MAT536 HW7

Due 11/5 in class. Each problem is worth 10 points

Problem 1. Let $m_0, \ldots, m_N \in k[x_0, \ldots, x_n]$ be all the monomials of degree d. The *Veronese embedding* is the map $v_d : \mathbb{P}^n \to \mathbb{P}^N$ defined by:

$$v_d(x_0 : \cdots : x_n) = (m_0(x_0 : \cdots : x_n) : \cdots : m_N(x_0 : \cdots : x_n)).$$

(a) Show that v_d is an isomorphism of $\mathbb{P}^n$ with a closed subvariety in $\mathbb{P}^N$.

(b) Let $S \subset \mathbb{P}^n$ be a hypersurface of degree d, i.e. $S = V_p(f)$, where $f \in k[x_0, \ldots, x_n]$ is a form of degree d. Show that $S = v_d^{-1}(H)$ for a unique hyperplane $H \subset \mathbb{P}^N$.

Problem 2. Let L_1, L_2 and L_3 be lines in $\mathbb{P}^3$ such that none of them meet.

(a) Prove that there exists a unique quadric surface $S \subset \mathbb{P}^3$ containing L_1, L_2 and L_3.

(b) Show that S is the disjoint union of all lines $L \subset \mathbb{P}^3$ meeting L_1, L_2 and L_3.

(c) Let $L_4 \subset \mathbb{P}^3$ be a fourth line which does not meet L_1, L_2 or L_3. Then the number of lines meeting L_1, L_2, L_3 and L_4 is equal to the number of points in $L_4 \cap S$ which is one, two or infinitely many.

Problem 3. An *algebraic group* is a pre-variety G together with morphisms $m : G \times G \to G$ and $i : G \to G$ and an identity element $e \in G$ such that (G, m) is a multiplicative group and i maps every element to its inverse.

(a) Show that $GL_n(k)$ is an algebraic group.

(b) Show that any algebraic group is separated.

(c) Show that $\mathbb{P}^1$ is not an algebraic group.

(d) Extra credit challenge: is $\mathbb{P}^n$ an algebraic group (for $n \geq 2$) ?

Problem 4. Let G be an irreducible algebraic group acting on a variety X.

(a) Show that each orbit in X is locally closed.

(b) Each orbit is a non-singular variety.

Problem 5. Let $GL_n(k)$ act on $Gr(d, n)$ by $g \cdot V = \{g(x) | x \in V\}$. Show that for any two points $V_1, V_2 \in Gr(d, n)$ there exists an element $g \in GL_n(k)$ such that both $g \cdot V_1$ and $g \cdot V_2$ are in $U_{\{1,\ldots,d\}} \subset Gr(d, n)$. Conclude that $Gr(d, n)$ is separated.