABC$ is similar to $\triangle A'B'C'$.

Proof. Without loss of generality we may assume that $A'B'$ is shorter than AB. Find a point D on AB such that $|BD| = |B'A'|$. Draw a segment DE parallel to AC. By Lemma A, $\frac{DB}{AB} = \frac{BE}{BC} = \frac{DE}{AC}$. Therefore $|BE| = |B'C'|$ and $|DE| = |A'C'|$. By SSS test for congruence of triangles, $\triangle A'B'C'$ is congruent to $\triangle DBE$. The homothety centered at B with ratio $\frac{DB}{AB}$ maps $\triangle ABC$ onto $\triangle DBE$. □