BOUNDEDNESS IN FAMILIES WITH APPLICATIONS TO ARITHMETIC HYPERBOLICITY

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ABSTRACT. Motivated by conjectures of Demailly, Green–Griffiths, Lang, and Vojta, we show that several notions related to hyperbolicity behave similarly in families. We apply our results to show the persistence of arithmetic hyperbolicity along field extensions for projective normal surfaces with nonzero irregularity. These results rely on the mild boundedness of semi-abelian varieties. We also introduce and study the notion of pseudo-algebraic hyperbolicity which extends Demailly's notion of algebraic hyperbolicity for projective schemes.

CONTENTS

1. INTRODUCTION

The aim of this paper is to provide evidence for the following conjecture due in part to Demailly, Green–Griffiths, Lang and Vojta. The following conjecture is a consequence of conjectures appearing in [1, §0.3], [12, Conj. XV.4.3], [15], [25], [32, §6], [44], and [65, Conj. 4.3].

Conjecture 1.1 (Demailly, Green–Griffiths, Lang, Vojta)**.** *Let X be a projective variety over a field k of characteristic zero. Then the following statements are equivalent.*

(i) The projective variety X is algebraically hyperbolic over k.
(ii) The projective variety X is bounded over k.
(iii) For all $n \geq 1$ and $m \geq 1$, the projective variety X is (n,m)-bounded over k.
(iv) Every integral closed subvariety of X is of general type.
(v) The projective variety X is groupless over k.

Here a projective variety X over an algebraically closed field k is algebraically hyperbolic over k if there is an ample line bundle $\mathcal{L}$ on X and a real number $\alpha_{X,\mathcal{L}}$ such that, for every

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smooth projective connected curve C over k of genus g and every morphism $f : C \to X$, the inequality

$$\deg_C f^*\mathcal{L} \leq \alpha_{X,L} \cdot (2g-2)$$

holds.

For $n \geq 1$ and $m \geq 0$ integers, we follow the terminology introduced in [34, §4] and refer to a projective variety as (n,m)-bounded over the algebraically closed field k if, for every projective normal integral scheme Y over k of dimension at most n, all pairwise distinct points $y_1, \ldots, y_m \in Y(k)$, and all $x_1, \ldots, x_m \in X(k)$, the scheme

$$\underline{\mathrm{Hom}}_k([Y, y_1, \ldots, y_m], [X, x_1, \ldots, x_m])$$

parametrizing morphisms $f : Y \to X$ with $f(y_1) = x_1, \ldots, f(y_m) = x_m$ is of finite type over k. We say that X is n-bounded if it is $(n,0)$-bounded and we say that X is bounded if it is n-bounded for every integer n. We refer the reader to [34] for a discussion of the relations between algebraic hyperbolicity, boundedness, and (n,m)-boundedness.

The above notions are defined for projective varieties over algebraically closed fields. More generally, if X is a projective variety over a field k, we say that X is algebraically hyperbolic over k if $X_{\overline{k}}$ is algebraically hyperbolic over $\overline{k}$, where $k \to \overline{k}$ is some algebraic closure. We define the notions of boundedness and (n,m)-boundedness over k in a similar manner.

We follow standard terminology and say that an integral proper scheme X over k is of general type if it has a desingularisation $X' \to X$ such that $\omega_{X'/k}$ is a big line bundle. Also, we will say that a proper scheme X over a field k is of general type if, for every irreducible component Y of X, the reduced closed subscheme Y_{red} is of general type. Finally, a proper variety X over k is groupless if, for every abelian variety A over $\overline{k}$, every morphism $A \to X_{\overline{k}}$ is constant; see [33, 34, 37, 38] for basic properties of groupless varieties.

Our starting point in this paper is the fact that the notion of being of general type is an open condition in families of projective varieties. This statement can be deduced from results of Siu, Kawamata, and Nakayama (see [53]).

Theorem 1.2 (Nakayama)**.** *Let $X \to S$ be a proper morphism of schemes. Then the set of s in S such that X_s is of general type is an open subscheme of S.*

Lang notes that "the extent to which hyperbolicity is open for the Zariski topology in families (of projective varieties)" is unclear [44, p. 176]. Our aim in this paper is to investigate how every notion of hyperbolicity appearing in Conjecture 1.11 behaves in families and to show that all these notions are "Zariski-countable open". It seems worth stressing that it is not known whether *any* notion of hyperbolicity appearing in Conjecture 1.11 is a Zariski open condition in families.

1.1. **Stable under generisation.** Nakayama's theorem implies that the locus of s in S such that every subvariety of X_s is of general type is stable under generisation. Our first result confirms that every notion appearing in Conjecture 1.1 is in fact stable under generisation.

Theorem 1.3 (Generisation)**.** *Let S be an integral noetherian scheme and let $X \to S$ be a projective morphism. Let $s \in S$ be a closed point with residue field k of characteristic 0. Let $X_{K(S)}$ be the geometric generic fibre of $X \to S$.*

(i) If every integral closed subvariety of X_s is of general type, then every integral closed subvariety of $X_{K(S)}$ is of general type.
(ii) If X_s is groupless, then $X_{K(S)}$ is groupless.
(iii) If X_s is an algebraically hyperbolic projective variety, then $X_{K(S)}$ is algebraically hyperbolic.

(iv) If X_s is a bounded projective projective variety over k, then $X_{K(S)}$ is bounded over $K(S)$.

(v) Let $n \geq 1$ and $m \geq 0$ be integers. If X_s is an (n,m)-bounded projective variety over k, then $X_{K(S)}$ is (n,m)-bounded over $K(S)$.

Let us also mention the complex-analytic analogue of Theorem 1.3. Namely, let $\mathfrak{X} \to S$ be a surjective holomorphic map of complex analytic spaces with compact fibres. If there is a point s in S such that the fibre $\mathfrak{X}_y$ is Kobayashi hyperbolic, then there is an analytic open neighbourhood $U \subset S$ of s such that, for every u in U, the fibre $\mathfrak{X}_u$ is Kobayashi hyperbolic; see [41, Theorem 3.11.1].

The first statement on varieties of general type in Theorem 1.3 follows from Nakayama's result stated above (see Section 4.1). The second statement on grouplessness is proven using non-archimedean methods in [37]. As we will explain below, the third statement on algebraic hyperbolicity follows from a mild generalisation of a theorem of Demailly (see Theorem 1.4 below). The last two statements are proven in Section 4. In fact, we deduce these two statements from the fact that the locus of s in S such that X_s is bounded (respectively (n,m)-bounded) is a Zariski-countable open in the sense defined below.

1.2. **Countable-openness of the hyperbolic locus.** Given a projective morphism $X \to S$ with S a complex algebraic variety, Demailly showed that the locus of s in $S(\mathbb{C})$ such that X_s is algebraically hyperbolic is an open subset of $S(\mathbb{C})$ in the countable-Zariski topology. Recall that, if $(X, \mathcal{T})$ is a noetherian topological space, then there exists another topology $\mathcal{T}^{\mathrm{cnt}}$, or $\mathcal{T}$-countable, on X whose closed sets are the countable unions of $\mathcal{T}$-closed sets (see Lemma 4.1). If S is a noetherian scheme, a subset $Z \subset S$ is a Zariski-countable closed if it is a countable union of closed subschemes $Z_1, Z_2, \ldots \subset S$. With this terminology at hand, Demailly essentially proved the following result.

Theorem 1.4 (Demailly)**.** *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Then, the set of s in S such that X_s is algebraically hyperbolic is Zariski-countable open in S.*

This is not the exact result proven by Demailly. Indeed, Demailly proved that, if $k = \mathbb{C}$ and $S^{\text{not-ah}}$ is the set of s in S such that X_s is not algebraically hyperbolic, then $S^{\text{not-ah}} \cap S(\mathbb{C})$ is closed in the countable topology on $S(\mathbb{C})$. This, strictly speaking, does not imply that $S^{\text{not-ah}}$ is closed in the countable topology on S. For example, if S is an integral curve over $\mathbb{C}$ and η is the generic point of S, then $\{\eta\}$ is not a Zariski-countable open of S, whereas $\{\eta\} \cap S(\mathbb{C}) = \emptyset$ is a Zariski-countable open of $S(\mathbb{C})$.

We give a proof of Theorem 1.4 which is similar to Demailly's proof, but more adapted to the scheme-theoretic setting. Moreover, note that Demailly's theorem as stated above actually implies that the notion of being algebraically hyperbolic is stable under generisation. Furthermore, to prove Theorem 1.4 we replace part of Demailly's line of reasoning by stack-theoretic arguments. Finally, using similar (but slightly more involved) arguments, we obtain similar results on boundedness and (n,m)-boundedness.

Theorem 1.5 (Countable-openness of boundedness)**.** *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Then, the set of s in S such that X_s is bounded is Zariski-countable open in S.*

Theorem 1.6 (Countable-openness of (n,m)-boundedness)**.** *Let S be a noetherian scheme over $\mathbb{Q}$, let $n \geq 1$ and $m \geq 0$ be integers. If $X \to S$ is projective, then the set of s in S such that X_s is (n,m)-bounded is Zariski-countable open in S.*

As algebraic hyperbolicity is conjecturally equivalent to every subvariety being of general type (Conjecture 1.1), one expects a similar Zariski-countable openness property to hold for the latter notion. We use Nakayama's theorem and the fact that the stack of proper schemes of general type is a countable union of finitely presented algebraic stacks to prove the following result.

Theorem 1.7 (Countable-openness of every subvariety being of general type)**.** *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Then, the set of s in S such that every integral closed subvariety of X_s is of general type is Zariski-countable open in S.*

Finally, in [37] it is shown that the set of s in S such that X_s is groupless is open in the Zariski-countable topology on S. Thus, for every property appearing in Conjecture 1.1, the locus of s in S such that X_s has this property is Zariski-countable open in S.

1.3. **Mildly bounded varieties.** The notions of algebraic hyperbolicity, boundedness, and grouplessness discussed above are expected to coincide. In [33], a "weak" notion of boundedness that suffices for certain arithmetic applications (see Theorem 7.1 below) is introduced. The precise definition of this notion reads as follows.

Definition 1.8 (Mildly bounded varieties)**.** Let k be a field with algebraic closure $k \to \overline{k}$. A finite type separated scheme X over k is *mildly bounded over k* if, for every smooth quasi-projective connected curve C over $\overline{k}$, there is an integer m and points $c_1, \ldots, c_m$ in $C(\overline{k})$ such that, for every $x_1, \ldots, x_m$ in $X(\overline{k})$, the set of morphisms $f : C \to X_{\overline{k}}$ with $f(c_1) = x_1, \ldots, f(c_m) = x_m$ is finite.

It is not hard to show that mildly bounded proper varieties have no rational curves. More generally, if X is a mildly bounded variety, then every morphism $\mathbb{A}^1_{\overline{k}} \to X$ is constant (see Proposition 5.1).

Quite surprisingly, we are able to prove that every semi-abelian variety over a field of characteristic zero is mildly bounded, so that the notion of mild boundedness is *strictly weaker* than any notion of hyperbolicity or boundedness discussed above (including grouplessness).

Proposition 1.9. *If k is a field of characteristic zero and X is a semi-abelian variety over k, then X is mildly bounded over k.*

Proposition 1.9 shows that mildly bounded varieties are not necessarily hyperbolic, nor even of general type. Related to this proposition we show the following global boundedness result for families of abelian varieties. Its proof relies on Silverman's specialisation theorem [61].

Theorem 1.10. *Let S be a hyperbolic integral curve over k, and let $\mathcal{X} \to S$ be a semi-abelian scheme over S. Then $\mathcal{X}$ is mildly bounded over k.*

We conjecture that, for projective varieties, the only obstruction to being mildly bounded is the presence of a rational curve.

Conjecture 1.11. *If k is a field of characteristic zero and X is a projective variety over k such that $X_{\overline{k}}$ has no rational curves, then X is mildly bounded over k.*

The following result says that our conjecture holds for surfaces, under a suitable assumption on the Albanese variety. Its proof crucially uses the mild boundedness of abelian varieties (Corollary 6.4).

Theorem 1.12. *Let X be a projective integral surface over a field k of characteristic zero such that $X_{\overline{k}}$ has no rational curves. If there is an abelian variety A and a morphism $X \to A$ which is generically finite onto its image, then X is mildly bounded over k.*

We can also prove the conjecture for groupless projective surfaces which admit a non-constant map to some abelian variety. In particular, the conjecture holds if X is a groupless projective normal surface with non-zero irregularity $q(X) := \mathrm{h}^1(X, \mathcal{O}_X)$.

Theorem 1.13. *Let X be a projective groupless surface over k. If X admits a non-constant map to some abelian variety over k, then X is mildly bounded over k.*

Our next result is also in accordance with Conjecture 1.11. To motivate this result, recall that, if $X \to S$ is a projective morphism of noetherian schemes over $\mathbb{Q}$, the set of s in S such that $X_{\overline{k(s)}}$ has no rational curve is Zariski-countable open in S (see for instance [13]). Conjecture 1.11 predicts that these two loci are in fact equal, and our next result verifies that the locus of s in S with X_s mildly bounded is Zariski-countable open.

Theorem 1.14. *Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. Then the set of s in S such that X_s is mildly bounded over $k(s)$ is Zariski-countable open in S.*

It seems worthwhile to stress that the proof of Theorem 1.14 follows a similar line of reasoning as the proofs of Theorems 1.4, 1.5, and 1.6. However, the proof of Theorem 1.14 is arguably the most involved, due to the fact that the condition of mild boundedness is *much weaker* than the conditions of being *algebraically hyperbolic, bounded, or (n,m)-bounded*, respectively.

As before, the Zariski-countable openness of the locus of s in S such that X_s is mildly bounded implies that this locus is stable under generisation.

Corollary 1.15. *Let S be an integral noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. If there is an s in S such that X_s is mildly bounded over $k(s)$, then the generic fibre $X_{K(S)}$ is mildly bounded.*

It is not clear that, given a mildly bounded variety X over k and a field extension $k \subset L$, the variety X_L is mildly bounded over L. Using the fact that being mildly bounded is stable under generisation in projective families, we are able to deduce the persistence of mild boundedness over field extensions of *finite* transcendence degree.

Corollary 1.16. *Let $k \subset L$ be an extension of fields of characteristic zero, and let X be a projective mildly bounded variety over k. If $k \subset L$ has finite transcendence degree, then X_L is mildly bounded over L.*

1.4. **Persistence of arithmetic hyperbolicity.** The notion of mildly bounded varieties was introduced in [33] with the aim of giving arithmetic applications; see for instance [33, Theorem 1.4]. In Section 8 we give such applications based on the results in Section 1.3 and [33]. For example, we prove the following new result on rational points on surfaces.

Theorem 1.17. *Let X be a projective integral surface over a number field K such that there is a non-constant morphism $X_{\overline{K}} \to A$ to an abelian variety A over $\overline{K}$. Assume that for every* ***number field*** *L over K, the set $X(L)$ is finite. Then, for every* ***finitely generated field*** *M of characteristic zero, the set $X(M)$ is finite.*

Note that a special case of Theorem 1.17 was already proven in [33, Corollary 1.6], under the additional assumption that X admits a *finite* morphism to an abelian variety. The proof in *loc. cit.* relies on Yamanoi's extension of Bloch-Ochiai-Kawamata's theorem to finite covers of abelian varieties [70]. However, as Yamanoi's theorem does not apply to surfaces with non-maximal Albanese dimension, we can not use his results to prove Theorem 1.17. Instead, to prove Theorem 1.17, we will rely on the fact that abelian varieties are mildly bounded (Proposition 1.9).

Theorem 1.17 verifies a prediction implied by the Lang–Vojta conjecture. To explain this, we recall that Lang introduced the notion of arithmetic hyperbolicity (sometimes also referred to as *Mordellicity*) over $\overline{\mathbb{Q}}$ to capture the property of having only finitely many rational points over number fields. This notion is studied for instance in [4, 5, 30, 31, 33, 35], [64, §2], and [68]. Let us start with extending Lang's notion to varieties over arbitrary algebraically closed fields k of characteristic zero.

Definition 1.18 (Arithmetic hyperbolicity)**.** A finite type separated scheme X over k is *arithmetically hyperbolic over k* if there is a $\mathbb{Z}$-finitely generated subring $A \subset k$ and a finite type separated A-scheme $\mathcal{X}$ with $\mathcal{X}_k \cong X$ over k such that, for all $\mathbb{Z}$-finitely generated subrings $A' \subset k$ containing A, the set of A'-points $\mathcal{X}(A') := \mathrm{Hom}_A(\mathrm{Spec}\, A', \mathcal{X})$ on $\mathcal{X}$ is finite.

With this definition at hand, let us recall that Lang–Vojta's arithmetic conjecture says that grouplessness (and thus also algebraic hyperbolicity) is equivalent to being arithmetically hyperbolic.

Conjecture 1.19 (Arithmetic Lang–Vojta)**.** *A projective variety X over k is groupless over k if and only if X is arithmetically hyperbolic over k.*

Faltings's theorems show that the Arithmetic Lang–Vojta conjecture holds if X is one-dimensional, or when X is a closed subvariety of an abelian variety; see [19, 20, 21, 22].

The Arithmetic Lang–Vojta conjecture has many interesting consequences. For example, in light of the aforementioned properties of grouplessness, it predicts that "being arithmetically hyperbolic" is stable under generisation and even a Zariski-countable open condition in projective families of varieties. Verifying these predictions of Lang–Vojta's conjecture seems currently out of reach. Slightly more reasonable seems to be the Arithmetic Persistence Conjecture stated below. Under the additional assumption that X is projective, the Arithmetic Persistence Conjecture is indeed a consequence of the Arithmetic Lang–Vojta conjecture. However, as the conjecture also seems to be reasonable in the quasi-projective setting, we state it in this generality.

Conjecture 1.20 (Arithmetic Persistence Conjecture)**.** *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. If X is an arithmetically hyperbolic finite type separated scheme over k, then X_L is arithmetically hyperbolic over L.*

To give a better idea of what this conjecture entails, let us consider an affine finite type scheme $\mathcal{X}$ over $\mathbb{Z}$ with only finitely many integral points, i.e., for every number field K and every finite set of finite places S of K, the set of $\mathcal{O}_{K,S}$-points of $\mathcal{X}$ is finite. Then, the Arithmetic Persistence Conjecture says that for every *finitely generated integral domain A of characteristic zero A*, the set $\mathcal{X}(A)$ is finite. This is not an unreasonable expectation, as it can be verified in many cases (see for instance [18] or [31]).

Theorem 1.17 can be reformulated as saying that the Arithmetic Persistence Conjecture holds for projective surfaces X over $\overline{\mathbb{Q}}$ which admit a non-constant map to some abelian variety over $\overline{\mathbb{Q}}$. Indeed, if X is a projective arithmetically hyperbolic surface over $\overline{\mathbb{Q}}$ which admits a non-constant morphism to an abelian variety and $\overline{\mathbb{Q}} \subset L$ is an extension of algebraically closed fields, then Theorem 1.17 says that X_L is arithmetically hyperbolic over L, and this can be shown to imply the finiteness of rational points $X(M)$ (see Section 8.1). The restriction to $\overline{\mathbb{Q}}$ in Theorem 1.17 is unnecessary, and was made above only for the sake of simplifying the statement; see Theorem 8.12 for a more general statement.

Our next result solves the Arithmetic Persistence Conjecture for varieties which admit a quasi-finite morphism to some semi-abelian variety.

Theorem 1.21. *Let A be a $\mathbb{Z}$-finitely generated integral domain of characteristic zero with fraction field K and let $\mathcal{X}$ be a finite type separated scheme over A such that $\mathcal{X}_{\overline{K}}$ admits a quasi-finite morphism to a semi-abelian variety over $\overline{K}$. Assume that, for every* ***finite*** *extension L/K and every $\mathbb{Z}$-finitely generated subring $A' \subset L$ containing A, the set $\mathcal{X}(A')$ is finite. Then, for every* ***finitely generated*** *field extension M/K and every $\mathbb{Z}$-finitely generated subring $B \subset M$ containing A, the set $\mathcal{X}(B)$ is finite.*

Theorem 1.21 is concerned with the finiteness of integral points on (not necessarily proper) varieties which map quasi-finitely (possibly surjectively) to some semi-abelian variety. As before, we note that Theorem 1.21 was proved using Yamanoi's results on abelian varieties in the case that $\mathcal{X}$ is proper and smooth in [33, Corollary 1.6]. Removing the properness condition in Yamanoi's work would require a substantial amount of new ideas. In this paper, we prove Theorem 1.21 by appealing to the simple fact that semi-abelian varieties are mildly bounded (Proposition 1.9).

The reader interested in a concrete application of Theorem 1.21 might find Remark 7.5, and especially Theorem 7.6, helpful.

1.5. **Predictions made by Vojta's geometric conjecture.** Part of the conjecture of Demailly, Green-Griffiths, and Lang as stated above (Conjecture 1.1) is implied by Vojta's conjecture that a projective variety is of general type if and only if it is pseudo-algebraically hyperbolic; see [66] or [67]. Here, a projective variety X over an algebraically closed field k is *pseudo-algebraically hyperbolic over k* if there is a proper closed subset Δ of X, an ample line bundle $\mathcal{L}$ on X and a real number $\alpha_{X,\Delta,\mathcal{L}}$ depending only on X, Δ, and $\mathcal{L}$ such that, for every smooth projective curve C over k and every non-constant morphism $f : C \to X$ with $f(C) \not\subset \Delta$, the inequality

$$\deg_C f^*\mathcal{L} \leq \alpha_{X,\Delta,\mathcal{L}} \cdot \mathrm{genus}(C)$$

holds. More generally, a projective variety X over a field k is *pseudo-algebraically hyperbolic over k* if $X_{\overline{k}}$ is pseudo-algebraically hyperbolic over $\overline{k}$. The word "pseudo" was first coined by Kiernan-Kobayashi [39] and also employed by Lang [44].

Conjecture 1.22 (Vojta's conjecture)**.** *A projective variety X over k is of general type over k if and only if it is pseudo-algebraically hyperbolic over k.*

To motivate our next result, let S be an integral variety over a field k and let $X \to S$ be a projective morphism whose generic fibre $X_{K(S)}$ is of general type. Then, for a general s in $S(k)$, the projective scheme X_s is of general type by Nakayama's theorem (Theorem 1.2). Vojta's conjecture (Conjecture 1.22) predicts that pseudo-algebraic hyperbolicity specialises in a similar way that varieties of general type do. Our next result verifies part of this prediction. This result is proven in Section 4.

Theorem 1.23 (Specialising pseudo-algebraic hyperbolicity)**.** *Let S be an integral variety over k and let $X \to S$ be a projective morphism whose generic fibre $X_{K(S)}$ is pseudo-algebraically hyperbolic. Then, for a very general s in $S(k)$, the projective scheme X_s is pseudo-algebraically hyperbolic.*

Moreover, since being of general type persists over field extensions, our next result is also in accordance with Vojta's conjecture (Conjecture 1.22).

Theorem 1.24 (Persistence of pseudo-algebraic hyperbolicity)**.** *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. If X is a pseudo-algebraically hyperbolic projective variety over k, then X_L is pseudo-algebraically hyperbolic over L.*

Similar results are proven in Section 9 for pseudo-bounded varieties, where a projective scheme X over an algebraically closed field k is said to be *pseudo-bounded* if there is a proper closed subscheme $\Delta \subsetneq X$ such that, for every smooth projective connected variety Y over k, the scheme

$$\underline{\mathrm{Hom}}_k(Y, X) \setminus \underline{\mathrm{Hom}}_k(Y, \Delta)$$

parametrizing morphisms $f : Y \to X$ with $f(Y) \not\subset \Delta$ is of finite type over k. In this case, we also say that X is *bounded modulo* Δ. The relation to pseudo-algebraic hyperbolicity is as follows.

Theorem 1.25 (From curves to varieties)**.** *Let k be an algebraically closed field of characteristic zero. If X is algebraically hyperbolic modulo Δ over k, then X is bounded modulo Δ over k.*

The following result shows that mere boundedness implies the existence of a uniform bound for maps from a fixed curve in the genus of that curve. Therefore, the *a priori* difference between boundedness and algebraic hyperbolicity is the *linearity* of the dependence on the genus in the definition.

Theorem 1.26 (From boundedness to uniformity)**.** *Let k be an algebraically closed field of characteristic zero. If X is bounded modulo Δ over k, then, for every ample line bundle $\mathcal{L}$ on X and every integer $g \geq 0$, there is a real number $\alpha(X, \Delta, \mathcal{L}, g)$ such that, for every smooth projective connected curve C of genus g over k and every morphism $f : C \to X$ with $f(C) \not\subset \Delta$, the inequality*

$$\deg_C f^*\mathcal{L} \leq \alpha(X, \Delta, \mathcal{L}, g)$$

holds.

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Conventions. Throughout this paper, we let k be an algebraically closed field of characteristic zero. A variety over k is a finite type separated reduced k-scheme.

Let X be a finite type separated scheme over k and let $A \subset k$ be a subring. A model for X over A is a pair $(\mathcal{X}, \phi)$ with $\mathcal{X} \to \operatorname{Spec} A$ a finite type separated scheme and $\phi : \mathcal{X} \times_A k \to X$ an isomorphism of schemes over k. We will often omit ϕ from our notation.

2. Characterising boundedness and algebraic hyperbolicity

Recall that a projective variety X over k is algebraically hyperbolic (over k) if there is an ample line bundle $\mathcal{L}$ on X and a real number $\alpha_{X,\mathcal{L}}$ such that, for every smooth projective connected curve C over k of genus g and every morphism $f : C \to X$, the inequality

$$\deg_C f^*\mathcal{L} \leq \alpha_{X,L}(2g - 2)$$

holds. We mention that one could also ask for a weaker bound on $\deg_C f^*\mathcal{L}$ which only depends on X, L and the genus g of C, but which is not necessarily linear in g. This leads to an a priori weaker notion of boundedness which is referred to as *weakly boundedness* by Kovács-Lieblich [43]. Their notion is the same as the notion of *boundedness* introduced in [34].

In this section we first record the fact that maps from a possibly singular curve into an algebraically hyperbolic projective variety also satisfy similar boundedness properties. We then prove similar statements for the notions of boundedness, (n,m)-boundedness and mildly boundedness.

2.1. **Testing algebraic hyperbolicity.** Our starting point is the following characterisation of algebraic hyperbolicity. The geometric genus $p_g(D)$ of C is the sum of the genera of the components D_i $(i = 1, \dots, n)$ of the normalisation $\widetilde{D}$ of D.

Lemma 2.1. *Let X be an algebraically hyperbolic projective scheme over k with an ample line bundle $\mathcal{L}$. Let $\alpha_{X,\mathcal{L}}$ be a real number such that, for every smooth projective connected curve C and every morphism $f\colon C \to X$, the inequality $\deg f^*\mathcal{L} \leq \alpha_{X,\mathcal{L}} \cdot g(C)$ holds. Then, for every reduced projective scheme D pure of dimension 1 over k and every morphism $f\colon D \to X$, the inequality $\deg f^*\mathcal{L} \leq \alpha_{X,\mathcal{L}} \cdot p_g(D)$ holds.*

Proof. Let $\pi\colon \widetilde{D} \to D$ be the normalisation. Then $\deg \pi^* f^*\mathcal{L} = \deg f^*\mathcal{L}$ by [49, Proposition 7.3.8]. Moreover, by assumption, the inequality

$$\deg \pi^* f^*\mathcal{L} \leq \sum_{i=1}^{n} \alpha_{X,\mathcal{L}} \cdot g(D_i)$$

holds. Also, by definition, we have that

$$\sum_{i=1}^{n} \alpha_{X,\mathcal{L}} \cdot g(D_i) = p_g(D).$$

This implies the statement. □

In some sense, most projective varieties should be algebraically hyperbolic, and this philosophy is confirmed by the work of many authors [6, 7, 10, 14, 15, 16, 17, 52, 57, 58].

2.2. **Testing boundedness.** The following lemma will allow us to test boundedness on reduced curves (Theorem 2.3).

Lemma 2.2. *Let X be a projective scheme over k. Let $g\colon C' \to C$ be a finite surjective birational morphism of projective reduced schemes pure of dimension one over k. Then, the natural morphism of schemes*

$$\underline{\mathrm{Hom}}_k(C, X) \to \underline{\mathrm{Hom}}_k(C', X)$$

is a closed immersion.

Proof. Let $H = \underline{\mathrm{Hom}}_k(C, X)$, and let $H' = \underline{\mathrm{Hom}}_k(C', X)$. Let $F' : H' \times C' \to X$ be the evaluation morphism. Note that the morphism

$$E' = (\mathrm{pr}_1, \mathrm{pr}_2, F') : H' \times C' \longrightarrow H' \times C' \times X$$

is a closed immersion. Let

$$E'^{\#} : \mathcal{O}_{H' \times C' \times X} \longrightarrow E'_* \mathcal{O}_{H' \times C'}$$

be the induced morphism. Let $G := (\mathrm{pr}_1, g \circ \mathrm{pr}_2, \mathrm{pr}_3) : H' \times C' \times X \to H' \times C \times X$. Let $\mathcal{A}$ be the image of the morphism $\mathcal{O}_{H' \times C \times X} \longrightarrow G_* E'_* \mathcal{O}_{H' \times C'}$ defined as the composition

$$\mathcal{O}_{H' \times C \times X} \xrightarrow{G^{\#}} G_* \mathcal{O}_{H' \times C' \times X} \xrightarrow{G_* E'^{\#}} G_* E'_* \mathcal{O}_{H' \times C'} .$$

Let $(\mathrm{pr}_1, \mathrm{pr}_2)\colon H' \times C \times X \to H' \times C$ be the projection, and note that $(\mathrm{pr}_1, g \circ \mathrm{pr}_2)\colon H' \times C' \to H' \times C$ is the composition $(\mathrm{pr}_1, \mathrm{pr}_2) \circ G \circ E'$. Then we consider $(\mathrm{pr}_1, \mathrm{pr}_2)_* \mathcal{A}$ as subsheaf of

$$(\mathrm{pr}_1, \mathrm{pr}_2)_* G_* E'_* \mathcal{O}_{H' \times C'} = (\mathrm{pr}_1, g \circ \mathrm{pr}_2)_* \mathcal{O}_{H' \times C'}.$$

Note that the sheaf $\mathrm{Coker}(\mathrm{pr}_1, g \circ \mathrm{pr}_2)^\#$ is supported on $H' \times C^{\mathrm{sing}}$. Moreover, the sheaf $\mathrm{Coker}(\mathrm{pr}_1, g \circ \mathrm{pr}_2)^\#$ is free of finite rank as $\mathcal{O}_{H'}$-module (through the projection on the first coordinate). By [27, Corollaire 7.7.78], the morphism of sheaves

$$(\mathrm{pr}_1, \mathrm{pr}_2)_* \mathcal{A} \hookrightarrow (\mathrm{pr}_1, g \circ \mathrm{pr}_2)_* \mathcal{O}_{H' \times C'} \twoheadrightarrow \mathrm{Coker}(\mathrm{pr}_1, g \circ \mathrm{pr}_2)^\#,$$

is equivalent to a certain morphism $\mathcal{N} \to \mathcal{O}_{H'}$ whose image is an ideal sheaf whose corresponding 0-scheme is H, i.e., $H \to H'$ is a closed immersion. □

We now use Lemma 2.2 to show that a projective scheme X is bounded (as defined in [34, §4]) if and only if, for every reduced curve C, the moduli space of maps from C to X is of finite type.

Theorem 2.3 (Testing boundedness on reduced objects)**.** *Let X be a projective scheme over k. Then the following are equivalent.*

1. *The projective scheme X is bounded over k.*
2. *For every smooth projective connected curve C over k, the scheme $\underline{\mathrm{Hom}}_k(C, X)$ is of finite type over k.*
3. *For every reduced projective (not necessarily irreducible nor smooth) scheme C pure of dimension one over k, the scheme $\underline{\mathrm{Hom}}_k(C, X)$ is of finite type over k.*

Proof. By definition, a bounded variety is 1-bounded, so that (1) $\Longrightarrow$ (2). Moreover, by [34, Theorem 9.2], we have that (2) $\Longrightarrow$ (1). It is clear that (3) $\Longrightarrow$ (2). Therefore, to prove the theorem, it suffices to show that (2) $\Longrightarrow$ (3).

Assume that X satisfies (2). Let C be a projective reduced scheme pure of dimension one over k. Let $C_1, \ldots, C_n$ be the irreducible components of C. Let C' be the normalisation of C in the product of function fields $K(C_1) \times \ldots \times K(C_n)$, and note that $C' \to C$ is a finite birational surjective morphism. Moreover, C' is a smooth projective curve over k. For $i = 1, \ldots, n$, let C_i' be the connected component of C' lying over C_i. By Lemma 2.2, the natural morphism of schemes

$$\underline{\mathrm{Hom}}_k(C, X) \to \underline{\mathrm{Hom}}_k(C', X) = \prod_{i=1}^{n} \underline{\mathrm{Hom}}_k(C_i', X)$$

is a closed immersion. Since X satisfies property (2), for all $i = 1, \ldots, n$, the scheme $\underline{\mathrm{Hom}}_k(C_i', X)$ is of finite type over k. Since closed immersions of schemes are of finite type, we conclude that $\underline{\mathrm{Hom}}_k(C, X)$ is of finite type over k. □

We now prove a similar result for "pointed boundedness".

Proposition 2.4 (Testing pointed boundedness on reduced objects)**.** *Let X be a projective variety over k, let $n \geq 1$ and let $m \geq 1$ be an integer. Then the following are equivalent.*

1. *The projective variety X is (n, m)-bounded over k.*
2. *For every projective connected variety C pure of dimension one over k, every c in $C(k)$, and every x in $X(k)$, the set $\mathrm{Hom}_k([C, c], [X, x])$ is finite.*

Proof. Note that (2) implies that X is $(1, 1)$-bounded. Therefore, by [34, Proposition 8.2], it follows that X is (n, m)-bounded. This shows that (2) $\Longrightarrow$ (1).

To prove that (1) $\Longrightarrow$ (2), we argue as follows. First, as X is (n, m)-bounded, it follows from [34, Lemma 4.6] that, for every smooth projective connected curve C' over k, every c' in $C'(k)$, and every x in $X(k)$, the set $\mathrm{Hom}_k([C', c'], [X, x])$ is finite. Now, let C be a projective connected variety pure of dimension one over k, let c be in $C(k)$, and let x be in $X(k)$. We prove that $\mathrm{Hom}([C, c], [X, x])$ is a finite set by induction on the number N of irreducible components of C not containing c. Let $C_1, \ldots, C_n$ be the irreducible components of C. Let $C' \to C$ be the normalisation of C in the product of the function fields $K(C_1) \times \ldots \times K(C_n)$. Let C'_i be the connected component of C' lying over C_i.

Assume that $N = 0$, i.e., c lies on every irreducible component of C. For every $i = 1, \ldots, n$, let c'_i in C'_i be a point mapping to c in C. As $C' \to C$ is surjective, the natural map of sets

$$\mathrm{Hom}_k([C, c], [X, x]) \to \prod_i \mathrm{Hom}_k([C'_i, c'_i], [X, x])$$

is injective. Since C'_i is a smooth projective connected curve over k, for every i, the set $\mathrm{Hom}_k([C'_i, c'_i], [X, x])$ is finite, so that $\mathrm{Hom}_k([C, c], [X, x])$ is finite, as required.

If $N > 0$, after renumbering if necessary, we may and do assume that c does not lie on C_n and that the projective reduced scheme $D := C_1 \cup \ldots \cup C_{n-1}$ is connected. By the induction hypothesis, the set

$$\mathrm{Hom}_k([D, c], [X, x])$$

is finite. Let d be a point in $D \cap C_n \subset C$. Then, the set

$$A := \{f(d) \mid f \in \mathrm{Hom}_k([D, d], [X, x])\}$$

is a finite subset of X. Therefore, as the map of sets

$$\mathrm{Hom}_k([C, c], [X, x]) \subset \mathrm{Hom}_k([D, c], [X, x]) \times \bigcup_{y \in A} \mathrm{Hom}_k([C_n, d], [X, y])$$

is injective and $\mathrm{Hom}_k([C_n, d], [X, y])$ is finite, we conclude that $\mathrm{Hom}_k([C, c], [X, x])$ is finite. □

2.3. **Testing mild boundedness.** We show that the notion of mild boundedness (Definition 1.8) can be tested on reduced curves. The curves do not even need to be irreducible in this case.

Lemma 2.5. *Let X be a mildly bounded variety over k and let C over k be a finite type separated reduced scheme over k whose irreducible components are one-dimensional. Then there exist an integer n and distinct points $c_1, \ldots, c_n \in C(k)$ such that for every $x_1, \ldots, x_n \in X(k)$ the scheme*

$$\mathrm{Hom}([C, (c_1, \ldots, c_n)], [X, (x_1, \ldots, x_n)])$$

is finite over k.

Proof. Let $C_1, \ldots, C_\ell$ be the irreducible components of C. For $i = 1, \ldots, \ell$, let D_i be the locus of points in C_i that are smooth as a point of C. As X is mildly bounded over k, there exist points $d_{1,1}, \ldots, d_{1,n_1} \in D_1(k)$, points $d_{2,1}, \ldots, d_{2,n_2} \in D_2(k)$, ..., and points $d_{\ell,1}, \ldots, d_{\ell,n_\ell} \in D_\ell(k)$ such that

$$H_i = \mathrm{Hom}([D_i, (d_{i,1}, \ldots, d_{i,n_i})], [X, (x_1, \ldots, x_{n_i})])$$

is finite for every $i = 1, \ldots, \ell$ and $x_1, \ldots, x_{n_i} \in X(k)$.

Let $c_1, \ldots, c_n$ be the points $d_{1,1}, \ldots, d_{\ell,n_\ell}$ considered as points in $C(k)$. Then we claim that

$$H = \mathrm{Hom}([C, (c_1, \ldots, c_n)], [X, (x_1, \ldots, x_n)])$$

is finite for all $x_1, \ldots, x_n \in X(k)$. For any morphism $f \colon C \to X$ such that $f(c_i) = x_i$ for all $i = 1, \ldots, n$, the composition $f_i \colon D_i \to C \to X$ lies in the finite scheme H_i. As a morphism is fixed by its restriction to the dense open $\bigcup_{i=1}^{\ell} D_i$, this gives an embedding $H \hookrightarrow H_1 \times \ldots \times H_\ell$, which proves that H is finite over k. □

3. Extending maps over Dedekind schemes

A morphism of schemes $\mathcal{C} \to S$ is a semi-stable curve over S if it is a proper flat morphism whose geometric fibres are connected semi-stable curves; see [49, Definition 10.3.1].

The following lemma is a mild generalisation of a well-known extension property for rational maps from big opens of *smooth* varieties to projective varieties with no rational curves.

Lemma 3.1. *Let S be a Dedekind scheme with function field K, and let $X \to S$ be a projective morphism of schemes. Let $\mathcal{C}$ be a semi-stable curve over S such that $\mathcal{C}_K$ is smooth projective connected. If every geometric closed fibre of $X \to S$ contains no rational curves, then every morphism $\mathcal{C}_K \to X_K$ extends to a morphism $\mathcal{C} \to X$.*

Proof. Since $\mathcal{C} \to S$ is semi-stable, the exceptional locus of the minimal resolution of singularities $\mathcal{C}' \to \mathcal{C}$ is a forest of $\mathbb{P}^1$'s; see [49, Corollary 10.3.25]. Also, as the geometric closed fibres of $X \to S$ do not contain any rational curves, every geometric fibre of $X \to S$ contains no rational curves [37]. In particular, the geometric fibres of $X \times_S \mathcal{C}' \to \mathcal{C}'$ do not contain rational curves. Therefore, since $\mathcal{C}'$ is an integral regular noetherian scheme, the rational section $\mathcal{C}' \dashrightarrow X \times_S \mathcal{C}'$ extends to a morphism [23, Proposition 6.2]. We now show that the induced morphism $\mathcal{C}' \to X$ factors over $\mathcal{C}$.

Indeed, note that the rational curves in the exceptional locus of $\mathcal{C}' \to \mathcal{C}$ are contracted to a point in X. This implies that the morphism $\mathcal{C}' \to X$ factors over a morphism $\mathcal{C} \to X$; see [26, Proposition 8.11.1]. $\square$

Corollary 3.2. *Let $X \to S$ be a projective morphism of noetherian schemes such that all geometric fibres do not contain a rational curve. Fix a relatively ample line bundle $\mathcal{L}$ on X. Let $g, n \geq 0$ be integers such that $(g, n) \notin \{(0,0), (0,1), (0,2), (1,0)\}$, and let $\overline{\mathcal{U}_{g,n}} \to \overline{\mathcal{M}_{g,n}}$ be the universal curve over the stack $\overline{\mathcal{M}_{g,n}}$ of n-pointed stable curves of arithmetic genus g over S. Let $d \geq 0$ be an integer and let $\mathcal{H}_d^{g,n} = \underline{\mathrm{Hom}}^d_{\overline{\mathcal{M}_{g,n}}}(\overline{\mathcal{U}_{g,n}}, X \times \overline{\mathcal{M}_{g,n}})$ be the algebraic stack of morphisms of degree d with respect to $\mathcal{L}$. Then the natural morphism $\rho\colon \mathcal{H}_d^{g,n} \to \overline{\mathcal{M}_{g,n}}$ is proper.*

Proof. It suffices to prove the existence part of the valuative criterion for properness. We prove this by induction on g and n. Let R be a discrete valuation ring with field of fractions K. Let $\mathcal{V}_d^{g,n} = \rho^{-1}(\mathcal{M}_{g,n})$. Suppose we have a point $\varphi \in \mathcal{H}_d^{g,n}(K)$ and a curve $\mathcal{C} \in \overline{\mathcal{M}_{g,n}}(R)$ such that $\mathcal{C}_K \cong \rho(\varphi)$.

In the case $\varphi \in \mathcal{V}_d^{g,n}(K)$, the morphism $\varphi\colon \mathcal{C}_K \to X_K$ extends to a morphism $\mathcal{C} \to X_R$ by Lemma 3.1. In the case $\varphi \notin \mathcal{V}_d^{g,n}(K)$, the point lies in the image of one of the clutching morphisms as described in [40, Definition 3.8]. As these clutching morphisms are finite and hence proper, the statement now follows from the induction hypothesis. $\square$

4. Zariski-countable openness of the hyperbolic locus

Let S be a noetherian scheme over $\mathbb{Q}$ and let $X \to S$ be a projective morphism. We start with presenting Demailly's proof of Theorem 1.4 by using the language of algebraic stacks. Following Demailly, we will make use of the following simple lemma.

Lemma 4.1. *Let $(X, \mathcal{T})$ be a noetherian topological space. Then there exists another topology $\mathcal{T}^{\mathrm{cnt}}$, or $\mathcal{T}$-countable, on X whose closed sets are the countable union of $\mathcal{T}$-closed sets.*

Proof. The only non-trivial thing to check is that an arbitrary intersection

$$V = \bigcap_{i \in I} \bigcup_{j=1}^{\infty} C_{ij},$$

where $C_{ij} \subset X$ are $\mathcal{T}$-closed sets, is $\mathcal{T}^{\mathrm{cnt}}$-closed. The proof goes by noetherian induction on $(X, \mathcal{T})$. If for every $i \in I$ there is a $j \in \mathbb{Z}_{>0}$ such that $C_{ij} = X$, then we are done. If this is not the case, we take an $i \in I$, such that $C_{ij} \subsetneq X$ for all $j \in \mathbb{Z}_{>0}$. Then for every $j \in \mathbb{Z}_{>0}$ the set $V \cap C_{ij}$ is $\mathcal{T}^{\mathrm{cnt}}$-closed in C_{ij} and hence in X. Then the union $V = \bigcup_{j=1}^{\infty} V \cap C_{ij}$ is also $\mathcal{T}^{\mathrm{cnt}}$-closed. □

Remark 4.2 (Varieties with no rational curves)**.** We follow [34, §3] and say that a proper variety X over an algebraically closed field F is *pure* if and only if, for every smooth variety T over F and every dense open subscheme $U \subset T$ with $\mathrm{codim}(T \setminus U) \geq 2$, we have that every morphism $U \to X$ extends to a morphism $T \to X$. Note that a proper variety X over F is pure if and only if it has no rational curves. This terminology will allow us to simplify some of the proofs below.

Stack-theoretic version of Demailly's proof of Theorem 1.4. Let $\mathcal{L}$ be a relatively ample line bundle on X over S. Let $\mathcal{M}_g$ be the stack of smooth projective curves of genus g over S, and let $\mathcal{U}_g \to \mathcal{M}_g$ be the universal curve. The Hom-stack $\underline{\mathrm{Hom}}_{\mathcal{M}_g}(\mathcal{U}_g, X \times \mathcal{M}_g)$ is the countable union over $d \geq 0$ of the finitely presented stacks $\mathcal{H}_{g,d} := \underline{\mathrm{Hom}}^d_{\mathcal{M}_g}(\mathcal{U}_g, X \times \mathcal{M}_g)$ of morphisms f which have (fibrewise) degree d with respect to $\mathcal{L}$. Let $S_{g,d}$ be the image of $\mathcal{H}_{g,d}$ in S under the structure map $\mathcal{H}_{g,d} \to S$. As the structure map $\mathcal{H}_{g,d} \to S$ is quasi-compact, locally of finite presentation and S is quasi-compact and quasi-separated, the set $S_{g,d}$ is a constructible subset of S; see [62, Tag 054J] or [46, Théorème 5.4.9].

For any β, consider

$$S_\beta = \bigcup_{\substack{(g,d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \\ d > \beta \cdot g}} S_{g,d};$$

this is the locus of s in S for which X_s is not algebraically hyperbolic with constant β. Note this is a countable union, over $j = 1, 2, 3, \ldots$, of locally closed subsets $U_i \cap V_i$ with $U_i \subset S$ open and $V_i \subset S$ closed. Without loss of generality, we will assume that V_i is irreducible and $\overline{U_i \cap V_i} = V_i$. In particular, for each i, the generic point η_i of V_i lies in S_β. Now, we claim that S_β is stable under specialisation, which implies that $S_\beta = \bigcup_{i=1}^{\infty} V_i$ is a Zariski-countable closed.

To prove the claim consider two points $s, t \in S$ such that s specialises to t and $s \in S_\beta$. If X_t is not pure (Remark 4.2), then t is clearly contained in S_β, so assume X_t is pure. As $s \in S_\beta$, there is a morphism from a smooth curve $C/k(s)$ of certain genus g to X_s of degree greater than $\beta \cdot g$. Then by repeated application of Lemma 3.1, we also get a morphism of the same degree from a curve $D/k(t)$ of the same genus to X_d. Hence, $t \in S_\beta$.

It now follows that the non-algebraically hyperbolic locus

$$S^{\mathrm{nh}} = \bigcap_{\beta=1}^{\infty} S_\beta$$

is also Zariski-countable closed, as the Zariski-countable topology is a topology on S by Lemma 4.1. □

Remark 4.3. Let us briefly assume that S is a positive-dimensional integral variety over an algebraically closed field k of characteristic zero. Let K be the function field of S, and suppose that $X_{\overline{K}}$ is algebraically hyperbolic over $\overline{K}$. If k is *uncountable*, then it follows from Theorem 1.4 that there is an s in $S(k)$ such that X_s is algebraically hyperbolic. If we assume Lang's conjecture, then the hypothesis on the cardinality of k is unnecessary. Indeed, to explain this, we assume for simplicity that $k \subset \mathbb{C}$. Since $\mathbb{C}$ is uncountable, it follows from Theorem 1.4

that there is an s in $S(\mathbb{C})$ such that X_s is algebraically hyperbolic over $\mathbb{C}$. In particular, by Lang's conjecture (which we are assuming to hold for now), the complex analytic space X_s^{an} is Kobayashi hyperbolic. Now, as the fibres of $X_{\mathbb{C}}^{\mathrm{an}} \to S_{\mathbb{C}}^{\mathrm{an}}$ are compact, it follows from a theorem of Brody that there is an analytic open neighbourhood $U \subset S^{\mathrm{an}}$ such that, for every u in U, the fibre X_u is Kobayashi hyperbolic (see [41, Theorem 3.11.1]). Now, since $S(k) \subset S(\mathbb{C})$ is a dense subset of $S_{\mathbb{C}}^{\mathrm{an}}$ with respect to the complex analytic topology, there is a point s in $U \cap S(k)$. We see that X_s is algebraically hyperbolic over k.

Remark 4.4. In the analytic setting, we cannot hope for the locus of points in the base with hyperbolic fibre to be Zariski open. For example, if we have a relative smooth proper curve $\mathcal{C} \to \mathbb{C}^*$ with precisely one non-hyperbolic fibre, then we can pull-back this family along the exponential map $\mathbb{C} \to \mathbb{C}^*$ to obtain a family $\mathcal{X} \to \mathbb{C}$ such that the set of s in $\mathbb{C}$ with $\mathcal{X}_s$ non-hyperbolic is a countably infinite subset of $\mathbb{C}$.

The notion of pseudo-algebraic hyperbolicity should be Zariski open in families in light of Vojta's conjecture (Conjecture 1.22). However, as we currently do not know whether pseudo-algebraic hyperbolicity is stable under generisation, we also do not know whether the locus of points in the base for which the fibre is pseudo-algebraically hyperbolic is in fact Zariski-countable open. Nonetheless, our next result shows that the locus of pseudo-algebraically hyperbolic fibres contains a Zariski-countable open. As the reader will notice, the proof of this result is similar to the proof of Theorem 1.4.

Proposition 4.5. *Assume S is integral with function field $K = K(S)$, and let $K \to \overline{K}$ be an algebraic closure. Let $\Delta \subset X$ be a closed subscheme such that $X_{\overline{K}}$ is algebraically hyperbolic modulo $\Delta_{\overline{K}}$ over $\overline{K}$. Then, for every algebraically closed field k of characteristic zero and a very general s in $S(k)$, the projective scheme X_s is algebraically hyperbolic modulo Δ_s over k.*

Proof. We may and do assume that k is uncountable. Let $\mathcal{L}$ be a relatively ample line bundle on X over S. Let

$$\beta = \alpha_{X_{\overline{K}}, \Delta_{\overline{K}}, \mathcal{L}_{\overline{K}}}$$

be the constant as in the definition of algebraic hyperbolicity for $X_{\overline{K}}$ modulo $\Delta_{\overline{K}}$.

Let $\mathcal{M}_g$ be the stack of smooth proper curves of genus g over S, and let $\mathcal{U}_g \to \mathcal{M}_g$ be the universal curve. The Hom-stack $\underline{\mathrm{Hom}}_{\mathcal{M}_g}(\mathcal{U}_g, X \times \mathcal{M}_g) \setminus \underline{\mathrm{Hom}}_{\mathcal{M}_g}(\mathcal{U}_g, \Delta \times \mathcal{M}_g)$ is the countable union over $d \geq 0$ of the finitely presented stacks $\mathcal{H}_{g,d} := \underline{\mathrm{Hom}}^d_{\mathcal{M}_g}(\mathcal{U}_g, X \times \mathcal{M}_g) \setminus \underline{\mathrm{Hom}}_{\mathcal{M}_g}(\mathcal{U}_g, \Delta \times \mathcal{M}_g)$ of morphisms f which have (fibrewise) degree d with respect to $\mathcal{L}$. Let $S_{g,d}$ be the image of $\mathcal{H}_{g,d}$ in S under the structure map $\mathcal{H}_{g,d} \to S$. Then $S_{g,d}$ is a constructible subset of S.

Hence, for any $d > \beta \cdot g$, the closure $\overline{S_{g,d}}$ of $S_{g,d}$ inside S does not equal S, as $X_{\overline{K}}$ is algebraically hyperbolic modulo $\Delta_{\overline{K}}$. As k is uncountable, we have

$$\bigcup_{\substack{(g,d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \\ d > \beta \cdot g}} \overline{S_{g,d}} \neq S.$$

Let s be a point in $S(k)$ such that, for every $g \geq 0$ and $d > \beta \cdot g$, the point s is not in $S_{g,d}$. Then, the projective scheme X_s is algebraically hyperbolic modulo Δ_s over k. This concludes the proof. $\square$

Proof of 1.23. This follows from Proposition 4.5. $\square$

Next, we will prove that the 1-bounded locus is Zariski-countable open. For this, we first need two intermediate results.

Lemma 4.6. *The set S^0 consisting of these s in S such that X_s contains a rational curve is Zariski-countable closed in S.*

Proof. This is well-known; see for instance [13, Lemma 3.7]. $\square$

Proposition 4.7. *The subset of s in S such that X_s is 1-bounded, is Zariski-countable open.*

Proof. Let $\mathcal{L}$ be a relatively ample line bundle on X. For $g > 1$, let $\overline{\mathcal{M}_g}$ be the stack of stable curve of genus g over S. Note that $\overline{\mathcal{M}_g}$ is noetherian (as S is noetherian), and that the natural morphism $\kappa\colon \overline{\mathcal{M}_g} \to S$ is proper.

Let $\overline{\mathcal{U}_g} \to \overline{\mathcal{M}_g}$ be the universal curve over $\overline{\mathcal{M}_g}$, and let $\mathcal{H}_d^g = \underline{\mathrm{Hom}}_{\overline{\mathcal{M}_g}}(\overline{\mathcal{U}_g}, X \times \overline{\mathcal{M}_g})$ and $\rho\colon \mathcal{H}_d^g \to \overline{\mathcal{M}_g}$ be the structure morphism. Let S^0 be as in Lemma 4.6. Let $T_d^g = \rho(\mathcal{H}_d^g) \cup \kappa^{-1}(S^0)$. A priori, this is a countable union of constructible subsets in $\overline{\mathcal{M}_g}$. Because of Corollary 3.2, ρ satisfies the existence part of the valuative criterion of properness outside of S^0, hence T_d^g is closed under specialisation and hence Zariski-countable closed.

For each $n \in \mathbb{Z}_{>0}$ consider $V_n^g = \bigcup_{d=n}^{\infty} T_d^g$; this is the locus of stable curves admitting a morphism of degree at least n to X, together with $\kappa^{-1}(S^0)$. Now we are interested in $V^g = \bigcap_{n=1}^{\infty} V_n^g$, which is the locus of curves admitting a morphism of arbitrary high degree, together with $\kappa^{-1}(S^0)$. Since S is an integral noetherian scheme, the subset $V^g = \bigcap_{n=1}^{\infty} V_n^g$ is Zariski-countable closed (Lemma 4.1). Hence, the image $S^g = \kappa(V^g)$ in S is Zariski-countable closed. For S^1, the locus where there are morphisms of arbitrary high degree from a genus 1 curve to X, we can use $\mathcal{M}_{1,1}$ instead to prove that this is Zariski-countable closed.

In particular $\bigcup_{g\geq 0} S^g$ is Zariski-countable closed. This concludes the proof, as the locus where X_s is not 1-bounded equals $\bigcup_{g\geq 0} S^g$ by Proposition 2.3. $\square$

Corollary 4.8. *The subset of s in S such that X_s is bounded is Zariski-countable open.*

Proof. Since a projective scheme over k is bounded if and only if it is 1-bounded over k (see [34, Theorem 9.2]), the corollary follows from Proposition 4.7. $\square$

Proposition 4.9. *The subset of s in S such that X_s is $(1,1)$-bounded, is Zariski-countable open.*

Proof. Let g, S^0, $\overline{\mathcal{M}_g}$, $\overline{\mathcal{U}_g}$ and $\mathcal{H}_d^g$ be as in the proof of Proposition 4.7. Now we consider the morphism

$$\tau\colon \overline{\mathcal{U}_g} \times_{\overline{\mathcal{M}_g}} \mathcal{H}_d^g \to \overline{\mathcal{U}_g} \times_S X$$
$$((C,c),\varphi) \mapsto ((C,c),\varphi(c)).$$

The fibre $\tau^{-1}\{((C,c),x)\}$ is exactly $\mathrm{Hom}^d([C,c],[X,x])$, the morphisms of degree d mapping c to x. Again we let $\kappa\colon \overline{\mathcal{U}_g} \times_S X \to S$ be the structure map, which is still proper. We proceed in the same way as in the proof of Proposition 4.7, taking $T_d^g = \mathrm{im}(\tau) \cup \kappa^{-1}(S^0)$, which is then proved to be a Zariski-countable closed set. The properness over S of all schemes involved causes the morphism τ to also satisfy the existence part of the valuative criterion for properness as in Corollary 3.2.

Then $V^g = \bigcap_{n=1}^{\infty} \bigcup_{d=n}^{\infty} T_d^g$ is again Zariski-countable closed, as $\overline{\mathcal{U}_g} \times_S X$ is still noetherian. Hence, $S^g = \kappa(V^g)$ is Zariski-countable closed in S. As also done in Proposition 4.7, S^1 can be defined and shown to be Zariski-countable closed in S using $\mathcal{M}_{1,1}$. Hence, $\bigcup_{g=0}^{\infty} S^g$ is Zariski-countable closed in S. The latter is exactly the locus of s in S such that X_s is not $(1,1)$-bounded by Proposition 2.4. $\square$

Corollary 4.10. *For $m, n > 0$, the subset of s in S such that X_s is (n,m)-bounded, is Zariski-countable open.*

Proof. Since $m \geq 1$, a projective scheme over k is (n, m)-bounded if and only if it is $(1, 1)$-bounded over k (see [34, §8]). Therefore, the corollary follows from Proposition 4.9. $\square$

4.1. **Varieties of general type.** Let k be an algebraically closed field of characteristic zero. Let $\mathcal{M}^{\mathrm{pol}}$ be the stack over k whose objects over a k-scheme S are pairs $(f : X \to S, \mathcal{L})$ with $f : X \to S$ a flat proper finitely presented morphism and $\mathcal{L}$ an f-relative ample line bundle on X; see [55]. Note that $\mathcal{M}^{\mathrm{pol}}$ is a locally finitely presented algebraic stack over k with affine diagonal [36, Section 2.1]. This follows from [62, Tag 0D4X] and [63]. The additional datum of a polarisation (i.e., the f-relative ample line bundle) is necessary to ensure the algebraicity of the stack $\mathcal{M}^{\mathrm{pol}}$.

Recall that a proper scheme X over an algebraically closed field K is of general type (over K) if, for every irreducible component X' of X, there is a resolution of singularities $\tilde{X} \to X'_{\mathrm{red}}$ such that $\omega_{\tilde{X}}$ is a big line bundle. In other words, a proper scheme over K is of general type if *every* irreducible component of X is of general type. We refer the reader to [47, 48] for basic properties of varieties of general type.

Let $\mathcal{M}^{\mathrm{gt}}$ be the substack of $\mathcal{M}^{\mathrm{pol}}$ whose objects over a k-scheme S are pairs $(f : X \to S, \mathcal{L})$ in $\mathcal{M}^{\mathrm{pol}}$ such that the geometric fibres of $f : X \to S$ are proper schemes of general type.

Theorem 4.11 (Nakayama)**.** *The substack $\mathcal{M}^{\mathrm{gt}}$ of $\mathcal{M}^{\mathrm{pol}}$ is an open substack.*

Proof. This is a consequence of Nakayama's theorem (Theorem 1.2). $\square$

For every polynomial h in $\mathbb{Q}[t]$, we let $\mathcal{M}_h^{\mathrm{pol}} \subset \mathcal{M}^{\mathrm{pol}}$ be the substack of pairs $(f : X \to S, \mathcal{L})$ such that, for every geometric point s of S, the Hilbert polynomial of the pair $(X_s, \mathcal{L}_s)$ over the algebraically closed field $k(s)$ equals h. Analogously, we define $\mathcal{M}_h^{\mathrm{gt}} = \mathcal{M}_h^{\mathrm{pol}} \times_k \mathcal{M}^{\mathrm{gt}}$. The following proposition is a well-known consequence of the theory of Hilbert schemes.

Proposition 4.12. *The stack $\mathcal{M}^{\mathrm{pol}}$ is a countable disjoint union of the finitely presented open and closed substacks $\mathcal{M}_h^{\mathrm{pol}}$, and its open substack $\mathcal{M}^{\mathrm{gt}}$ is a countable disjoint union of the finitely presented open and closed substacks $\mathcal{M}_h^{\mathrm{gt}}$.*

Proof. The first statement follows from [62, Tag 0D4X], and the second statement follows from the first by Theorem 4.11. $\square$

Proof of Theorem 1.7. Let S be a noetherian scheme over $\mathbb{Q}$, and let $f : X \to S$ be a projective morphism. To show that the set of s in S such that every subvariety of X_s is of general type is Zariski-countable open, we fix an f-relative ample line bundle $\mathcal{L}$ on X. Let $\mathrm{Hilb}_{X/S} \to S$ be the Hilbert scheme of the projective morphism $X \to S$. Consider the forgetful morphism $\mathrm{Hilb}_{X/S}^h \to \mathcal{M}_h^{\mathrm{pol}}$ which associates to a closed S-flat subscheme $Z \subset X$ with Hilbert polynomial h (with respect to $\mathcal{L}$) the corresponding object $(Z \to S, \mathcal{L}|_Z)$ of $\mathcal{M}_h^{\mathrm{pol}}$. Let $\mathrm{Hilb}_{X/S}^{\mathrm{gt},h}$ be the inverse image of the open substack $\mathcal{M}_h^{\mathrm{gt}}$ in $\mathcal{M}_h^{\mathrm{pol}}$ (Theorem 4.11) .

For h in $\mathbb{Q}[t]$, we let $\mathrm{Hilb}_{X/S}^{\text{n-gt},h} = \mathrm{Hilb}_{X/S}^h \setminus \mathrm{Hilb}_{X/S}^{\mathrm{gt},h}$. Note that $\mathrm{Hilb}_{X/S}^{\text{n-gt},h}$ is a closed subscheme of the quasi-projective S-scheme $\mathrm{Hilb}_{X/S}^h$. For h in $\mathbb{Q}[t]$, let $S_h \subset S$ be the image of $\mathrm{Hilb}_{X/S}^{\text{n-gt},h} \to S$. Since $\mathrm{Hilb}_{X/S}^h \to S$ is a proper morphism [54, Theorem 5.1] and $\mathrm{Hilb}_{X/S}^{\text{n-gt},h}$ is closed in $\mathrm{Hilb}_{X/S}$, we see that S_h is a closed subscheme of S. Note that the locus of s in S such that X_s has an integral subvariety which is not of general type is

$$\bigcup_{h \in \mathbb{Q}[t]} S_h.$$

We conclude that it is a countable union of closed subschemes of S, as required. $\square$

The fact that the locus of algebraic hyperbolicity is Zariski-countable open implies that this locus is stable under generisation. This allows us to easily prove Theorem 1.3.

Proof of Theorem 1.3. Note that (*ii*) is proven in [37]. To prove (*iii*), note that the set S^{ah} of s in S with X_s algebraically hyperbolic is Zariski-countable open in S by Theorem 1.4. By assumption, S^{ah} is non-empty, so that the generic point of S lies in S^{ah}. This proves (*iii*). To prove (*iv*) and (*v*), we argue in a similar manner employing Theorems 1.5 and 1.6 instead of Theorem 1.4, respectively. Similarly, to prove (*i*), we argue in a similar manner employing instead Nakayama's theorem (or rather its consequence Theorem 1.7). □

4.2. **Mildly bounded varieties in families.** In this section we prove that the set of s in S such that X_s is mildly bounded (Definition 1.8) is Zariski-countable open (Theorem 1.14). To do this, we use the following lemma.

Lemma 4.13. *Let $f\colon T \to S$ be a finite type flat morphism of noetherian schemes. If $Y \subset T$ is a Zariski closed (resp. Zariski-countable closed) subset of T, then the locus of s in S such that $Y_s = T_s$ is a Zariski closed (resp. Zariski-countable closed) subset of T.*

Proof. First consider the case $Y \subset T$ is Zariski closed. Since f is a finite type flat morphism of noetherian schemes, it follows that f is an open map [62, Tag 01UA]. In particular, if $U := T \setminus Y$, then $f(U)$ is open in S. This is exactly the complement of points s in S such that $Y_s = T_s$.

Now suppose $Y = \bigcup_{i=1}^{\infty} Y_i$ is a Zariski-countable closed and $Y_i \subset T$ is closed. Without loss of generality we may and do assume that $Y_1 \subset Y_2 \subset \cdots$. Then, as T is noetherian, we have that $Y_s = T_s$ if and only if there exists an i such that $Y_{i,s} = T_s$. Let $S_i \subset S$ be the (closed) locus of s in S satisfying $Y_{i,s} = T_s$. Then, the locus of s in S for which $Y_s = T_s$, equals $\bigcup_{i=1}^{\infty} S_i$. Since $\bigcup_{i=1}^{\infty} S_i$ is Zariski-countable closed (by definition), this concludes the proof. □

To prove Theorem 1.14 we will use the characterisation of mildly bounded varieties stated in Lemma 2.5. This equivalent definition of mildly boundedness allows us to use moduli-theoretic arguments similar to those employed in the proof of Theorem 1.4, 1.5, and 1.6, respectively.

Proof of Theorem 1.14. Let g, $\overline{\mathcal{M}_{g,n}}$, and $\mathcal{H}^d_{g,n}$ be as in Corollary 3.2. Without loss of generality we will assume that every geometric fibre of $X \to S$ is pure (Remark 4.2). The proof below can be adjusted for the non-pure case by adjoining the inverse image of the non-pure locus of S to each of the Zariski(-countable) closed sets appearing in this proof, in the same way as we did in the proof of Proposition 4.7.

In the notion of mildly boundedness, there are possibly non-projective (smooth connected) curves C appearing. We will consider each such smooth connected curve C as an open subset of its smooth projective closure $\overline{C}$. Then $\overline{C} \setminus C$ consists of finitely many points, say m points, and in this way we can consider C as a (possibly non-unique) point of $\overline{\mathcal{M}_{g,m}}$.

Consider the evaluation morphism

$$\tau\colon \mathcal{H}^d_{g,n+m} \to X^n \times \overline{\mathcal{M}_{g,n+m}}$$

$$([C,(c_1,\ldots,c_n,d_1,\ldots,d_m)], \varphi\colon C \to X) \mapsto ((\varphi(c_1),\ldots,\varphi(c_n)), [C,(c_1,\ldots,c_n,d_1,\ldots,d_m)])$$

As X is proper over S, the properness of τ follows immediately from Corollary 3.2. Hence, the image $A^d_{g,n,m}$ of τ in $X^n \times \overline{\mathcal{M}_{g,n+m}}$ is closed. This is the locus of n-tuples of possibly non-distinct points in X and n-pointed (non-projective) curves C admitting an n-pointed morphism of degree d to X. Let $B_{g,n,m} = \bigcap_{d'=1}^{\infty} \bigcup_{d=d'}^{\infty} A^d_{g,n,m}$ be the locus of n-tuples of points in X and n-pointed curves C admitting n-pointed morphisms of unbounded degree to X. This locus is Zariski-countable closed by Lemma 4.1.

Now let $C_{g,n,m}$ be the projection of $B_{g,n,m}$ under the projection $X^n \times \overline{\mathcal{M}_{g,n+m}} \to \overline{\mathcal{M}_{g,n+m}}$, which is Zariski-countable closed in $\overline{\mathcal{M}_{g,n+m}}$ as X is proper over S.

Let $f\colon \overline{\mathcal{M}_{g,n+m}} \to \overline{\mathcal{M}_{g,m}}$ be the forgetful map and $D_{g,m} \subset \overline{\mathcal{M}_{g,m}}$ the locus of $s \in \overline{\mathcal{M}_{g,m}}$ such that $f^{-1}(s) = (C_{g,n,m})_s$. Then, it follows from Lemma 4.13 that $D_{g,m,n}$ is Zariski-countable closed in $\overline{\mathcal{M}_{g,m}}$. Similarly, we have that $E_{g,m} := \bigcap_{n=1}^{\infty} D_{g,m,n}$ is Zariski-countable closed by Lemma 4.1. The latter is the locus of (non-projective, non-irreducible) curves obtained by removing m points from a projective curve of genus g for which the mildly boundedness condition fails.

Let $\rho_{g,m}\colon \overline{\mathcal{M}_{g,m}} \to S$ be the proper structure morphism. As mildly boundedness can be tested on curves which are not irreducible (Lemma 2.5), the locus where $X \to S$ is not mildly bounded is $\bigcup_{g=1}^{\infty} \bigcup_{m=0}^{\infty} \rho_{g,m}(E_{g,m})$. This is now Zariski-countable closed. $\square$

Proof of Corollary 1.15. This follows from Theorem 1.14 and the fact that a Zariski-countable closed set of (the scheme) S contains the generic point of S if and only if this set equals S itself. $\square$

Proof of Corollary 1.16. Let K be a finitely generated subfield of L containing k with $\overline{K} = L$ and let S be an integral variety over k with function field equal to K and let $s \in S(k)$ be a k-point of S. Consider $\mathcal{X} = X \times S \to S$, and note that $\mathcal{X}_s = X$ is mildly bounded over k. Since the locus of mildly bounded varieties is stable under generisation (Corollary 1.15), it follows that X_K (hence X_L) is mildly bounded. $\square$

Corollary 4.14. *Let k be an* ***uncountable*** *algebraically closed field of characteristic zero, and let $k \subset L$ be an algebraically closed field of finite transcendence degree over k. If X is a projective variety over k, then X is mildly bounded over k if and only if X_L is mildly bounded over L.*

Proof. If X is mildly bounded over k, then X_L is mildly bounded over L by Corollary 1.16. Now, assume X_L is mildly bounded over L. Let S be an integral variety over k whose dimension equals the transcendence degree of L over k and whose function field $K(S)$ is contained in L. Consider $\mathcal{X} := X \times S$ as a projective scheme over S. Then, as the set of s in S such that X_s is mildly bounded is Zariski-countable open (Theorem 1.14) and k is uncountable, it follows that there is an s in $S(k)$ such that the variety $X = \mathcal{X}_s$ is mildly bounded. $\square$

5. Mildly boundedness

Let k be an algebraically closed field of characteristic zero. In this section we study mildly bounded varieties (Definition 1.8). We start by showing that $\mathbb{A}^1_k$ (hence $\mathbb{P}^1_k$) is not mildly bounded over k.

Proposition 5.1. *The curve $\mathbb{A}^1_k$ over k is not mildly bounded over k.*

Proof. Indeed, if $c_1, \ldots, c_n \in \mathbb{A}^1_k(k)$ are distinct points and $x_1, \ldots, x_n \in \mathbb{A}^1_k(k)$ are arbitrary, then there exist morphisms $\varphi\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ of arbitrary high degree such that $\varphi(c_i) = x_i$ by using Lagrange interpolation. $\square$

Proposition 5.2. *Assume k is uncountable. Let X be a projective scheme over k, and let $\mathcal{L}$ be an ample line bundle on X. If X is* not *mildly bounded over k, then there is a smooth projective connected curve C over k such that, for every $d \geq 1$, the moduli scheme $\underline{\mathrm{Hom}}_k^{\geq d}(C, X)$ of morphisms $f : C \to X$ with $\deg_C f^*\mathcal{L} \geq d$ is positive-dimensional.*

Proof. If X has a rational curve, then we can take $C = \mathbb{P}^1_k$. (We do not use that k is uncountable here.) Thus, we may and do assume that X has no rational curves. Let C be a smooth projective connected curve over k which does not satisfy the mildly boundedness condition. Note that, as X has no rational curves, for every integer $d \geq 1$, the scheme $\underline{\mathrm{Hom}}_k^{\leq d}(C, X)$ is a proper scheme over k and that, for every c in C and x in X, the set $\mathrm{Hom}_k^{\leq d}([C, c], [X, x])$ is finite; see [34, §3]. Therefore, for every $d \geq 1$, the set $\mathrm{Hom}_k^{\geq d}(C, X)$ must have infinitely many k-points (otherwise C would satisfy the mildly boundedness condition). Suppose that this scheme is zero-dimensional. Then its k-points form a countable infinite set. Let $f_1, \ldots$ be the elements of $\mathrm{Hom}_k^{\geq d}(C, X)$. For every $i \neq j$, the locus $C_{ij} \subset C(k)$ where f_i and f_j agree consists of finitely many points, as X is separated over k. Since $C(k)$ is uncountable, there exists a point $c \in C(k)$ such that $c \notin C_{ij}$ for all $i \neq j$. As X over k does not satisfy the mildly boundedness condition with respect to C, there must be an $x \in X(k)$ such that

$$H = \mathrm{Hom}_k^{\geq d}([C, c], [X, x])$$

is infinite. However, by the choice of c, the set H can only contain at most one element of $\{f_1, f_2, \ldots\}$. Hence, we obtain a contradiction and we conclude that $\mathrm{Hom}_k^{\geq d}(C, X)$ must be uncountable, so that the scheme $\underline{\mathrm{Hom}}_k^{\geq d}(C, X)$ has (a component of) dimension at least one. ☐

Remark 5.3. Let k be an uncountable algebraically closed field and suppose that X is a variety over k which is *not* mildly bounded. Then, the argument used in the proof of Proposition 5.2 shows that there is a smooth quasi-projective connected curve C over k such that the set of non-constant morphisms is uncountable.

Lemma 5.4. *Let X be a projective variety over k. Consider the following statements.*

(1) The variety X is mildly bounded over k.
(2) The variety X has no rational curves.
(3) For every curve C, there is an integer $d \geq 1$ such that $\mathrm{Hom}^{\geq d}(C, X)$ is zero-dimensional, i.e., $\mathrm{Hom}(C, X)$ has only finitely many positive-dimensional components.
(4) X is groupless.

Then we have (1) $\implies$ (2), (4) $\implies$ (2), *and* (3) $\implies$ (4). *If k is uncountable, then we also have* (3) $\implies$ (1).

Proof. The implication (1) $\implies$ (2) follows from Proposition 5.1. The implication (4) $\implies$ (2) is an immediate consequence of the definitions. For the implication (3) $\implies$ (4), assume that X is not groupless. Then, by [34, Lemma 2.7], there is a non-zero abelian variety A and a non-constant morphism $\phi\colon A \to X$. Let $\iota\colon C \hookrightarrow A$ be a curve which is not contracted by ϕ. Then ι can be composed with any endomorphism of A. In particular, it can be composed with multiplication by $n \in \mathbb{Z}_{>0}$ and any translation with a point of $A(k)$. This gives infinitely many components of (strictly) positive dimension in $\mathrm{Hom}(C, X)$. This shows that (3) $\implies$ (4). Finally, to conclude the proof, we may assume that k is uncountable. Now, the implication (3) $\implies$ (1) follows from Proposition 5.2. ☐

Note that Conjecture 1.11 predicts the equivalence of (1) and (2). In fact, we conjecture something stronger.

Conjecture 5.5. *In the situation of Lemma 5.4,* (1) $\iff$ (2) *and* (3) $\iff$ (4).

Example 5.6. Let X be a smooth projective curve of genus at least two over k. Then we claim that X is mildly bounded. Indeed, if $C \to X$ is a morphism from a smooth irreducible quasi-projective curve to X, then it extends to a morphism from its smooth projectivisation $\overline{C}$ to X, and the set of non-constant morphisms $\overline{C} \to X$ is finite by De Franchis's theorem. In particular, we see that X is mildly bounded over k. Below, we will give a simpler (more direct) proof of the mild boundedness of any smooth quasi-projective connected curve X which is neither isomorphic to $\mathbb{A}^1_k$ nor $\mathbb{P}^1_k$.

To prove that abelian varieties are mildly bounded we will use that most smooth curves inject into their generalised Jacobian (also referred to as the semi-Albanese variety). This will be explained in more detail in the rest of this section. Definitions and constructions of this (semi-)Albanese variety can be found in [51, 60]. If X is a geometrically integral variety over k, we let $\mathrm{Alb}(X)$ be its semi-Albanese variety over k; this exists by [51, Corollary A.11.(i)].

Given a point x in $X(k)$, we will refer to the universal morphism $X \to \mathrm{Alb}(X)$ associated to (X, x) as an Abel-Jacobi morphism. We will usually suppress the choice of a base point in X.

Lemma 5.7. *Let $\overline{C}$ be a smooth projective connected curve of genus g and let $C \subset \overline{C}$ be a dense open such that $\#(\overline{C} \setminus C) = r > 0$. Fix an Abel-Jacobi map $\iota : C \to \mathrm{Alb}(C)$ to the Albanese variety of C (associated to the choice of a base point in $C(k)$). For any integer $N \geq 0$, let $\rho_N \colon C^N \to \mathrm{Alb}(C)$ be given by $(c_1, \ldots, c_N) \mapsto \iota(c_1) + \ldots + \iota(c_N)$. Then $\rho_{2g+2r-2}$ is surjective.*

Proof. In [51, Proposition A.3], it is already claimed that there is an N such that ρ_N is surjective. As we are considering curves, we know more about the structure of $\mathrm{Alb}(C)$. Let D be the divisor of $\overline{C}$ obtained by taking each of the points of $\overline{C} \setminus C$ once. Then there is a generalised Jacobian in the sense of [60, Chap. V] associated to D. Using [59, Lemme 6], we see that this generalised Jacobian is isomorphic to $\mathrm{Alb}(C)$.

Hence, $\mathrm{Alb}(C)$ is an extension of $\mathrm{Jac}(\overline{C})$ by $\mathbb{G}_{\mathrm{m}}^{r-1}$ and ρ_{g+r-1} is birational. In particular, the image $\mathrm{Im}(\rho_{g+r-1})$ contains a dense open subset of $\mathrm{Alb}(C)$. Therefore, for any point $P \in \mathrm{Alb}(C)$, the intersection of $\mathrm{Im}(\rho_{g+r-1})$ with $P - \mathrm{Im}(\rho_{g+r-1})$ must be non-empty. Hence, the morphism $\rho_{2g+2r-2}$ is surjective. $\square$

Proposition 5.8. *Let $r > 0$ be an integer. Let $\overline{C}$ be a smooth projective connected curve of genus g and let $C \subset \overline{C}$ be a dense open subscheme such that $\#(\overline{C} \setminus C) = r$. Then, there are points $c_0, \ldots, c_{2(g+r-1)^2} \in C(k)$ such that, for every semi-abelian variety X over k and every $x_0, \ldots, x_{2(g+r-1)^2} \in X(k)$, the set*

$$\mathrm{Hom}_k\big([C, (c_0, \ldots, c_{2(g+r-1)^2})], [X, (x_0, \ldots, x_{2(g+r-1)^2})]\big)$$

has at most one element.

Proof. Let X be any semi-abelian variety over k. Let $c_0 \in C(k)$ be a point and let A be the Albanese variety of C over k together with the associated Abel-Jacobi map $\iota : C \to A$ mapping c_0 to the identity in A.

Note that a morphism $C \to X$ mapping c_0 to the identity in X, gives rise to a morphism $A \to X$ of group schemes (see [51, Proposition A.3]). We will prove that such morphisms are determined by the image of at most $g + r - 1$ carefully chosen points on A. In order to find these points, note that there is an exact sequence $1 \to \mathbb{G}_{\mathrm{m}}^{r-1} \to A \to J \to 0$, where J is the Jacobian of $\overline{C}$, cf. the description of the generalised Jacobian given in the proof of Lemma 5.7.

We will first look at morphisms $J \to X$. Suppose we have an isogeny $J_1 \times \ldots \times J_n \to J$, such that $J_1, \ldots, J_n$ are simple abelian varieties over k. Then $n \leq g$ and by composition we get an injection

$$\mathrm{Hom}_{\mathrm{Grp}/k}(J, X) \hookrightarrow \mathrm{Hom}_{\mathrm{Grp}/k}(J_1 \times \ldots \times J_n, X).$$

For $1 \leq i \leq n$, let $g_i \in J_i(k)$ be a point of infinite order and note that g_i generates a (Zariski) dense subgroup of J_i. Therefore, any morphism of group schemes $\psi : J_i \to X$ is determined by the image $\psi(g_i)$ of g_i. Now, for $1 \leq i \leq n$, we let j_i be the image of

$(0,\ldots,0,g_i,0,\ldots,0) \in J_1 \times \ldots \times J_n$ in J. Then, a morphism of semi-abelian varieties $\psi : J \to X$ is determined by the images $\psi(j_1),\ldots,\psi(j_n)$ of $j_1,\ldots,j_n$.

On the other hand, group morphisms $\mathbb{G}_{\mathrm{m}}^{r-1} \to X$ are determined by the images of the points $\ell_1 = (2,1,\ldots,1), \ell_2 = (1,2,1,\ldots,1),\ldots,$ and $\ell_{r-1} = (1,\ldots,1,2)$ in $\mathbb{G}_{\mathrm{m}}^{r-1}(k)$.

In particular, if $\ell_r,\ldots,\ell_{r+n-1} \in J(k)$ are points mapping to $j_1,\ldots,j_n$, then we claim that a morphism of group schemes $\varphi\colon A \to X$ is determined by $\varphi(\ell_1),\ldots,\varphi(\ell_{r+n-1})$. Indeed, if φ' is another morphism with the same images, then $\varphi - \varphi'$ is trivial on $\mathbb{G}_{\mathrm{m}}^{r-1}$ and factors through J, where it is also trivial.

By Lemma 5.7, for each $1 \leq i \leq n+r-1$, there exist points $c_{i,1},\ldots,c_{i,2g+2r-2} \in C(k)$ such that $\iota(c_{i,1})+\ldots+\iota(c_{i,2g+2r-2}) = \ell_i$. Now, we claim that, for any $1+(n+r-1)(2g+2r-2)$-tuple of points $x_0, x_{1,1}, x_{1,2},\ldots,x_{n+r-1,2g+2r-2} \in X(k)$, the set

$$H = \mathrm{Hom}([C,(c_0,c_{1,1},\ldots,c_{n+r-1,2g+2r-2})],[X,(x_0,x_{1,1},\ldots x_{n+r-1,2g+2r-2})])$$

is finite. Indeed, if we change the group structure on X such that x_0 becomes the identity, then any morphism $f\colon C \to X$ in H gives rise to a homomorphism of group schemes $h\colon A \to X$. Now, by definition, for each $i = 1,\ldots,n+r-1$, we have that

$$h(\ell_i) = h(\iota(c_{i,1})) + \ldots + h(\iota(c_{i,2g+2r-2})) = f(c_{i,1}) + \ldots + f(c_{i,2g+2r-2}) = x_{i,1} + \ldots + x_{i,2g+2r-2}$$

is fixed. This implies that H can have at most one element. The proposition is now proven by relabelling the points in C and appending some extra points in case $n < g$. $\square$

We obtain the following uniform finiteness statement for (not necessarily hyperbolic) curves.

Lemma 5.9. *Let C be a smooth affine curve over k. Then, there is an integer $m \geq 1$ and points $c_1,\ldots,c_m \in C(k)$ such that, for every projective variety X over k of dimension at most one without rational curves over k and every $x_1,\ldots,x_m \in X(k)$, the set*

$$\mathrm{Hom}_k([C,(c_1,\ldots,c_m)],[X,(x_1,\ldots,x_m)])$$

is finite.

Proof. It suffices to prove the required finiteness statement for smooth projective connected varieties X of dimension at most one without rational curves. However, since a smooth projective connected curve X over k with no rational curves embeds into an abelian variety (e.g., the Jacobian of X), the result follows from Proposition 5.8. $\square$

Proof of Proposition 1.9. The mildly boundedness of semi-abelian varieties follows immediately from Proposition 5.8. $\square$

In fact, we can prove another (slightly less effective) finiteness result for semi-abelian varieties. We briefly explain this in the following remark.

Remark 5.10. If Y is a variety over k, then there are $y_1,\ldots,y_m$ in $Y(k)$ such that, for every semi-abelian variety X and every $x_1,\ldots,x_m$ in $X(k)$, the set

$$\mathrm{Hom}([Y,(y_1,\ldots,y_m)],[X,(x_1,\ldots,x_m)])$$

is finite. To prove this, replacing Y by a suitable dense open subscheme if necessary, we may and do assume that Y is a smooth affine integral variety which maps injectively into its Albanese variety $\mathrm{Alb}(Y)$. Then, we pick a high power Y^n which generates $\mathrm{Alb}(Y)$ and we use the above line of reasoning to construct an integer $m \geq 1$ and points $y_1,\ldots,y_m$ with the desired property. The integer m and the points $y_1,\ldots,y_m$ only depend on Y (i.e., they do not depend on X).

Remark 5.11. If $\mathbb{F}$ is an algebraically closed field of positive characteristic with positive transcendence degree over its prime field and A is an abelian variety over $\mathbb{F}$, then the proof above shows that A is mildly bounded over $\mathbb{F}$. However, if E is an elliptic curve over $\overline{\mathbb{F}_p}$, then E is not mildly bounded over $\overline{\mathbb{F}_p}$, so that the assumption that $\mathbb{F}$ has transcendence degree at least one over its prime field is necessary in the previous statement.

Corollary 5.12. *Let X be a variety over k which admits a quasi-finite morphism to a semi-abelian variety over k. Then, for any field extension $k \subset L$, the variety X_L is mildly bounded over L.*

Proof. Let G be a semi-abelian variety over k and let $X \to G$ be a quasi-finite morphism of varieties over k. Note that the induced morphism $X_L \to G_L$ is quasi-finite. Since G_L is mildly bounded over L (Proposition 5.8), the quasi-finiteness of $X_L \to G_L$ implies that X_L is mildly bounded over L. □

Corollary 5.13. *Let X be an integral one-dimensional variety over k whose normalisation is not isomorphic to $\mathbb{A}^1_k$ nor $\mathbb{P}^1_k$. Then X is mildly bounded over k.*

Proof. Let $\widetilde{X}$ be the normalisation of X. As $\widetilde{X}$ is neither $\mathbb{A}^1_k$ nor $\mathbb{P}^1_k$, the Abel-Jacobi map $\widetilde{X} \to \mathrm{Alb}(\widetilde{X})$ is injective. Hence, by Corollary 5.12, the curve $\widetilde{X}$ is mildly bounded over k. As any non-constant map from a smooth curve C over k to X factors uniquely through $\widetilde{X}$, it follows that X is also mildly bounded over k. □

The results in this section are motivated by Conjecture 1.11 which predicts that a projective variety with no rational curves is mildly bounded. Note that Corollary 5.13 proves Conjecture 1.11 for one-dimensional projective varieties. We now prove Theorem 1.12 which says that our conjecture holds for projective surfaces, under suitable assumptions on the Albanese map.

Proof of Theorem 1.12. Let X be a projective surface with no rational curves (as in the statement), let A be an abelian variety, and let $f\colon X \to A$ be a morphism which is generically finite onto its image. Let $X \to B \to A$ be the Stein factorisation of f, where $X \to B$ is a morphism with connected fibres and $B \to A$ is finite. Since A is mildly bounded over k (Proposition 1.9) and $B \to A$ has finite fibres, it follows that B is mildly bounded over k. Let $B' \subset B$ be the image of the morphism $X \to B$. Since the morphism $X \to B$ is generically finite onto its image B' with geometrically connected fibres, we have that the induced morphism $p : X \to B'$ is birational. Let $Z \subset B'$ be a proper closed subset such that p induces an isomorphism $X \setminus \Delta \to B' \setminus Z$, where $\Delta = p^{-1}(Z)$. Now, to show that X is mildly bounded, let C be a smooth connected curve over k. Choose an integer $n \geq 1$ and points $c_1, \ldots, c_n$ in $C(k)$ such that, for every $b_1, \ldots, b_n$ in $B'(k)$, the set

$$\mathrm{Hom}_k([C, (c_1, \ldots, c_n)], [B', (b_1, \ldots, b_n)])$$

is finite. Let $x_1, \ldots, x_n$ be points in $X(k)$. To show that the set

$$\mathrm{Hom}_k([C, (c_1, \ldots, c_n)], [X, (x_1, \ldots, x_n)])$$

is finite, we note that it is the union of its subsets

$$\mathrm{Hom}_k([C, (c_1, \ldots, c_n)], [X, (x_1, \ldots, x_n)]) \setminus \mathrm{Hom}_k(C, \Delta)$$

and

$$\mathrm{Hom}_k([C, (c_1, \ldots, c_n)], [\Delta, (x_1, \ldots, x_n)]).$$

The former set maps injectively to $\mathrm{Hom}([C,(c_1,\ldots,c_n)],[B',(p(x_1),\ldots,p(x_n))])$, and is therefore finite. Moreover, since $\Delta\subset X$ is at most one-dimensional and does not admit a non-constant morphism from $\mathbb{P}^1_k$ (by our assumption that X has no rational curves), it follows from Corollary 5.13 that

$$\mathrm{Hom}_k([C,(c_1,\ldots,c_n)],[\Delta,(x_1,\ldots,x_n)])$$

is finite. (This set is by definition empty if there is an i with $x_i\notin\Delta$.) We conclude that X is mildly bounded, as required. □

6. Applying Silverman's specialisation theorem

For a regular extension of fields $K\subset L$ and an abelian variety A over L, the trace of A with respect to $K\subset L$ is a universal map $T_L\to A$ from an abelian variety T over K in the sense that all such other maps factor through it, see for example [45, chap. VII] or [11]. The trace can be viewed as a measure of how close A is to being a constant abelian variety.

In this section, as usual, we let k be an algebraically closed field of characteristic zero.

Lemma 6.1. *Let C be an integral curve over k and let $X\to C$ be an abelian scheme such that the $K(C)/k$-trace of the abelian variety $X_{K(C)}$ is trivial. Then there is a point $c\in C(k)$ such that for every $x\in X(k)$ there are only finitely many sections $\sigma\colon C\to X$ with $\sigma(c)=x$.*

Proof. Since $X_{K(C)}$ has trivial $K(C)/k$-trace, by the theorem of Lang-Néron [11], the group $X(C)$ of sections of $X\to C$ is finitely generated. We now descend all the necessary data (including the elements of the finitely generated group $X(C)$) to a finitely generated subfield of k.

Let L be a finitely generated field over $\mathbb{Q}$ contained inside k, let $\mathcal{C}$ be an integral curve over L, let $\mathcal{C}_k\cong C$ be an isomorphism over k, and let $\mathcal{X}\to\mathcal{C}$ be an abelian scheme such that $\mathcal{X}_k\to\mathcal{C}_k$ is isomorphic to $X\to C$ over k and such that the group of sections $\mathcal{X}(\mathcal{C})$ of $\mathcal{X}\to\mathcal{C}$ equals $X(C)$.

Note that the $K(\mathcal{C})/L$-trace of $\mathcal{X}_{K(\mathcal{C})}$ is zero. Let $\{\sigma_1,\ldots,\sigma_r\}$ be a basis for the free part of $\mathcal{X}(\mathcal{C})=\mathcal{X}_{K(\mathcal{C})}(K(\mathcal{C}))$. Since L is a finitely generated field over $\mathbb{Q}$, by Silverman's specialisation theorem [69, Theorem 1], there is a finite field extension L'/L contained in k and a point $c\in\mathcal{C}(L')\subset C(k)$ such that $\sigma_1(c),\ldots,\sigma_r(c)\in X_c(k)$ are still independent.

Let $c\in C(k)$ be such a point. Then, for a fixed $x\in X(k)$, there are only finitely many $\sigma\in X(C)$ with $\sigma(c)=x$. Indeed, for each of the finitely many torsion sections $\tau\in X(C)^{\mathrm{tors}}$, there is at most one tuple $(n_1,\ldots,n_r)$ of integers, such that $(\tau+n_1\sigma_1+\ldots+n_r\sigma_r)(c)=x$, due to the independence of $\sigma_1(c),\ldots,\sigma_r(c)$. □

We now combine Lemma 6.1 with the "uniform" mildly boundedness of abelian varieties (Proposition 5.8) to prove the following result.

Lemma 6.2. *Let k be an algebraically closed field of characteristic zero. Let C be an integral curve over k and let $\mathcal{X}\to C$ be an abelian scheme. Then there is an $n\geq 1$ and $c_1,\ldots,c_n\in C(k)$ such that, for every $x_1,\ldots,x_n\in\mathcal{X}(k)$, there are only finitely many sections $\sigma:C\to\mathcal{X}$ of $\mathcal{X}\to C$ with $\sigma(c_i)=x_i$.*

Proof. The proof goes by induction on the relative dimension of $\mathcal{X}\to C$. The case of relative dimension 0 is trivial. Now assume the statement is true for all abelian schemes of smaller relative dimension.

Let $T\to\mathcal{X}_{K(C)}$ be the $K(C)/k$-trace of $\mathcal{X}_{K(C)}$. In case, $T=0$, the result follows from Lemma 6.1, so assume $T\neq 0$. Let W be the cokernel. Shrinking C if necessary, we can spread this out to an exact sequence

$$\mathcal{T}\to\mathcal{X}\to\mathcal{W}$$

of abelian varieties over C, such that $\mathcal{T}$ is the base change of an abelian variety over k and the relative dimension of $\mathcal{W} \to C$ is smaller than that of $\mathcal{X} \to C$. It now suffices to prove that $\mathcal{T}$ and $\mathcal{W}$ have the property as stated in the statement of the lemma.

Indeed, suppose $c_1, \ldots, c_{n_t}, d_1, \ldots, d_{n_w} \in C(k)$ are points such that

$$\mathrm{Hom}_C([C, (c_1, \ldots, c_{n_t})], [\mathcal{T}, (t_1, \ldots, t_{n_t})])$$

$$\text{and} \qquad \mathrm{Hom}_C([C, (d_1, \ldots, d_{n_w})], [\mathcal{W}, (w_1, \ldots, w_{n_w})])$$

are finite for any $t_1, \ldots, t_{n_t} \in \mathcal{T}(k)$ and $w_1, \ldots, w_{n_w} \in \mathcal{W}(k)$. Now consider morphisms

$$\varphi, \varphi' \in \mathrm{Hom}_C([C, (c_1, \ldots, c_{n_t}, d_1, \ldots, d_{n_w})], [\mathcal{X}, (x_1, \ldots, x_{n_t+n_w})]),$$

where $x_1, \ldots, x_{n_t+n_w} \in \mathcal{X}(k)$ are arbitrary points. For these morphisms, there are only finitely many possibilities for the composed map $C \to \mathcal{X} \to \mathcal{W}$. If we suppose that φ and φ' give rise to the same composed map $C \to \mathcal{W}$, then their difference $\nu = \varphi - \varphi'$ factors through a morphism $\rho\colon C \to \mathcal{T}$. Moreover, $\rho(c_i)$ maps to $\nu(c_i) = 0 \in \mathcal{X}$ for each $i \in \{1, \ldots, n_t\}$. As the kernel of $\mathcal{T} \to \mathcal{X}$ is finite, this means that there are only finitely many candidates for the points $\rho(c_i)$. For each of the finitely many choices for $\rho(c_i) \in \mathcal{T}(k)$, there are only finitely many possibilities for ρ. Hence, there are indeed finitely many possibilities for φ as well.

To conclude the proof, let us show that $\mathcal{T}$ and $\mathcal{W}$ have the property as in the statement of the lemma. In the case of $\mathcal{T}$, we write $\mathcal{T}$ as $A \times_k K(C)$ for a certain abelian variety A over k. Then there is a bijection

$$\mathrm{Hom}_C(C, A \times_k C) = \mathrm{Hom}_k(C, A),$$

and the result follows from Proposition 1.9. In the case of $\mathcal{W}$, as the relative dimension of $\mathcal{W}$ over C is smaller than the relative dimension of $\mathcal{X}$ over C, the result follows from the induction hypothesis. ☐

The following lemma generalises the above lemma (Lemma 6.2) for abelian schemes to semi-abelian schemes.

Lemma 6.3. *Let k be an algebraically closed field of characteristic zero. Let C be an integral curve over k and let $\mathcal{X} \to C$ be a semi-abelian scheme. Then there is an $n \geq 1$ and $c_1, \ldots, c_n \in C(k)$ such that, for every $x_1, \ldots, x_n \in \mathcal{X}(k)$, there are only finitely many sections $\sigma : C \to \mathcal{X}$ with $\sigma(c_i) = x_i$.*

Proof. Note that the generic fibre $\mathcal{X}_{K(C)}$ of $\mathcal{X} \to C$ is an extension by a torus over $K(C)$ and an abelian variety over $K(C)$. Therefore, we may choose a smooth integral curve D over k and a quasi-finite (flat dominant) morphism $D \to C$ such that $\mathcal{X}_D = \mathcal{X} \times_C D$ is an extension of $\mathbb{G}^{\ell}_{m,D}$ and an abelian scheme $\mathcal{A}$ over D, i.e., there is an exact sequence

$$1 \to \mathbb{G}^{\ell}_{\mathrm{m},D} \to \mathcal{X}_D \xrightarrow{\nu} \mathcal{A} \to 0.$$

By Lemma 6.2, we find $d_1, \ldots, d_m \in D(k)$ such that for every $a_1, \ldots, a_m \in \mathcal{A}(k)$ the set

$$\mathrm{Hom}_D([D, (d_1, \ldots, d_m)], [\mathcal{A}, (a_1, \ldots, a_m)])$$

of m-pointed sections is finite. Using Proposition 5.8, we find points $d_{m+1}, \ldots, d_n \in D(k)$ such that, for every $g_{m+1}, \ldots, g_n \in \mathbb{G}^{\ell}_{\mathrm{m},D}(k)$, the set

$$\mathrm{Hom}_D([D, (d_{m+1}, \ldots, d_n), [\mathbb{G}^{\ell}_{\mathrm{m},D}, (g_{m+1}, \ldots, g_n)])$$

of $(n-m)$-pointed sections is finite. Let $c_1, \ldots, c_n \in C(k)$ be the images of the points $d_1, \ldots, d_n$. These points have the desired property using arguments similar to those used in the proof of Lemma 6.2.

Indeed, let $x_1,\dots,x_n \in \mathcal{X}(k)$ be arbitrary. Then any

$$\varphi \in \mathrm{Hom}_C([C,(c_1,\dots,c_n)],[\mathcal{X},(x_1,\dots,x_n)])$$

lifts to a morphism

$$\varphi_D \in H_{y_1,\dots,y_n} := \mathrm{Hom}_D([D,(d_1,\dots,d_n)],[\mathcal{X}_D,(y_1,\dots,y_n)]),$$

where $y_1,\dots,y_n \in \mathcal{X}_D(k)$ are lifts of $x_1,\dots,x_n$. Since there are only finitely many such lifts, it suffices to prove that $H_{y_1,\dots,y_n}$ is finite. Let $\varphi_D,\varphi'_D \in H_{y_1,\dots,y_n}$ be two elements, and let $\sigma,\sigma'\colon D\to\mathcal{A}$ be the composition of φ_D and $\varphi_{D'}$ with ν. Then σ and σ' lie in the finite set

$$\mathrm{Hom}_D([D,(d_1,\dots,d_m)],[\mathcal{A},(\nu(y_1),\dots,\nu(y_n))])$$

Assuming σ and σ' are equal, we will now prove that there are only finitely many possibilities for $\varphi_D-\varphi'_D$. Indeed, this difference $\varphi_D-\varphi'_D$ factors through $\mathbb{G}^{\ell}_{\mathrm{m},D}$ and lies in the finite set

$$\mathrm{Hom}_D([D,(d_{m+1},\dots,d_n)],[\mathbb{G}^{\ell}_{\mathrm{m},D},(1,\dots,1)]).$$

This proves that there are only finitely many elements in $H_{y_1,\dots,y_n}$, which concludes the proof of the lemma. □

We now combine the above results with the fact that semi-abelian varieties are mildly bounded in a "uniform" sense (Proposition 5.8).

Corollary 6.4. *Let k be an algebraically closed field of characteristic zero. Let S be an integral variety over k and let $f:\mathcal{X}\to S$ be a semi-abelian scheme. Assume that for every smooth curve C over k, the set of non-constant morphisms $C\to S$ is finite. Then the variety $\mathcal{X}$ is mildly bounded over k.*

Proof. Let C be a smooth connected curve over k. Choose a point c_0 in $C(k)$. Since there are only finitely many non-constant morphisms $C\to S$ (by assumption), for any x_0 in $X(k)$ and every $\rho: C\to X$ with $\rho(c_0)=x_0$, there are only finitely many possibilities for the composed morphism $\nu: C\to X\to S$. We now choose x_0 in $X(k)$ and consider such morphisms $\rho: C\to X$ such that $\rho(c_0)=x_0$.

If ν maps C onto a point (which has to be $f(x_0)$), then we already saw in Proposition 5.8 that there exists an integer $m\geq 1$ and points $c_1,\dots,c_m\in C(k)$, only depending on C (not depending on x_0), such that

$$\mathrm{Hom}_k([C,(c_0,c_1,\dots,c_m)],[\mathcal{X}_{f(x_0)},(x_0,x_1,\dots,x_m)])$$

is finite for any $x_1,\dots,x_m\in\mathcal{X}_{f(x_0)}(k)$.

In the case that ν is non-constant, we consider the base change $\mathcal{X}_\nu=\mathcal{X}\times_{f,S,\nu}C$ which is a semi-abelian scheme over C. Then by Lemma 6.3, there are points $c_{\nu,1},\dots,c_{\nu,n_\nu}\in C(k)$, such that

$$\mathrm{Hom}_C([C,(c_{\nu,1},\dots,c_{\nu,n_\nu})],[\mathcal{X}_\nu,(x_1,\dots,x_{n_\nu})])$$

is finite for any $x_1,\dots,x_{n_\nu}\in\mathcal{X}_\nu(k)$. Note that

$$\{c_0,c_1,\dots,c_m\}\cup\bigcup_{\nu}\{c_{\nu,1},\dots,c_{\nu,n_\nu}\}$$

is a finite subset of $C(k)$ by our assumption that the set of non-constant morphisms $C\to S$ is finite. In particular, if $c_0,\dots,c_n$ are its elements, then

$$\mathrm{Hom}([C,(c_0,\dots,c_n)],[\mathcal{X},(x_0,\dots,x_n)])$$

is finite for any $x_0,\dots,x_n\in\mathcal{X}(k)$. This proves that $\mathcal{X}$ is mildly bounded over k. □

Proof of Theorem 1.10. By De Franchis-Severi's theorem, the hyperbolic curve S satisfies the finiteness statement required to apply Corollary 6.4. $\square$

We now show that an analogue of Theorem 1.10 also holds for the universal Jacobian. To prove this statement, one replaces the use of De Franchis-Severi in the proof of Theorem 1.10 by the finiteness theorem of Arakelov-Parshin.

Corollary 6.5. *Let k be an algebraically closed field of characteristic zero, let $N \geq 3$ be an integer, let $g \geq 2$, and let M be the fine moduli space of smooth proper geometrically connected genus g curves with level N structure over k. Let $X \to M$ be the universal curve (with level N structure) and let $\mathrm{Jac}(X) \to M$ be its Jacobian. Then the quasi-projective variety $\mathrm{Jac}(X)$ is mildly bounded over k.*

Proof. By Arakelov-Parshin's theorem [2, 56] (*formerly* the Shafarevich conjecture for curves), for every curve C over k, the set of C-isomorphism classes of non-isotrivial smooth proper curves $X \to C$ of genus g (with $g \geq 2$) is finite. In particular (by the definition of the moduli space M), for every curve C over k, the set of non-constant morphisms $C \to M$ is finite. Therefore, the variety M satisfies the finiteness property required to apply Corollary 6.4 (with $S := M$ and $\mathcal{X} := \mathrm{Jac}(X)$). $\square$

To prove the final result of this section, we will need the following well-known (and very simple) lemma which says that every variety is "locally" hyperbolic (in any sense of the word).

Lemma 6.6. *Let S be an integral variety over k. Then, there is a dense open subscheme $U \subset S$ such that U is arithmetically hyperbolic over k and, for every smooth curve C over k, the set of non-constant morphisms $C \to U$ is finite.*

Proof. Note that $X := \mathbb{A}^1_k \setminus \{0, 1\}$ is arithmetically hyperbolic over k (by Siegel-Mahler-Lang's theorem). Moreover, for an integral curve C over k, the De Franchis-Severi theorem implies that the set of non-constant morphisms $C \to X$ is finite. Now, to prove the lemma, we may and do assume that S is affine, say $S \subset \mathbb{A}^N_k$. Choose a, b in $\mathbb{A}^1(k)$ such that $S \cap (\mathbb{A}^1_k \setminus \{a, b\})^N$ is non-empty (hence dense and open in S). Define $U := S \cap (\mathbb{A}^1_k \setminus \{a, b\})^N$ and note that U maps with finite fibres to X^N. In particular, U is arithmetically hyperbolic over k and, for every integral curve C over k, the set of non-constant morphisms $C \to U$ is finite. This proves the lemma. $\square$

We now record the following application of the above results. Our motivation for proving the following result is that it is a first step towards reducing part of Lang–Vojta's conjecture concerning rational points on hyperbolic projective varieties to $\overline{\mathbb{Q}}$.

Theorem 6.7. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero, and let X be a semi-abelian variety over L. Then there exists an integral variety S over k whose function field $K(S)$ is a subfield of L, a semi-abelian scheme $\mathcal{X}$ over S such that $\mathcal{X} \times_k L \cong X$ over L and $\mathcal{X}$ is mildly bounded over k.*

Proof. By spreading out arguments, there exists a finitely generated k-algebra $A \subset L$ with $S := \operatorname{Spec} A$ an integral variety over k, and a semi-abelian scheme $\mathcal{X} \to S$ such that $\mathcal{X}_L \cong X$. Replacing S by a dense open subscheme if necessary, by Lemma 6.6, the variety S is arithmetically hyperbolic over k, and for every curve C over k, the set of non-constant morphisms $C \to S$ is finite. The result now follows from Corollary 6.4. $\square$

7. A criterion for persistence of arithmetic hyperbolicity

Recall that a variety over k is arithmetically hyperbolic over k if every model for X over a $\mathbb{Z}$-finitely generated subalgebra of k has only finitely many sections (Definition 1.18); see also [30, 33, 35, 38]. The Lang–Vojta conjecture predicts that arithmetic hyperbolicity persists over field extensions, i.e., if X is arithmetically hyperbolic over k and L is an algebraically closed field containing k, then X_L should be arithmetically hyperbolic over L; this is the Arithmetic Persistence Conjecture (Conjecture 1.20). Although the Arithmetic Persistence Conjecture is not known to hold in general, the following result is useful for verifying it in many cases.

Theorem 7.1 (Criterion for Persistence)**.** *Let X be an arithmetically hyperbolic variety over k such that X_K is mildly bounded for all subfields $k \subset K \subset L$. Then X_L is arithmetically hyperbolic over L.*

Proof. See [33, Theorem 4.4]. □

As is shown in [33], Theorem 7.1 follows from the following result.

Theorem 7.2. *Let X be an arithmetically hyperbolic mildly bounded variety over k. If L/k is an extension of algebraically closed fields of transcendence degree one, then X_L is arithmetically hyperbolic over L.*

Proof. See [33, Lemma 4.2]. □

Applications of the above two results are given in [3], [31], [33, §4.2], and further results are also obtained in [38]. As a first application of the results of this paper and the above criterion for persistence of arithmetic hyperbolicity, we obtain the following corollary for projective varieties.

Corollary 7.3. *Let X be a projective arithmetically hyperbolic mildly bounded variety over k. Then, for every field extension $k \subset L$, the projective variety X_L is arithmetically hyperbolic over L.*

Proof. It follows from the definition of arithmetic hyperbolicity that we may assume L has finite transcendence degree over k. Then, by Corollary 1.16, as X is mildly bounded and projective over k, for every subfield $k \subset K \subset L$, the variety X_K is mildly bounded over K. Thus, by the above criterion for persistence (Theorem 7.1), we conclude that X_L is arithmetically hyperbolic over L. □

As another application of the results of this paper, we can show that the Arithmetic Persistence Conjecture holds for varieties which admit a quasi-finite morphism to some semi-abelian variety.

Theorem 7.4. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let X be a variety over k which admits a quasi-finite morphism to some semi-abelian variety over k. Then X is arithmetically hyperbolic over k if and only if X_L is arithmetically hyperbolic over L.*

Proof. Note that, by Corollary 5.12, for every subfield $k \subset K \subset L$, the variety X_K is mildly bounded over K. Therefore, the result follows from the criterion for persistence (Theorem 7.1). □

Proof of Theorem 1.21. This follows from Theorem 7.4. □

In the following remark we explain why our results (and definitions) simplify proofs of Faltings's finiteness theorem for finitely generated $\mathbb{Z}$-algebras on affine opens of abelian varieties.

Remark 7.5 (Faltings's theorem over finitely generated fields via specialisation)**.** In [22] Faltings proved that, if A is an abelian variety over $\overline{\mathbb{Q}}$ and $D \subset A$ is an ample divisor, the affine variety $A - D$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$. That is, for every number field K, every finite set of finite places S of K, every model $\mathcal{A}$ for A over $\mathcal{O}_{K,S}$ and every model $\mathcal{D} \subset \mathcal{A}$ for $D \subset A$ over $\mathcal{O}_{K,S}$, the set $(\mathcal{A} \setminus \mathcal{D})(\mathcal{O}_{K,S})$ is finite. One sometimes also says that the set of (D, S)-integral points on A is finite. We will now use our results to prove the following extension of Faltings's theorem which does not seem to appear explicitly in the literature, although related results are mentioned in [21, §5].

Theorem 7.6 (Faltings + ϵ)**.** *Let L be an algebraically closed field of characteristic zero, let A be an abelian variety over L, and let $D \subset A$ be an ample divisor. Then $A - D$ is arithmetically hyperbolic over L.*

Proof. (We use Faltings's finiteness theorem over $\overline{\mathbb{Q}}$, the fact that abelian varieties are mildly bounded, and a specialisation argument.)

First, we may and do assume that L has finite transcendence degree over $\mathbb{Q}$ and contains $\overline{\mathbb{Q}}$. If the transcendence degree $n = \mathrm{trdeg}_{\mathbb{Q}}(L)$ of L over $\mathbb{Q}$ is zero, then $L = \overline{\mathbb{Q}}$ so that the statement follows from Faltings's theorem [22, Corollary 6.2].

Now, assume that $n > 0$ and choose an algebraically closed subfield $k \subset L$ such that $\mathrm{trdeg}_k(L) = 1$, so that $\mathrm{trdeg}_{\mathbb{Q}}(k) = n - 1$. We now apply Theorem 6.7 to the abelian variety A over L and the extension $k \subset L$. Thus, we choose an arithmetically hyperbolic integral variety S over k whose function field $K(S)$ is a subfield of L with $\overline{K(S)} = L$, an abelian scheme $\mathcal{A}$ over S such that $\mathcal{A}$ is mildly bounded over k and $\mathcal{A} \times_k L$ is isomorphic to A over L. Now, shrinking S if necessary, we may and do assume that the ample divisor D on A extends to a relatively ample divisor $\mathcal{D} \subset \mathcal{A}$ on $\mathcal{A}$. Define $\mathcal{X} := \mathcal{A} \setminus \mathcal{D}$. Note that, by the induction hypothesis, for every s in $S(k)$, the fibre $\mathcal{X}_s$ of $\mathcal{X} \to S$ over s is arithmetically hyperbolic over k (as it is the complement of an ample divisor in an abelian variety over k). Therefore, as S is arithmetically hyperbolic over k and every fibre of $\mathcal{X} \to S$ is arithmetically hyperbolic, it is straightforward to see that the variety $\mathcal{X}$ is arithmetically hyperbolic over k (see [35, Lemma 4.11] for details). Since $\mathcal{X}$ is mildly bounded over k and arithmetically hyperbolic over k, it follows from Theorem 7.2 that $X = \mathcal{X}_{\overline{K(S)}}$ is arithmetically hyperbolic over L. This concludes the proof. $\square$

7.1. **Two remarks on integral points on abelian varieties.** It is well-known that the set of rational points on an abelian variety over a number field is potentially dense, and there are "many" different proofs of this fact. In the following remark, we explain how the mild boundedness of abelian varieties implies the potential density of rational points.

Remark 7.7 (Potential density of rational points on abelian varieties)**.** Let A be an abelian variety over k. It is well-known that there is a finitely generated subfield $K \subset k$ and a model $\mathcal{A}$ for A over K such that the subset $\mathcal{A}(K)$ of $A(k)$ is dense in A. Indeed, this is due to Hassett–Tschinkel [28] (see also [33, Corollary 3.10]). In this remark, we explain how to reprove this result. Firstly, using standard arguments (see for instance [33, §3.1]), we may and do assume that A is simple. Let L be an uncountable algebraically closed field containing k. Then, as the torsion in $A(L)$ is countable and L is an uncountable algebraically closed field, we see that $A(L)$ contains a point P of infinite order. Choose a finitely generated subfield $K_1 \subset L$ and a model $\mathcal{A}$ for A_L over K_1 such that P lies in the subset $\mathcal{A}(K_1)$ of $A(L)$. Then $\mathcal{A}(K_1)$ is infinite, so that in particular A_L is not arithmetically hyperbolic over L. Since A_K is mildly bounded over K for any subfield $k \subset K \subset L$ by Corollary 5.12 and A_L is not arithmetically hyperbolic over L, it follows that A is not arithmetically hyperbolic over k. In particular, there is a finitely generated field $K \subset k$ and a model $\mathcal{A}$ for A over K such that $\mathcal{A}(K)$ is infinite. In particular, $\mathcal{A}(K)$ is dense, as A is simple.

Another application of the mildly boundedness of abelian varieties is given in the following remark in which we briefly discuss Hassett-Tschinkel's arithmetic puncture problem [29, Problem 5.3]. This conjecture is also a consequence of conjectures of Campana [8, 9].

Remark 7.8 (Hassett-Tschinkel puncture problem)**.** Let A be a simple abelian variety over $\overline{\mathbb{Q}}$ and let $D \subset A$ be a closed subset with $\mathrm{codim}(A \setminus D) \geq 2$. In [29, Problem 5.3] Hassett and Tschinkel conjecture that there exists a number field K, a finite set of finite places S of K, a model $\mathcal{A}$ for A over $\mathcal{O}_{K,S}$ and a model $\Delta \subset \mathcal{A}$ for $D \subset A$ over $\mathcal{O}_{K,S}$ such that $(\mathcal{A} \setminus \Delta)(\mathcal{O}_{K,S})$ is dense in A. In particular, Hassett and Tschinkel conjecture that $A \setminus D$ is **not** arithmetically hyperbolic over $\overline{\mathbb{Q}}$. We note that Hassett-Tschinkel's conjecture is in fact a special case of Campana's more general conjectures on potential density of integral points on log-special varieties [8, 9]. Now, since abelian varieties are mildly bounded over any algebraically closed field of characteristic zero (Corollary 5.12), we can make the following observation.

> *Suppose there is an algebraically closed field k containing $\overline{\mathbb{Q}}$ such that $(A \setminus D)_k$ is not arithmetically hyperbolic over k. Then, $A \setminus D$ is not arithmetically hyperbolic over $\overline{\mathbb{Q}}$.*

Thus, in other words, to solve Hassett-Tschinkel's arithmetic puncture problem for the variety $A \setminus D$ it suffices to show the infinitude of integral points on $A \setminus D$ over some finitely generated subring of $\mathbb{C}$. This is arguably (most likely) easier to achieve, as can be seen in the case that $D = \emptyset$. The existence of a point of infinite order in $A(\mathbb{C})$ follows from the countability of the group of torsion points on A and the uncountability of $\mathbb{C}$.

8. Application to irregular surfaces

The aim of this section is to prove the mild boundedness of certain surfaces. We then use this to prove the Arithmetic Persistence Conjecture 1.20 for surfaces which admit a non-constant map to some abelian variety. The most general result we obtain on the Persistence Conjecture in this section is Theorem 8.4.

Lemma 8.1 (Grauert-Manin + ϵ)**.** *Let C be a smooth integral curve over k, and let $X \to C$ be a quasi-projective flat morphism of integral schemes whose fibres are groupless of dimension at most one. Let $c \in C(k)$ and let $x \in X(k)$. Then the set of sections $\sigma : C \to X$ of $X \to C$ with $\sigma(c) = x$ is finite.*

Proof. Replacing C by a dense open if necessary, we may (and do) assume that $X \to C$ is smooth of relative dimension one and that X is connected, so that X is a smooth integral surface over k. Since the fibres of $X \to C$ are groupless smooth quasi-projective connected curves, there is a finite étale morphism $Y \to X$ such that the geometric generic fibre of $Y \to C$ is of genus at least two. Replacing X by Y, we may assume that the geometric fibres of $X \to C$ are smooth projective curves with genus at least two. Moreover, to prove the lemma, we may assume that the set of sections $X(C)$ is infinite. In this case, by Grauert-Manin's theorem [24, 50], there is a smooth projective connected curve X_0 over k and an isomorphism $X \cong X_0 \times_k C$ over C. Furthermore, the set $X(C) \setminus X_0(k)$ is finite. (This latter finiteness statement follows from the theorem of De Franchis-Severi.) As there is precisely one element σ in the subset $X_0(k)$ of $X(C)$ with $\sigma : C \to X$ with $\sigma(c) = x$, this concludes the proof. $\square$

Lemma 8.2. *For every smooth connected curve C over k, there is an integer $n \geq 1$ and points $c_1, \ldots, c_n$ having the following property: for any quasi-projective variety X over k which admits a morphism $\varphi : X \to G$ to a semi-abelian variety G over k with groupless fibres of dimension* *at most one and every $x_1, \ldots, x_n$ in $X(k)$, the set*

$$\mathrm{Hom}_k([C,(c_1,\ldots,c_n)],[X,(x_1,\ldots,x_n)])$$

is finite. In particular, the variety X is mildly bounded over k.

Proof. (We will apply Grauert-Manin's theorem in the form of Lemma 8.1. We could avoid appealing to Grauert-Manin's theorem and instead use our results on the mild boundedness of total spaces of abelian schemes. However, appealing to the Grauert-Manin theorem allows us to give a shorter proof.)

Let C be a smooth affine connected curve over k. Choose $n \geq 1$ and $c_1, \ldots, c_n$ in $C(k)$ as in Proposition 5.8, so that for every semi-abelian variety A over k and every $a_1, \ldots, a_n$ in $A(k)$ the set

$$\mathrm{Hom}_k([C,(c_1,\ldots,c_n)],[A,(a_1,\ldots,a_n)])$$

is finite.

Let H be the subset of $\mathrm{Hom}_k(C,X)$ of morphisms $f : C \to X$ such that the composed morphism $C \to X \to G$ is constant. If the set $\mathrm{Hom}_k([C,(c_1,\ldots,c_n)],[X,(x_1,\ldots,x_n)]) \cap H$ is non-empty, then $x_1, \ldots, x_n$ lie in $X_{\varphi(x_1)}$ and

$$\begin{aligned}&\mathrm{Hom}_k([C,(c_1,\ldots,c_n)],[X,(x_1,\ldots,x_n)]) \cap H \\ &\qquad = \mathrm{Hom}_k([C,(c_1,\ldots,c_n)],[X_{\varphi(x_1)},(x_1,\ldots,x_n)])\end{aligned}$$

which is finite by the defining property of $c_1, \ldots, c_n$, and the fact that $X_{\varphi(x_1)}$ is at most one-dimensional and groupless. Thus, to prove the lemma, it suffices to show that

$$\mathrm{Hom}_k([C,(c_1,\ldots,c_n)],[X,(x_1,\ldots,x_n)]) \setminus H$$

is finite.

Now, let $g_i := \varphi(x_i)$ and note that

$$\mathrm{Hom}_k^{\mathrm{nc}}([C,(c_1,\ldots,c_n)],[G,(g_1,\ldots,g_n)])$$

is finite. Let $\nu_1, \ldots, \nu_r$ be its elements. Then, for every $1 \leq i \leq r$, we consider the morphism $\varphi_i : X_i \to C$, where $X_i := X \times_{\varphi,G,\nu_i} C$ is the pull-back of $X \to G$ along $\nu_i : C \to G$. As the morphism φ_i has groupless fibres of dimension at most one, it follows from Grauert-Manin's theorem (Lemma 8.1) that there are only finitely many sections which map c_1 to x_1.

In particular,

$$\begin{aligned}&\mathrm{Hom}_k([C,(c_1,\ldots,c_n)],[X,(x_1,\ldots,x_n)]) \setminus H \\ &\qquad = \bigcup_{i=1}^{r} \mathrm{Hom}_C([C,(c_1,\ldots,c_n)],[X_i,(x_1,\ldots,x_n)])\end{aligned}$$

is finite. This finishes the proof of the statement. □

Proof of Theorem 1.13. This follows from Lemma 8.2. □

Lemma 8.3. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero and let X be a quasi-projective variety over k which admits a morphism to some semi-abelian variety over k with fibres of dimension at most one. If L/k has transcendence degree one and X is arithmetically hyperbolic over k, then X_L is arithmetically hyperbolic over L.*

Proof. (We adapt the proof of [33, Lemma 4.2], and use both Grauert-Manin's function field Mordell conjecture and the fact that abelian varieties are mildly bounded.)

Let G be an abelian variety over k and let $f : X \to G$ be a non-constant morphism. Let $A \subset k$ be a $\mathbb{Z}$-finitely generated subring, let $\mathcal{X}$ be a projective model for X over A, let $\mathcal{G}$ be an abelian scheme over A, and let $F : \mathcal{X} \to \mathcal{G}$ be a morphism of A-schemes such that $F_k = f$. Let $B \subset L$ be a $\mathbb{Z}$-finitely generated subring containing A. To prove that X_L is arithmetically hyperbolic over L, we show that the set $\mathcal{X}(B)$ of morphisms $\operatorname{Spec} B \to \mathcal{X}$ over A is finite. Replacing B by a larger $\mathbb{Z}$-finitely generated subring of L if necessary, we may and do assume that the affine scheme $\mathcal{C} := \operatorname{Spec} B$ is smooth over A.

If $\dim B = \dim A$, then B is contained in k, so that the finiteness of $\mathcal{X}(B)$ follows from the assumption that X is arithmetically hyperbolic over k. Therefore, we may and do assume that $\dim \mathcal{C} = \dim A + 1$, i.e., the "arithmetic scheme" $\mathcal{C}$ is a "curve" over A. Define $C := \mathcal{C} \times_A k$, and note that C is a smooth affine one-dimensional scheme over k.

Note that, as X is arithmetically hyperbolic over k, it follows that X is groupless over k (see [33, §3]). In particular, as X is groupless over k, the fibres of $f : X \to G$ are groupless. Therefore, as the fibres of $f : X \to G$ are groupless of dimension at most one, we can apply Lemma 8.2. Thus, we choose an integer $n \geq 1$ and points $c_1, \ldots, c_n$ in $C(k)$ such that, for every $x_1, \ldots, x_n$ in $X(k)$, the set

$$\operatorname{Hom}_k([C, (c_1, \ldots, c_n)], [X, (x_1, \ldots, x_n)])$$

is finite. Next, we choose a $\mathbb{Z}$-finitely generated subring $A' \subset k$ containing A such that the points $c_1, \ldots, c_n$ in $C(k)$ descend to sections $\sigma_1, \ldots, \sigma_n$ of $\mathcal{C}' = \mathcal{C} \times_A A'$ over A'. Since $\mathcal{X}(\mathcal{C}) \subset \mathcal{X}(\mathcal{C}') = \operatorname{Hom}_A(\mathcal{C}', \mathcal{X})$, it suffices to show that $\mathcal{X}(\mathcal{C}')$ is finite.

By assumption, the variety X is arithmetically hyperbolic over k, so that $\mathcal{X}(A')$ is finite. Also, we have the following inclusion of sets

$$\mathcal{X}(\mathcal{C}') \subset \bigcup_{(x_1,\ldots,x_n) \in \mathcal{X}(A')^n} \operatorname{Hom}_{A'}([\mathcal{C}', (\sigma_1, \ldots, \sigma_n)], [\mathcal{X}_{A'}, (x_1, \ldots, x_n)]),$$

where $\operatorname{Hom}_{A'}([\mathcal{C}', (\sigma_1, \ldots, \sigma_n)], [\mathcal{X}_{A'}, (x_1, \ldots, x_n)])$ is the set of morphisms $P : \mathcal{C}' \to \mathcal{X}$ such that $P(\sigma_1) = x_1, \ldots, P(\sigma_n) = x_n$. Therefore, it suffices to show that, for any choice of (not necessarily pairwise distinct) elements $x_1, \ldots, x_n$ in the finite set $\mathcal{X}(A')$, the set

$$\operatorname{Hom}_{A'}([\mathcal{C}', (\sigma_1, \ldots, \sigma_n)], [\mathcal{X}_{A'}, (x_1, \ldots, x_n)])$$

is finite. Thus, let us fix sections $x_1, \ldots, x_n$ of $\mathcal{X}(A')$. As there are more morphisms over k than over A', we have the following inclusion of sets

$$\operatorname{Hom}_A([\mathcal{C}', (\sigma_1, \ldots, \sigma_n)], [\mathcal{X}_{A'}, (x_1, \ldots, x_n)]) \subset \operatorname{Hom}_k([C, (c_1, \ldots, c_n)], [X, (x_{1,k}, \ldots, x_{n,k})]).$$

By our choice of $c_1, \ldots, c_n$ in $C(k)$, the latter is set is finite. We conclude that X_L is arithmetically hyperbolic over L, as required. □

Theorem 8.4. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero and let X be a quasi-projective variety over k which admits a morphism to some semi-abelian variety over k whose fibres are of dimension at most one. If X is arithmetically hyperbolic over k, then X_L is arithmetically hyperbolic over L.*

Proof. We may and do assume that L has finite transcendence degree over k, say $n \geq 0$. We now argue by induction on n. By our assumption that X is arithmetically hyperbolic over k, we may assume that $n > 0$. Let $k \subset K \subset L$ be an algebraically closed subfield such that K has transcendence degree $n - 1$ over k. Then, the induction hypothesis implies that X_K is arithmetically hyperbolic over K. Now, the theorem follows from Lemma 8.3, as L has transcendence degree one over K. □

Corollary 8.5. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero and let X be an integral projective surface over k which admits a non-constant morphism* *to some abelian variety. If X is arithmetically hyperbolic over k, then X_L is arithmetically hyperbolic over L.*

Proof. If A is an abelian variety over k and $X \to A$ is a non-constant morphism, then the fibres of $X \to A$ are at most one-dimensional, so that the result follows from Theorem 8.4. $\square$

8.1. **Proof of Theorem 1.17.** We will prove Theorem 1.17 using Corollary 8.5. However, deducing Theorem 1.17 from Corollary 8.5 involves understand a subtle difference between the a priori finiteness of rational points on projective varieties and the finiteness of integral points; we refer the reader to [30, §7] for a discussion of this phenomenon which does not occur when one studies rational points on projective varieties over number rings, but occurs (in some situations) when studying rational points valued in finitely generated fields with positive transcendence degree over $\mathbb{Q}$. To deal with this subtlety in a systematic manner, we introduce the notion of a "pure model".

Definition 8.6 (Pure model)**.** Let X be a variety over k. Let $A \subset k$ be a subring. A model $\mathcal{X}$ for X over A is *pure over A* (or: *satisfies the extension property over A*) if, for every smooth finite type integral scheme T over A, every dense open subscheme $U \subset T$ with $T \setminus U$ of codimension at least two in T, and every morphism $f : U \to \mathcal{X}$, there is a (unique) morphism $\overline{f} : T \to \mathcal{X}$ extending the morphism f. (The uniqueness of the extension $\overline{f}$ follows from our convention that a model for X over A is separated.)

Note that X is pure over k (in the sense of Remark 4.2) if it has a pure model over k (in the sense of Definition 8.6).

Definition 8.7. A variety X over k has an *arithmeticallly-pure model* if there is a $\mathbb{Z}$-finitely generated subring $A \subset k$ and a pure model $\mathcal{X}$ for X over A.

Remark 8.8. Let X be a proper variety over k which has an arithmetically-pure model. Then X has no rational curves.

Remark 8.9. Let X be a proper variety over k. A model $\mathcal{X}$ for X over A with no rational curves in any geometric fibre is pure over A by [23, Proposition 6.2]. On the other hand, a pure model for X over A might have rational curves in every special fibre (of positive characteristic), as can be seen by considering the moduli space of principally polarised abelian surfaces over $\mathbb{Z}$.

Theorem 8.10. *Let X be an arithmetically hyperbolic proper variety over k which has an arithmetically-pure model. Then, for every finitely generated subfield $K \subset k$ and every model $\mathcal{X}$ for X over K, the set $\mathcal{X}(K)$ is finite.*

Proof. Let $A \subset k$ be a smooth $\mathbb{Z}$-finitely generated subring and let $\mathcal{X}$ be a pure proper model for X over A. It suffices to show that for any finitely generated subfield $K \subset k$ containing A, the set $\mathcal{X}(K)$ is finite. To do so, let $B \subset k$ be a $\mathbb{Z}$-finitely generated subring containing A with fraction field K such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is smooth. Let $T := \operatorname{Spec} B$ and note that, for every x in $\mathcal{X}(K)$, by the valuative criterion for properness, there is a dense open subscheme $U \subset T$ with $\operatorname{codim}(T \setminus U) \geq 2$ such that $x : \operatorname{Spec} K \to \mathcal{X}$ extends to a morphism $U \to \mathcal{X}$. Since $\mathcal{X}$ is a pure model over A, this morphism extends uniquely to a morphism $T \to U$. This shows that $\mathcal{X}(B) = \mathcal{X}(K)$. Since X is arithmetically hyperbolic over k, we have that $\mathcal{X}(B)$ is finite, so that $\mathcal{X}(K)$ is finite, as required. $\square$

Lemma 8.11. *Let X be a projective integral groupless surface over k. If there is an abelian variety G and a non-constant morphism $f : X \to G$, then X has an arithmetically-pure model.*

Proof. Let Y be the image of f. Since G is an abelian variety, it follows Y has a projective model whose geometric fibres do not have any rational curves, so that Y has an arithmetically-pure model. Let $X \to Y' \to Y$ be the Stein factorisation. Since $Y' \to Y$ is finite, it follows that Y' has an arithmetically-pure model.

The fibres of $f : X \to G$ are of dimension at most one and groupless. In particular, since being groupless is a Zariski open property for curves, we may choose a $\mathbb{Z}$-finitely generated subring $A \subset k$, a projective flat model $\mathcal{X}$ for X over A, a pure projective model $\mathcal{Y}$ for Y over A, a model $\mathcal{X} \to \mathcal{Y}$ whose geometric fibres are groupless. In particular, the fibres of $\mathcal{X} \to \mathcal{Y}$ do not contain rational curves. Therefore, the geometric fibres of $\mathcal{X} \to \operatorname{Spec} A$ do not contain rational curves. This implies that the model $\mathcal{X}$ is pure (Remark 8.9). □

Theorem 8.12. *Let K be a finitely generated field over $\mathbb{Q}$ and let X be a projective integral surface over K. Assume that there is an abelian variety A over $\overline{K}$ and a non-constant morphism from $X_{\overline{K}}$ A. If, for every* ***finite*** *extension L over K, the set $X(L)$ is finite, then, for every* ***finitely generated*** *extension M of K, the set $X(M)$ is finite.*

Proof. Let $k := \overline{K}$ and note that the assumption implies that X_k is arithmetically hyperbolic over k. Now, since X_k admits a non-constant morphism to some abelian variety A over k, for every algebraically closed field k; containing k, the projective variety $X_{k'}$ is arithmetically hyperbolic over k' by Corollary 8.5. Moreover, the variety $X_{k'}$ also has an arithmetically-pure model over k' by Lemma 8.11, so that the result follows from Theorem 8.10. □

Proof of Theorem 1.17. This follows from Theorem 8.12 with $k = \overline{\mathbb{Q}}$. □

Remark 8.13. The *a priori* difference between the finiteness of rational points and the finiteness of integral points on a projective variety over a finitely generated field K naturally leads to Vojta's notion of "near-integral points"; see [68] and also [30].

9. Pseudo-algebraic hyperbolicity

We have followed Demailly in our definition of algebraic hyperbolicity. We now extend Demailly's notion of algebraic hyperbolicity by allowing for an "exceptional locus" on which the desired property fails. Lang and Kobayashi use the term "pseudo" for such more general notions (see [41, 44]).

Definition 9.1. Let X be a projective scheme over k and let Δ be a closed subscheme of X. We say that X is *algebraically hyperbolic modulo* Δ if, there is an ample line bundle $\mathcal{L}$ on X and a real number $\alpha_{X,\mathcal{L}}$ depending only on X and $\mathcal{L}$ such that, for every smooth projective curve C over k and every morphism $f : C \to X$ with $f(C) \not\subset \Delta$, the inequality

$$\deg_C f^*\mathcal{L} \leq \alpha_{X,\mathcal{L}} \cdot \operatorname{genus}(C)$$

holds.

With the definitions given in the introduction, a projective scheme over k is pseudo-algebraically hyperbolic over k if and only if there is a proper closed subscheme Δ of X such that X is algebraically hyperbolic modulo Δ. Also, needless to stress, a projective scheme is algebraically hyperbolic over k if it is algebraically modulo the empty subset.

We start by showing that the pseudo-algebraic hyperbolicity of X persists over field extensions.

Lemma 9.2. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let X be a projective scheme over k and let $\Delta \subset X$ be a closed subscheme. If X is algebraically hyperbolic modulo Δ, then X_L is algebraically hyperbolic modulo Δ_L.*

Proof. Let $\mathcal{L}$ be an ample line bundle on X. We follow the proof of [34, Theorem 7.1]. Assume that X_L is not algebraically hyperbolic modulo Δ_L over L. Then, we may choose, for every $n \geq 1$, a smooth projective connected hyperbolic curve C_n over L and a morphism $f_n : C_n \to X_L$ with $f_n(C_n) \not\subset \Delta_L$ such that the slope $s(f_n) := \frac{\deg(f_n^*\mathcal{L})}{\mathrm{genus}(C_n)} > n$; note that X_L does not contain any rational curves, except those in Δ_L, as X has this property, so that $\mathrm{genus}(C_n) \neq 0$.

For every $n \geq 1$, we choose a finitely generated k-algebra $A_n \subset L$ with $U_n := \operatorname{Spec} A_n$, a smooth projective morphism $\mathcal{C}_n \to U_n$ with geometrically connected fibres, an isomorphism $\mathcal{C}_{n,L} \cong C_L$ over L, a model $F_n : \mathcal{C}_n \to X \times U_n$ for $f_n : C_n \to X_L$ over U_n, and a point u_n in $U_n(k)$ such that the image of F_{n,u_n} is not contained in Δ. Note that, for every $n \geq 1$, the slope $s(F_{n,u_n})$ of the morphism $F_{n,u_n} : \mathcal{C}_{n,u_n} \to X \times \{u_n\} \cong X$ equals the slope $s(f_n)$.

For every $n \geq 1$, we write $D_n := \mathcal{C}_{n,u_n}$ and note that D_n is a smooth projective connected curve over k. Note that the slope of the morphisms $F_{n,u_n} : D_n \to X$ tends to infinity as n grows, and that the image of F_{n,u_n} is not contained in Δ. This implies that X is not algebraically hyperbolic modulo Δ, as required. □

Proof of Theorem 1.24. Let $\Delta \subset X$ be a proper closed subset such that X is algebraically hyperbolic modulo Δ over k. Then it follows from Lemma 9.2 that X_L is algebraically hyperbolic modulo Δ_L over L. As $\Delta_L \subset X_L$ is a proper closed subscheme, we conclude that the projective variety X_L is pseudo-algebraically hyperbolic over L, as required. □

Remark 9.3. The proof of Lemma 9.2 also shows the following presumably useful fact. Let $\psi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{>0}$ be a map. Let X be a projective variety over k and let $\Delta \subset X$ be a proper closed subscheme such that, there is an ample line bundle $\mathcal{L}$ on X and a real number $\alpha_{X,\Delta,\mathcal{L}}$ such that, for every smooth projective connected curve C over k and every $f \in \mathrm{Hom}(C, X) \setminus \mathrm{Hom}(C, \Delta)$, the inequality

$$\deg_C f^*\mathcal{L} \leq \alpha_{X,\Delta,\mathcal{L}} \cdot \psi(\mathrm{genus}(C))$$

holds. Then the following holds. If $k \subset L$ is an extension of algebraically closed fields, C is a smooth projective connected curve over L and $f \in \mathrm{Hom}_L(C, X_L) \setminus \mathrm{Hom}_L(C, \Delta_L)$, the inequality

$$\deg_C f^*\mathcal{L}_L \leq \alpha_{X,\Delta,\mathcal{L}} \cdot \psi(\mathrm{genus}(C)).$$

holds.

9.1. **Pseudo-boundedness.** We extend the notion of boundedness introduced in [34] to the pseudo-setting.

Definition 9.4. Let X be a projective scheme over k and let Δ be a closed subscheme of X. We say that X is *N-bounded modulo* Δ if for every normal projective variety V of dimension at most N over k the scheme $\underline{\mathrm{Hom}}(V, X) \setminus \underline{\mathrm{Hom}}(V, \Delta)$ is of finite type over k, i.e. if there are only finitely many polynomials occurring as the Hilbert polynomial of a morphism $V \to X$ not mapping into Δ.

It is obvious that algebraically hyperbolic varieties are 1-bounded. We record this in the following lemma.

Lemma 9.5. *Let X be a projective scheme over k and let Δ be a closed subscheme. If X is algebraically hyperbolic modulo Δ, then X is 1-bounded modulo Δ.*

Proof. For every smooth projective connected curve C over k, the scheme $\underline{\mathrm{Hom}}_k(C, X) \setminus \underline{\mathrm{Hom}}_k(C, \Delta)$ is an open subscheme of scheme $\underline{\mathrm{Hom}}_k(C, X)$. Therefore the statement follows from the definitions. □

Lemma 9.6. *Let S be an integral normal variety over k, let N be a positive integer, and let $X \to S$ be a projective morphism. Let $\Delta \subset X$ be a closed subscheme. Let $A \subset S(k)$ be a subset not contained in any countable union of proper closed subsets of S. If X_s is N-bounded modulo Δ_s for all $s \in A$, then $X_{\overline{K(S)}}$ is N-bounded modulo $\Delta_{\overline{K(S)}}$.*

Proof. Write $M = \overline{K(S)}$. Suppose X_M is not N-bounded modulo Δ_M. Then there exists a normal projective integral variety Y over M of dimension at most N and a sequence of morphisms $f_n \colon Y \to X_M$ with pairwise distinct Hilbert polynomials such that $f(Y) \not\subset \Delta_M$. Taking a finite extension of $K(S)$ if necessary, there is a dense open subscheme $U \subset S$ and a projective flat geometrically integral model $\mathcal{Y} \to U$ for Y over U whose geometric fibres are normal (and of dimension at most N).

For every $n \geq 1$, by standard spreading out arguments, there is a dominant étale morphism $V_n \to U$ with image $U_n \subset U$ and a morphism $F_n \colon \mathcal{Y}_{V_n} \to X_{V_n}$ extending the morphism $f_n : Y \to X_M$ such that, for every v in $V_n(k)$, the Hilbert polynomial of the morphism $F_{n,v} \colon \mathcal{Y}_{V_n,v} \to X_{V_n,v}$ equals the Hilbert polynomial of f_n and the image of $F_{n,v}$ is not contained in Δ_{v_n}.

By our assumption on the set A, the intersection $\bigcap_{n=1}^{\infty} U_n \cap A$ is non-empty. Let s in $S(k)$ be an element of this intersection. For every $n \geq 1$, let $v_n \in V_n$ be a point lying over s (via $V_n \to U_n$). Note that $\mathcal{Y}_s \cong \mathcal{Y}_{V_n,v_n}$ for all $n \geq 1$. Moreover, the morphisms $F_{n,v_n} : \mathcal{Y}_s \cong \mathcal{Y}_{V_n,v_n} \to X_{V_n,v_n} \cong X_s$ have pairwise distinct Hilbert polynomials and their image is not contained in Δ_s. This implies that X_s is not N-bounded modulo Δ_s, as required. □

Corollary 9.7. *Let $k \subset L$ be an extension of* ***uncountable*** *algebraically closed fields of characteristic zero. Let X be a projective scheme over k and let $\Delta \subset X$ be a closed subscheme. If X is N-bounded modulo Δ over k, then X_L is N-bounded modulo Δ_L over L.*

Proof. This follows from Lemma 9.6. □

Remark 9.8. A pseudo-bounded variety is not necessarily birational to a bounded variety.

Lemma 9.9. *Assume k is* ***uncountable***. *Let X be a projective scheme over k and let Δ be a closed subscheme. If X is 1-bounded modulo Δ over k, then X is bounded modulo Δ over k.*

Proof. (We adapt the proof of [34, Theorem 9.2].) We show by induction on $n \geq 1$ that X is n-bounded modulo Δ over k. Thus, let $n \geq 2$ and assume that X is $(n-1)$-bounded. Note that, for V a projective normal variety over k, the Hilbert polynomial of a morphism $f : V \to X$ is uniquely determined by the numerical equivalence class of $f^*\mathcal{O}(1)$ in $\mathrm{NS}(V)$; see the proof of [34, Theorem 9.2] for an argument.

Assume that X is not n-bounded, so that there is a projective normal variety Y of dimension n over k and morphisms $f_1, f_2, f_3, \ldots$ from Y to X with pairwise distinct Hilbert polynomials. Note that the numerical equivalence classes of $f_1^*\mathcal{O}(1), f_2^*\mathcal{O}(1), \ldots$ are pairwise distinct. Define $\Delta_i := f_i^{-1}(\Delta) \subset Y$, and note that Δ_i is a proper closed subscheme of Y. Since k is uncountable, there is a smooth ample divisor D in Y such that, for all i, we have that D is not contained in Δ_i. By [34, Lemma 9.1], it follows that

$$f_i^*(\mathcal{O}(1)) \cdot D^{\dim Y - 1} \to \infty, \quad i \to \infty.$$

In particular, we have that $f_i^*(\mathcal{O}(1))|_D \cdot D|_D^{\dim Y-2} \to \infty$. Therefore, the morphisms $f_i|_D : D \to X$ have pairwise distinct Hilbert polynomials and satisfy $f_i|_D(D) \not\subset \Delta$. Since D is a smooth projective variety with $\dim D = n - 1 < n$, this contradicts the fact that X is $(n-1)$-bounded modulo Δ. We conclude that X is n-bounded modulo Δ. □

Lemma 9.10. *Let $k \subset L$ be an extension of algebraically closed fields. Let X be a projective scheme over k and let Δ be a closed subscheme. If X is algebraically hyperbolic modulo Δ over k, then X_L is bounded modulo Δ_L over L.*

Proof. We may and do assume that L is uncountable. By Theorem 1.24, we have that X_L is algebraically hyperbolic modulo Δ. In particular, it follows that X_L is 1-bounded modulo Δ (Lemma 9.5). Therefore, as L is uncountable, it follows from Lemma 9.9 that X_L is bounded modulo Δ_L. □

Proof of Theorem 1.25. This follows from Lemma 9.10 (with $k = L$). □

Proof of Theorem 1.26. We follow the arguments used to prove [34, Theorem 1.14].

Let $\mathcal{M} := \mathcal{M}_{g,k}$ be the stack of smooth proper genus $g \geq 2$ curves over k and let $\mathcal{U} \to \mathcal{M}$ be the universal curve of genus g. We claim that the morphism

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{U}, X \times \mathcal{M}) \setminus \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{U}, \Delta \times \mathcal{M}) \to \mathcal{M}$$

is of finite type.

To do so, let $\overline{\mathcal{U}} \to \overline{\mathcal{M}}$ be the universal stable curve of genus g over the stack $\overline{\mathcal{M}}$ of stable curves of genus g. Let P be a smooth projective scheme and let $P \to \overline{\mathcal{M}}$ be a smooth surjective morphism. Now, let Z be a normal projective bounded scheme over k and let $Z \to P$ be a finite flat surjective morphism of schemes. Let $Y := \overline{\mathcal{U}} \times_{\overline{M}} Z$ and consider the morphism $Y \to Z$.

Note that $X \times Z$ is bounded modulo $\Delta \times Z$. Therefore, the scheme $\underline{\mathrm{Hom}}_k(Y, X \times Z) \setminus \underline{\mathrm{Hom}}_k(Y, \Delta \times Z)$ is of finite type over k. In particular, the morphism $\underline{\mathrm{Hom}}_Z(Y, X \times Z) \setminus \underline{\mathrm{Hom}}_k(Y, \Delta \times Z) \to Z$ is of finite type. By descent, the morphism $\underline{\mathrm{Hom}}_{\overline{\mathcal{M}}}(\overline{\mathcal{U}}, X \times \overline{\mathcal{M}}) \setminus \underline{\mathrm{Hom}}_{\overline{\mathcal{M}}}(\overline{\mathcal{U}}, \Delta \times \overline{\mathcal{M}}) \to \overline{\mathcal{M}}$ is of finite type. By base-change, this proves our claim that the morphism $\underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{U}, X \times \mathcal{M}) \setminus \underline{\mathrm{Hom}}_{\mathcal{M}}(\mathcal{U}, \Delta \times \mathcal{M}) \to \mathcal{M}$ is of finite type. As the latter morphism is of finite type (for every $g \geq 2$) we see that, for every ample line bundle $\mathcal{L}$ on X and every integer $g \geq 2$, there is an integer $\alpha(X, \mathcal{L}, g)$ such that, for every smooth projective connected curve C of genus g over k and every morphism $f : C \to X$ with $f(C) \not\subset \Delta$, the inequality

$$\deg_C f^*\mathcal{L} \leq \alpha(X, \mathcal{L}, g)$$

holds. This implies the desired statement and concludes the proof. □

Corollary 9.11. *Let X be a projective scheme over k and let Δ be a closed subscheme of X such that X is 1-bounded modulo Δ over k. If k is* ***uncountable****, for every ample line bundle $\mathcal{L}$ on X and every integer $g \geq 0$, there is a real number $\alpha(X, \mathcal{L}, \Delta, g)$ such that, for every smooth projective connected curve C of genus g over k and every morphism $f : C \to X$ with $f(C) \not\subset \Delta$, the inequality*

$$\deg_C f^*\mathcal{L} \leq \alpha(X, \mathcal{L}, \Delta, g)$$

holds.

Proof. Since the ground field k is assumed to be uncountable, this follows from Lemma 9.9 and Theorem 1.26. □

Corollary 9.12. *Let $k \subset L$ be an extension of algebraically closed fields. Let X be a projective scheme over k and let Δ be a closed subscheme. If X is bounded modulo Δ over k, then X_L is bounded modulo Δ_L over L.*

Proof. By Remark 9.3, this follows from Theorem 1.26. □

Remark 9.13. The notion of pseudo-algebraic hyperbolicity is further studied in [38]. For example, motivated by Vojta's conjecture and the finiteness theorem of Kobayashi–Ochiai [42] for varieties of general type, it is shown in [38] that if X is a projective pseudo-algebraically hyperbolic scheme over k and Y is a projective integral variety over k, then the set of surjective morphisms $Y \to X$ is finite.

9.2. **Pointed boundedness.** We extend the notion of pointed boundedness introduced in [34, §4] to the pseudo-setting. Note that this notion is referred to as "geometric hyperbolicity" in [30].

Definition 9.14. Let X be a projective scheme over k and let $\Delta \subset X$ be a closed subscheme. We say that X is $(n,1)$-*bounded modulo* Δ if, for every smooth projective connected variety Y of dimension at most n over k, every y in $Y(k)$, and every x in $X(k) \setminus \Delta$, the scheme $\underline{\mathrm{Hom}}_k([Y,y],[X,x])$ is of finite type over k.

Remark 9.15. Let $k \subset L$ be an extension of **uncountable** algebraically closed fields of characteristic zero, let X be a projective scheme over k and let $\Delta \subset X$ be a closed subscheme. If X is $(n,1)$-bounded modulo Δ over k, then X_L is $(n,1)$-bounded modulo Δ_L over L. This is proven in a similar manner as Corollary 9.7.

Lemma 9.16. *Let X be a projective variety over k, and let $\Delta \subset X$ be a closed subscheme, and let $n \geq 1$ be an integer. If X is $(1,1)$-bounded modulo Δ, then X is $(n,1)$-bounded modulo Δ.*

Proof. As the argument is similar to the proof of Lemma 9.9, we will be brief on the details. (Note that we do not require k to be uncountable.)

We argue by induction on n. Assume that X is not $(n,1)$-bounded, so that there is a projective smooth variety Y of dimension n over k, a point y in $Y(k)$, a point $x \in X(k) \setminus \Delta$, and morphisms $f_1, f_2, f_3, \ldots$ from Y to X with pairwise distinct Hilbert polynomials and $f_i(y) = x$. Note that the numerical equivalence classes of $f_1^*\mathcal{O}(1), f_2^*\mathcal{O}(1), \ldots$ are pairwise distinct. Define $\Delta_i := f_i^{-1}(\Delta) \subset Y$, and note that Δ_i is a proper closed subscheme of Y. Let D be a smooth ample divisor in Y which contains y. Then, the morphisms $f_i|_D$ send y to x. In particular, the image of $f_i|_D$ is not contained in Δ and, by [34, Lemma 9.1], the morphisms $f_i|_D : D \to X$ have pairwise distinct Hilbert polynomials. Since D is a smooth projective variety with $\dim D = n-1 < n$, we obtain a contradiction. We conclude that X is $(n,1)$-bounded modulo Δ. □

Proposition 9.17. *Let X be a projective scheme over k and let Δ be a closed subscheme of X. Then the following are equivalent.*

1. *The projective variety X is $(1,1)$-bounded modulo Δ.*
2. *For every smooth projective connected curve C over k, every c in $C(k)$, and every x in $X(k) \setminus \Delta$, the set $\mathrm{Hom}([C,c],[X,x])$ is finite.*

Proof. Clearly, it suffices to show that (1) $\implies$ (2). Thus, let us assume that there is a sequence $f_1, f_2, \ldots$ of pairwise distinct elements of $\mathrm{Hom}_k([C,c],[X,x])$, where C is a smooth projective connected curve, $c \in C(k)$ and $x \in X(k) \setminus \Delta$. Since $\underline{\mathrm{Hom}}_k([C,c],[X,x])$ is of finite type (by assumption), the degree of all the f_i is bounded by some real number (depending only on C, c, X, x, and Δ). Then, it follows that $\underline{\mathrm{Hom}}_k([C,c],[X,x])$ is positive-dimensional, so that by bend-and-break [13, Proposition 3.5] there is a rational curve in X containing x. This contradicts the fact that every rational curve in X is contained in Δ. □

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