MAT536 HW4

Due 10/8 in class. Each problem is worth 10 points

Problem 1. (a) Any subspace of a separated space with functions is separated.
(b) A product of separated spaces with functions is separated.

Problem 2. Let X be a pre-variety such that for each pair of points $x, y \in X$ there is an open affine subvariety $U \subset X$ containing both x and y.
(a) Show that X is separated.
(b) Show that $\mathbb{P}^n$ has this property.

Problem 3. Let X be any variety and $f \in k[X]$ a regular function.
(a) If h is a regular function on $D(f) \subset X$, then $f^n h$ can be extended to a regular function on all of X for some $n > 0$.
(b) $k[D(f)] = k[X]_f$.
(c) Suppose $f_1, \dots, f_r \in k[X]$ satisfy $(f_1, \dots, f_r) = k[X]$ and $D(f_i)$ is affine for each i. Then X is affine.

Problem 4. Let E be the elliptic curve $X_P(y^2z - x^3 + xz^2) \subset \mathbb{P}^2$ and let $f, g : E \dashrightarrow \mathbb{P}^1$ be the rational maps defined by $f(x : y : z) = (x : z)$ and $g(x : y : z) = (y : z)$.
(a) Find the maximal open sets in E where f and g are defined as morphisms.
(b) Find the degrees of the field extensions $k(t) \subset k(E)$ induced by f and g.
(c) Find the cardinality of $f^{-1}(p)$ and $g^{-1}(p)$ when $p \in \mathbb{P}^1$ is a typical point (part of the exercise is to define what "typical" means).

Problem 5. Let X be a projective variety and $\phi : \mathbb{P}^1 \dashrightarrow X$ any rational map. Show that ϕ is defined as a morphism on all of $\mathbb{P}^1$.

Problem 6. Let X and Y be varieties.
(a) if X has components $X_1, \dots, X_m$, then $\dim(X) = \max \dim(X_i)$.
(b) $\dim(X \times Y) = \dim(X) + \dim(Y)$.