DEGENERATIONS OF 2-DIMENSIONAL TORI

LJUDMILA KAMENOVA

Abstract. In this paper we classify the possible degenerate fibers which can occur in a semistable degeneration of two-dimensional tori under the assumption that the canonical bundle of the total space of the family is trivial.

1. Preliminaries

Let $\pi : X \to \Delta$ be a proper map of a Kähler manifold X onto the unit disk $\Delta = \{t \in \mathbb{C} :| t |< 1\}$, such that the fibers X_t are nonsingular compact complex manifolds for every $t \neq 0$. We call π a *degeneration* and the fiber $X_0 = \pi^{-1}(0)$ - the *degenerate fiber.*

Definition 1.1. *A map $\psi : Y \to \Delta$ is called a modification of a degeneration π if there exists a birational map $f : X \to Y$ such that $\psi = \pi \circ f$ and ψ is an isomorphism outside of the degenerate fiber.*

A degeneration is called *semistable* if the degenerate fiber is a reduced divisor with normal crossings. Not every degeneration can be modified to a semistable one. Nonetheless, it is possible to reduce any degeneration to a semistable one after a base change according to Mumford's theorem ([1]).

Definition 1.2. *The polyhedron $\Pi(V)$ of a variety with normal crossings $V = V_1 + \cdots + V_n$, $dimV_i = d$ is the polyhedron whose vertices correspond to the irreducible components V_i and the vertices $V_{i_1}, \cdots, V_{i_k}$ form a $(k-1)-$simplex if $V_{i_1} \cap \cdots \cap V_{i_k} \neq 0$.*

2. Basic Tools

Let $\pi : X \to \Delta$ be a semistable degeneration of surfaces whose degenerate fiber is $X_0 = V_1 + V_2 + \cdots + V_n$. We will state some results from [2].

Lemma 2.1. *([2]) Let $C = V_i \cap V_j$ be a double curve of a semistable degeneration of surfaces. Then $(C^2)_{V_i} + (C^2)_{V_j} = -T_C$, where T_C is the number of triple points of the fiber X_0 on C.*

Lemma 2.2. *([2]) Let T be the number of all triple points of π, then*

$$\chi(X_t) = \sum_{i=1}^{n} \chi(V_i) - \sum_{i<j} \chi(C_{i,j}) + T,$$

where $C_{i,j} = V_i \cap V_j$

Remark 2.1. *([2]) For a variety with normal crossing X_0 there is a natural mixed Hodge structure with weight filtration W and $W_0 H^m(X_0) \cong H^m(\Pi(X_0))$.*

Theorem 2.1. *(Kulikov [2], Persson [4]) Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of surfaces, then*

$$h^1(X_t) = \sum_{i=1}^{n} h^1(V_i) - \sum_{i<j} h^1(C_{i,j}) + 2h^1(\Pi) + ckh^1,$$

$$p_g(X_t) = \sum_{i=1}^{n} p_g(V_i) + h^2(\Pi) + \frac{1}{2} ckh^1,$$

where $ckh^1 = dim\ Coker\ (\oplus H^1(V_i) \to \oplus H^1(C_{i,j}))$, d is the number of double curves of the fiber X_0 and Π is its polyhedron.

Lemma 2.3. *A surface V is ruled or $\mathbb{CP}^2$ if and only if $H^0(V, nK_V) = 0$ for every $n > 0$.*

3. Main Theorem

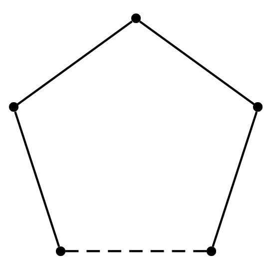
We prove the following theorem which is analogous to the classification theorems in [2] which Kulikov gives for K3-surfaces and Enriques surfaces.

Theorem 3.1. *Let $\pi : X \to \Delta$ be a semistable Kähler degeneration of two-dimensional tori such that K_X is trivial. Then the degenerate fiber X_0 is one of the following four types:*

(i) $X_0 = V_1$ is a nonsingular torus;

(ii) $X_0 = V_1 + V_2 + \cdots + V_n, n > 1$, all V_i are elliptic ruled surfaces, the double curves $C_{1,2}, \cdots, C_{n-1,n}$ are elliptic curves and the polyhedron Π is a simple path.

(iii) $X_0 = V_1 + V_2 + \cdots + V_n, n > 1$, all V_i are elliptic ruled surfaces, the double curves $C_{1,2}, \cdots, C_{n-1,n}$, $C_{n,1}$ are elliptic curves and the polyhedron Π is a cycle.



(iv) $X_0 = V_1 + V_2 + \cdots + V_n, n > 1$, all V_i are rational surfaces, and all the double curves $C_{i,j}$ are rational. The polyhedron Π is a triangulation of the real 2-dimensional tori T^2.

In the first case the monodromy M is trivial, i.e. $N = \log M = 0$. In the second and the third cases $N^2 = 0$. And, in the fourth case the monodromy is of maximal rank.

Proof. Case (i) is when X_0 has a single component.

Let $n > 1$. If $D \in Pic(X)$ and V is a component of the fiber, then let $D_V = i^*(D) = D \cdot V$, where $i : V \hookrightarrow X$ is the inclusion. For $D, D' \in Pic(X)$ the intersection index on V is defined: $D \cdot D' \cdot V = D_V \cdot D'_V$. The fibers X_t and X_0 are linearly equivalent and in addition $X_0 = V_1 + \cdots + V_n \sim 0$, hence by the adjunction formula,

$$K_{V_i} = K_X \otimes [V_i] \mid_{V_i} = O_{V_i}(\sum_{j \neq i} -V_j) = -\sum_{j \neq i} C_{i,j}$$

because K_X is trivial. Then K_{V_i} is anti-effective, and thus all of V_i are ruled surfaces. Consider a double curve $C_{i,j}$ on V_i. We have:

$$2g(C_{i,j}) - 2 = (K_{V_i} + C_{i,j}, C_{i,j})_{V_i} = -\sum_{k \neq i,j} (C_{i,k}, C_{i,j})_{V_i} = -T_{C_{i,j}},$$

where $T_{C_{i,j}}$ is the number of triple points of X_0 on $C_{i,j}$. Since $T_{C_{i,j}} \geq 0$ and $g(C_{i,j}) \geq 0$, there are two possibilities:

(A) $g(C_{i,j}) = 0$ and $T_{C_{i,j}} = 2$, so $C_{i,j}$ is a rational curve and there are exactly two triple points on $C_{i,j}$.

(B) $g(C_{i,j}) = 1$ and $T_{C_{i,j}} = 0$, so $C_{i,j}$ is an elliptic curve and $C_{i,j}$ does not intersect any other double curve.

In the case (A) we see that $C_{i,j}$ intersects some other double curves which must be rational as well and also contains two triple points. Thus every V_i is a ruled surface and the set of double curves on V_i consists of a disjoint union of a finite union of elliptic curves and a finite number of cycles of rational curves.

Let $V = V_{i_0}$ be one of the components, let $\phi : V \to \bar{V}$ be a morphism onto the minimal model $\bar{V}$ (ϕ is a composition of monoidal transforms) and let L be an exceptional curve on V such that $L \cong \mathbb{P}^1, (L^2)_V = -1$ and L is blown down to a point by the morphism ϕ. Then $(L, K_V)_V = -1$, so $(L, \sum_{j\neq i_0} C_{i_0,j})_V = 1$. Thus, either L intersects only one of the connected components of the divisor $\sum_{j\neq i_0} C_{i_0,j}$ or L coincides with one of $C_{i_0,j}$. It follows that the number of connected components of the divisor $\sum_{j\neq i_0} \phi_* C_{i_0,j}$ equals the number of connected components of the divisor $\sum_{j\neq i_0} C_{i_0,j}$ since $K_{\bar{V}} = \phi_* K_V$.

In Lemma 2.18 from [2] Kulikov gives a list of possible components of an effective divisor linearly equivalent to $-K_{\bar{V}}$, where $\bar{V}$ is either a minimal ruled surface or $\mathbb{CP}^2$. Since the reduced divisor

$$\sum_{j\neq i_0} \phi_* C_{i_0,j} \sim -K_{\bar{V}},$$

we have the following possibilities for V:

(a) V is a rational surface and $\sum_{j\neq i} C_{i,j}$ is a cycle of rational curves;

(b) V is a rational or an elliptic ruled surface and $\sum_{j\neq i} C_{i,j} = C$ is a single elliptic curve;

(c) V is a ruled elliptic surface and $\sum_{j\neq i} C_{i,j} = C_1 + C_2$ consists of two disjoint elliptic curves.

Case 1: One of V_i is of type (a). Then the double curves on the components adjacent to V_i also form a cycle, hence the components adjasent to V_i are also of type (a). Since X_0 is connected, it follows that all V_i are rational surfaces and their double curves form cycles. Therefore, the polyhedron Π is a triangulation of a compact real surface without a boundary. There is no boundary, because there are exactly two triple points on each double curve.

Since V_i and $C_{i,j}$ are rational, $p_g(V_i) = 0, h^1(V_i) = 0, h^1(C_{i,j}) = 0$ and from the second equality, $ckh^1 = 0$ (see [4]). Then the first formula in Theorem 2.1 says that $h^1(\Pi) = 2$ and the second formula says that $h^2(\Pi) = p_g(X_t) = 1$. We also know that $h^0(\Pi) = 1$ (from Remark 2.1). There is only one real surface without boundary with these cohomology numbers, namely the torus T^2. In this case the degenerate fiber falls into type (*iv*) in the statement of the theorem.

Case 2: All of the V_i have types (b) or (c). Then X_0 has no triple points ($T = 0$) and thus Π is 1-dimensional, so $h^2(\Pi) = 0$.

Let the number of rational surfaces be r. For a ruled elliptic surface V_i the Euler characteristic $\chi(V_i)$ is 0, while for a rational surface $\chi(V_i) = 1$. Also, $\chi(C_{i,j}) = 0$ for an elliptic curve $C_{i,j}$. Therefore, after we apply Lemma 2.2, we get $r = \chi(X_t) = 0$. In other words, there are no rational surfaces in the fiber X_0.

Since for an elliptic ruled surface V_i we have $p_g(V_i) = 0$, then from the second formula in Theorem 2.1 we get $ckh^1 = 2$. Now we substitute it in the first formula in this theorem and use that $h^1(V_i) = 2, h^1(C_{i,j}) = 2$ to obtain that

$$n = 1 + d - h^1(\Pi),$$

where n is the number of components and d is the number of double curves.

If $h^1(\Pi) = 0$, then $n = d + 1$ and Π is a tree. Moreover, on each component V_i there are at most two double curves, therefore Π is a simple path described in the case (*ii*) of the theorem.

If $h^1(\Pi) = 1$, then $n = d$ and there is one loop in the graph, hence Π is a simple cycle from case (*iii*) (because again there are at most two edges coming out of every vertex).

If $h^1(\Pi) \geq 2$, then there will be at least one vertex in which there are at least three edges meeting which is a contradiction.

The claims about the monodromy follow from the fact that $N = 0$ if and only if $h^2(\Pi) = 0$ and $ckh^1 = 0$; and $N^2 = 0$ if and only if $h^2(\Pi) = 0$ (see Theorem 2.7 in the paper [2]). □

References

[1] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B., *Toroidal embeddings*, LNM **339**, Springer-Verlag (1973),

[2] Kulikov, V, *Degenerations of K3 surfaces and Enriques surfaces,* Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977) 1008 - 1042,

[3] Parshin, A; Shafarevich, I.R.(eds.), *Algebraic geometry III: Complex algebraic varieties, algebraic curves and their Jacobians,* Encyclopaedia of Mathematical Sciences **36** Berlin: Springer (1998),

[4] Persson, U, *On degenerations of algebraic surfaces,* Mem. Amer. Math. Soc. **11** (1977), no. 189.

E-mail address: kamenova@math.mit.edu

Department of Mathematics, MIT