MAT536 HW5

Due 10/15 in class. Each problem is worth 10 points

Problem 1. The cummutative algebra result *lying over* states that if $R \subset S$ is an integral extension of commutative rings and $P \subset R$ is a prime ideal, then there is some prime $Q \subset S$ such that $Q \cap R = P$.

(a) Use lying over to show that if $\phi : X \to Y$ is a dominant morphism of irreducible varieties, then $\phi(X)$ contains a dense open subset of Y.

(b) If $\phi : X \to Y$ is any morphism of varieties, then its image $\phi(X)$ is constructible.

Problem 2. Assume that $\text{char}(k) \neq 2$. Find the singular points of the surfaces $X = V(xy^2 - z^2)$, $Y = V(x^2 + y^2 - z^2)$ and $Z = V(xy + x^3 + y^3)$ in $\mathbb{A}^3$.

Problem 3. Assume that $\text{char}(k) = 0$. Let $X = V_p(f) \subset \mathbb{P}^n$ be a hypersurface given by a square-free homogeneous polynomial $f \in k[x_0, \dots, x_n]$.

(a) Show that $X_{sing} = V_p(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$.

(b) Show that $X_{sing} \neq X$.

Problem 4. (a) Show that an intersection of r hypersurfaces in $\mathbb{P}^r$ is never empty.

(b) Let $X \subset \mathbb{P}^n$ be a hypersurface of degree at least two, such that X contains a linear subspace $L \subset \mathbb{P}^n$ of dimension $r \geq \frac{n}{2}$. Prove that X is singular.

Problem 5. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree three. If X has two different singular points, then X contains the line joining them.

Problem 6. If X is a variety and $x \in X$, we define the Zariski cotangent space to X at x to be m_x/m_x^2. The Zariski tangent space is the dual vector space $(m_x/m_x^2)^\star$. Show that if $f : X \to Y$ is a morphism of varieties with $f(x) = y$, then f induces linear maps $m_y/m_y^2 \to m_x/m_x^2$ and $(m_x/m_x^2)^\star \to (m_y/m_y^2)^\star$.