DEMAILLY'S NOTION OF ALGEBRAIC HYPERBOLICITY: GEOMETRICITY, BOUNDEDNESS, MODULI OF MAPS

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Abstract. Demailly's conjecture, which is a consequence of the Green–Griffiths–Lang conjecture on varieties of general type, states that an algebraically hyperbolic complex projective variety is Kobayashi hyperbolic. Our aim is to provide evidence for Demailly's conjecture by verifying several predictions it makes. We first define what an algebraically hyperbolic projective variety is, extending Demailly's definition to (not necessarily smooth) projective varieties over an arbitrary algebraically closed field of characteristic zero, and we prove that this property is stable under extensions of algebraically closed fields. Furthermore, we show that the set of (not necessarily surjective) morphisms from a projective variety Y to a projective algebraically hyperbolic variety X that map a fixed closed subvariety of Y onto a fixed closed subvariety of X is finite. As an application, we obtain that $\mathrm{Aut}(X)$ is finite and that every surjective endomorphism of X is an automorphism. Finally, we explore "weaker" notions of hyperbolicity related to boundedness of moduli spaces of maps, and verify similar predictions made by the Green–Griffiths–Lang conjecture on hyperbolic projective varieties.

1. Introduction

The aim of this paper is to provide evidence for Demailly's conjecture which says that a projective algebraically hyperbolic variety over $\mathbb{C}$ is Kobayashi hyperbolic.

We first define the notion of an algebraically hyperbolic projective scheme over an algebraically closed field k of characteristic zero which is not assumed to be $\mathbb{C}$, and could be $\overline{\mathbb{Q}}$, for example. Then we provide indirect evidence for Demailly's conjecture by showing that algebraically hyperbolic schemes share many common features with Kobayashi hyperbolic complex manifolds. Furthermore, we also investigate "weaker" variants of algebraic hyperbolicity, and prove similar properties.

Definition 1.1. A projective scheme X over k is *algebraically hyperbolic over k* if there is an ample line bundle $\mathcal{L}$, a positive real number α, and a positive real number β such that, for every smooth projective connected curve C over k and every k-morphism $f\colon C \to X$ we have that

$$\deg_C f^*\mathcal{L} \leq \alpha \cdot (2 \operatorname{genus}(C) - 2) + \beta = -\alpha \cdot \chi(C) + \beta.$$

In [8] Demailly defines this notion for *smooth* projective schemes over $\mathbb{C}$ (and more generally, for compact complex manifolds and for projective directed manifolds). Note that the above definition makes sense for (not necessarily smooth) projective schemes over k. Moreover, in our definition we ask for a bound of the form $-\alpha\chi(C) + \beta$. In Demailly's definition, one demands $\beta = 0$. These different choices do not lead to different notions. In fact, it is not hard to see that X is algebraically hyperbolic over k if, and only if, there is an integer $g_0 \geq 0$ and a positive real number α such that, for every smooth projective connected curve C over k

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of genus at least g_0 and every morphism $f : C \to X$, the inequality $\deg_C f^*\mathcal{L} \leq \alpha \cdot \mathrm{genus}(C)$ holds.

Examples of algebraically hyperbolic projective varieties are given in [3, 4, 7, 9, 10, 27, 30, 31]. Also, a logarithmic analogue of algebraic hyperbolicity (for quasi-projective varieties) was introduced and studied in [5].

A finite type scheme X over $\mathbb{C}$ is Kobayashi hyperbolic if Kobayashi's pseudometric on the reduced complex analytic space $X^{\mathrm{an}}_{\mathrm{red}}$ is a metric; see [21]. The relation between algebraic hyperbolicity and Kobayashi hyperbolicity is provided by the following theorem of Demailly.

Theorem 1.2 (Demailly). *If X is a Kobayashi hyperbolic projective scheme over $\mathbb{C}$, then X is algebraically hyperbolic over $\mathbb{C}$.*

In [8, Theorem 2.1] Demailly shows that a Kobayashi hyperbolic *smooth* projective variety over $\mathbb{C}$ is algebraically hyperbolic (see also [2, Theorem 2.13]). The smoothness assumption is however not used in Demailly's proof.

Recall that a variety X over $\mathbb{C}$ is Brody hyperbolic if every holomorphic map $\mathbb{C} \to X^{\mathrm{an}}$ is constant, where X^{an} is the complex analytic space associated to X [14, Exposé XII]. Since Brody hyperbolic projective varieties are Kobayashi hyperbolic [21], we see that Brody hyperbolic projective varieties over $\mathbb{C}$ are algebraically hyperbolic over $\mathbb{C}$. Similarly, as Borel hyperbolic projective varieties over $\mathbb{C}$ (as defined in [19]) are Brody hyperbolic, it follows that they are also algebraically hyperbolic over $\mathbb{C}$. In particular, roughly speaking, every "complex-analytically" hyperbolic variety is algebraically hyperbolic.

One can show that a projective Kobayashi hyperbolic variety X over $\mathbb{C}$ is *groupless* (Definition 2.1), i.e., for every connected complex algebraic group G, every morphism of varieties $G \to X$ is constant. In [25, page 160] Lang conjectured the converse, i.e., a groupless projective variety X over $\mathbb{C}$ is Kobayashi hyperbolic.

Lang's aforementioned conjecture is a variant of a similar conjecture of Green–Griffiths [12]. Indeed, Green and Griffiths conjectured that, if X is a projective variety of general type over $\mathbb{C}$, then there are no entire curves $\mathbb{C} \to X^{\mathrm{an}}$ with Zariski dense image. Consequently, combining the conjectures of Lang and Green–Griffiths, we are led to the following conjecture (which we will refer to as the Green–Griffiths–Lang conjecture).

Conjecture 1.3 (Green–Griffiths–Lang). *Let X be a projective variety over $\mathbb{C}$. Then the following are equivalent.*

(1) The projective variety X is groupless over $\mathbb{C}$.
(2) The complex analytic space X^{an} is Kobayashi hyperbolic.
(3) Every closed subvariety of X is of general type.

We now explain the relation of algebraically hyperbolic varieties to the Green–Griffiths–Lang conjecture. In fact, we follow (and simplify) a strategy of Demailly to show that projective algebraically hyperbolic varieties are groupless (Corollary 4.5). Therefore, as Demailly notes [8, p. 14], the converse of the statement of Theorem 1.2 is in fact a consequence of the Green–Griffiths–Lang conjecture. In other words, the following conjecture is a consequence of the Green–Griffiths–Lang conjecture (and we will refer to it as Demailly's conjecture throughout this paper).

Conjecture 1.4 (Demailly, consequence of Green–Griffiths–Lang conjecture). *If X is an algebraically hyperbolic projective variety over $\mathbb{C}$, then X is Kobayashi hyperbolic.*

In the next section we present our main results. We emphasize that all of our results are in accordance with Conjecture 1.4 in the sense that they allow one to verify some of the predictions one can make *assuming* Demailly's conjecture (Conjecture 1.4) holds.

1.1. **Properties of algebraically hyperbolic varieties.** Our first result illustrates that algebraic hyperbolicity is a geometric property. The proof is contained in Theorem 7.1.

Theorem 1.5. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let X be a projective algebraically hyperbolic scheme over k. Then the projective scheme X_L is algebraically hyperbolic over L.*

The Green–Griffiths–Lang conjecture says that a projective variety over k is algebraically hyperbolic over k if (and only if) it is groupless over k (Definition 2.1). Now, it is not hard to see that for $k \subset L$ an extension of algebraically closed fields of characteristic zero, a variety X over k is groupless over k if and only if X_L is groupless over L (see Lemma 2.3). In particular, Theorem 1.5 is in accordance with Green–Griffiths–Lang's aforementioned conjecture.

The fact that the moduli of maps from *any* given curve to an algebraically hyperbolic variety X is bounded (by definition) has consequences for the moduli of maps from *any* given variety to X, and also for the endomorphisms of X. The precise result we obtain reads as follows.

Theorem 1.6. *Let X be a projective algebraically hyperbolic variety over k. The following statements hold.*

1. *If Y is a projective reduced scheme over k, then the set of surjective morphisms $Y \to X$ is finite. If Y is normal, then every rational dominant map $Y \dashrightarrow X$ is a morphism.*
2. *Assume that X is reduced. Then, the group $\mathrm{Aut}(X)$ is finite, every surjective endomorphism $X \to X$ of X is an automorphism, and X has only finitely many surjective endomorphisms.*

The analogue of the first statement of Theorem 1.6 for Kobayashi hyperbolic varieties was obtained by Noguchi; see [29] or [21, Theorem 6.6.2]. This latter statement (for Kobayashi hyperbolic varieties) was (also) conjectured by Lang and has a long history; see [21, § 6] for a discussion. For instance, earlier results were obtained by Horst [16]. An analogue of the statement about automorphisms for Kobayashi hyperbolic varieties is contained in [21, Theorem 5.4.4], and an analogue of the statements about endomorphisms for Kobayashi hyperbolic varieties is an application of [21, Theorem 6.6.20] and [21, Theorem 5.4.4]. Thus, needless to emphasize, we see that Theorem 1.6 is in accordance with Demailly's conjecture (Conjecture 1.4).

Remark 1.7. In [2, Theorem 3.5], the finiteness of $\mathrm{Aut}(X)$ is proven when X is a *smooth* projective algebraically hyperbolic variety over $\mathbb{C}$. We stress that we do not impose smoothness. Moreover, we allow for the base field to be any algebraically closed field of characteristic zero. Our proof of Theorem 1.6 is different than the proof in *loc. cit.* and allows for a more general result (see Theorem 1.12).

The finiteness results in Theorem 1.6 for surjective morphisms from a projective scheme to an algebraically hyperbolic projective scheme can in fact be subsumed into the following statement (which we prove using Theorem 1.6).

Theorem 1.8. *Let X be an algebraically hyperbolic projective scheme over k. Then, for every projective scheme Y over k, every non-empty reduced closed subscheme $B \subset Y$, and every reduced closed subscheme $A \subset X$, the set of morphisms $f : Y \to X$ with $f(B) = A$ is finite.*

The analogue of Theorem 1.8 for Kobayashi hyperbolic varieties when $\dim B = \dim A = 0$ is Urata's theorem (see [21, Theorem 5.3.10] or the original paper [33]). Also, the analogue of the statement of Theorem 1.8 for Kobayashi hyperbolic varieties is contained in [21, Corollary 6.6.8]. Thus, needless to stress, Theorem 1.8 is also in accordance with Demailly's conjecture (Conjecture 1.4).

To state our following result, for X and Y projective schemes over k, we let $\underline{\mathrm{Hom}}_k(X, Y)$ be the associated Hom-scheme (see Section 2). Moreover, we let $\underline{\mathrm{Hom}}_k^{nc}(X, Y)$ be the subscheme parametrizing non-constant morphisms $X \to Y$.

Roughly speaking, our next result verifies that moduli spaces of maps to a projective algebraically hyperbolic variety are (also) projective and algebraically hyperbolic.

Theorem 1.9. *Let X be a projective algebraically hyperbolic variety over k. If Y is a projective scheme over k, then the scheme $\underline{\mathrm{Hom}}_k(Y, X)$ is a projective algebraically hyperbolic scheme over k. Moreover, we have that $\dim \underline{\mathrm{Hom}}_k^{nc}(Y, X) < \dim X$.*

The analogue of Theorem 1.9 for Kobayashi hyperbolic projective varieties over $\mathbb{C}$ is provided by [21, Theorem 5.3.9] and [21, Theorem 6.4.1]. Thus, like the two results above, Theorem 1.9 is also in accordance with Demailly's conjecture (Conjecture 1.4).

In the hope of understanding what properties of a projective scheme are sufficient for the conclusions of Theorems 1.5, 1.6, 1.8, and 1.9 to hold, we also investigate "weaker" notions of hyperbolicity.

1.2. Weaker notions of boundedness.

The results in the previous section were motivated by Demailly's conjecture (Conjecture 1.4). With a view towards Green–Griffiths–Lang's more general conjecture, we seek for analogues of the results in Section 1.1 for "weaker" notions of (algebraic) hyperbolicity.

Definition 1.10. A projective scheme X over k is 1-*bounded over* k if for every smooth projective connected curve C over k, the scheme $\underline{\mathrm{Hom}}_k(C, X)$ is of finite type over k.

Note that, if $\mathcal{L}$ is an ample line bundle on a projective scheme X over k, then X is 1-bounded over k if and only if, for every smooth projective connected curve C over k, there is a real number α_C such that, for every morphism $f : C \to X$ the inequality $\deg_C f^*\mathcal{L} \leq \alpha_C$ holds.

Clearly, an algebraically hyperbolic projective scheme over k is 1-bounded over k. The "difference" between algebraic hyperbolicity and 1-boundedness is in the uniformity of the bound we demand on the degree of a morphism $f : C \to X$. For X to be algebraically hyperbolic, we demand $\deg_C f^*L$ to be bounded **linearly** in the **genus** of C. For X to be 1-bounded, we ask the latter to be bounded by a real number depending only on C.

Despite the clear difference in the definitions, it seems reasonable to suspect that a 1-bounded projective variety is algebraically hyperbolic over k. As we explain in Section 10, the Green–Griffiths–Lang conjecture in fact predicts that 1-bounded projective schemes over k are algebraically hyperbolic over k. The results in this section are motivated by this latter observation.

We first show that 1-boundedness is also a "geometric" property, i.e., it persists over any algebraically closed field extension of k.

Theorem 1.11. *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Let X be a projective 1-bounded scheme over k. Then the projective scheme X_L is 1-bounded over L.*

Note that Theorems 1.6 and 1.8 follow from the following more general result.

Theorem 1.12. *Let X be a 1-bounded projective scheme over k. Then, for every projective scheme Y over k, every non-empty reduced closed subscheme $B \subset Y$, and every reduced closed subscheme $A \subset X$, the set of morphisms $f : Y \to X$ with $f(B) = A$ is finite.*

Finally, analogous to Theorem 1.9, we prove the following statement.

Theorem 1.13. *Let X be a projective 1-bounded scheme over k. If Y is a projective scheme over k, then the scheme $\underline{\mathrm{Hom}}_k(Y, X)$ is a projective 1-bounded scheme over k. Moreover, we have that $\dim \underline{\mathrm{Hom}}_k^{nc}(Y, X) < \dim X$.*

Building on the results of [20], the results of this paper are used in [18] to prove certain properties of "arithmetically hyperbolic" varieties. Moreover, in *loc. cit.*, certain arithmetic analogues of the results we obtain in this paper are also established.

1.3. **Outline of paper.** In Section 2 we introduce the notion of groupless varieties, and note that a proper groupless variety has a countable discrete group of automorphisms. We combine this with a theorem of Hwang-Kebekus-Peternell to prove that the scheme $\underline{\mathrm{Sur}}(Y, X)$ parametrizing surjective morphisms from a projective variety Y to a projective groupless variety X over k is a countable union of zero-dimensional projective schemes over k; see Theorem 2.7.

In Section 3 we explore basic properties of projective varieties with no rational curves. We show that finite type components of certain Hom-schemes of such varieties are proper, and that the "evaluation maps" defined on these Hom-schemes are finite morphisms; see Corollary 3.11.

In Section 4 we study various notions of boundedness and we explore some relations between boundedness, grouplessness and purity of projective schemes. Then, in Section 5 we show that algebraic hyperbolicity, Kobayashi hyperbolicity, and all the various notions of boundedness introduced in Section 4 behave in a similar way along finite maps. Furthermore, the finiteness of the set of dominant rational maps, surjective endomorphisms and automorphisms of a bounded scheme are proven in Section 6.

The geometricity of algebraic hyperbolicity (as predicted by the Green–Griffiths–Lang conjecture) is verified in Section 7. In fact, we also prove that every notion of boundedness introduced in this paper is "geometric" (i.e., persists over any algebraically closed field extension of the base field).

In Section 8 we show that finiteness of pointed Hom-sets from curves implies finiteness of pointed Hom-sets from varieties; see Theorem 8.4 for a precise statement. Similarly, in Section 9 we prove that boundedness of Hom-schemes from curves implies boundedness of Hom-schemes from varieties (Theorem 9.2). As an application of this result, we deduce that projective algebraically hyperbolic schemes are bounded (Theorem 9.2). This latter result implies, in particular, that all the properties proven for bounded schemes in Sections 4, 5 and 6 hold for algebraically hyperbolic schemes.

We prove all the results stated in the introduction in Section 10. In Section 11 we conclude the paper with several conjectures related to Demailly's and Green–Griffiths–Lang's conjectures. These conjectures relate the various notions of boundedness introduced in this paper.

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Conventions. Throughout this paper, we let k be an algebraically closed field of characteristic zero. A variety over k is a finite type separated reduced k-scheme.

2. Grouplessness

A variety is "groupless" if it does not admit any non-trivial map from an algebraic group. The precise definition reads as follows.

Definition 2.1. A finite type scheme X over k is *groupless (over k)* if, for every finite type connected group scheme G over k, every morphism of k-schemes $G \to X$ is constant.

Note that Kovács [23] and Kobayashi [21, Remark 3.2.24] refer to groupless varieties as being "algebraically hyperbolic". We avoid this unfortunate mix of terminology, and only use the term "algebraically hyperbolic" in the sense of Demailly (Definition 1.1).

The main result of this section is that the moduli space of surjective maps from a given projective variety to a given groupless projective variety is zero-dimensional; see Theorem 2.7 for a precise statement. We also take the opportunity to prove certain basic properties of groupless varieties.

Lemma 2.2. *Let S be an integral variety over k with function field K. Let $K(S) \subset L$ be an algebraically closed field extension. Let $X \to S$ be a morphism of varieties over k. Suppose that the set of s in $S(k)$ such that X_s is groupless over k is Zariski-dense in S. Then X_L is groupless over L.*

Proof. Suppose that X_L is not groupless. Then, we may choose a K-finitely generated subfield $K(S) \subset K \subset L$, a finite type connected group scheme G over K, and a a non-constant morphism $G \to X_K$. Let U be an integral variety with $K(U) = K$ and let $U \to S$ be a smooth dominant morphism of varieties over k extending the inclusion $K(S) \subset K$. Let $\mathcal{G}$ be a finite type geometrically connected group scheme over U, and let $\mathcal{G} \to X \times_S U$ be a morphism of U-schemes which extends the morphism $G \to X_K$ on the generic fibre. Our assumption that the set of s in $S(k)$ with X_s groupless is Zariski dense in S implies that the set of s in $U(k)$ such that X_s is groupless is Zariski-dense in U. In particular, for a dense set of s in $U(k)$, the morphism $\mathcal{G}_s \to (X \times_S U)_s = X_s$ is constant. This implies that the morphism $G \to X_K$ is constant, contradicting our assumption. We conclude that X_L is groupless over L. $\square$

Lemma 2.3 (Groaplessness is a geometric property)**.** *Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero and X a finite type scheme over k. If X is groupless over k, then X_L is groupless over L.*

Proof. This follows from Lemma 2.2 with $S = \operatorname{Spec} k$. □

Lemma 2.4. *Let X be a finite type scheme over k. The following statements are equivalent.*

1. *The finite type scheme X is groupless over k.*
2. *Every morphism to X from either $\mathbb{G}_{m,k}$ or an abelian variety over k is constant.*

Proof. That (1) implies (2) is clear. The other implication is a consequence of the structure theory of connected finite type (smooth quasi-projective geometrically connected) group schemes over k [6]. Indeed, assume (2) holds. Let G be a connected finite type group scheme over k. Let H be the unique normal connected affine (closed) subgroup of G such that G/H is an abelian variety. Since any morphism $G/H \to X$ is constant, it suffices to show that any morphism $H \to X$ is constant. Let $U \subset H$ be the unipotent radical. Since any morphism $\mathbb{G}_{m,k} \to X$ is constant, we see that every morphism $\mathbb{G}_{a,k} \to X$ is constant. Therefore, any morphism $U \to X$ is constant. Thus, we may replace by H by H/U, so that H is reductive. However, since H is the union of its Borel subgroups, we may and do assume that H is a solvable group in which case it is clear that $H \to X$ is constant by (2). □

Lemma 2.5. *Let X be a proper variety over k. Then X is groupless over k if and only if, for every abelian variety A over k, every morphism $A \to X$ is constant.*

Proof. Suppose that, for every abelian variety A over k, every morphism $A \to X$ is constant. To show that X is groupless, it suffices to show that every morphism $\mathbb{G}_{m,k} \to X$ is constant (Lemma 2.4). However, any such map extends to a morphism $\mathbb{P}^1_k \to X$. Let E be an elliptic curve over k and let $E \to \mathbb{P}^1_k$ be a surjective morphism. Then the composed morphism $E \to \mathbb{P}^1_k \to X$ is constant (by assumption), so that the morphism $\mathbb{P}^1_k \to X$ is constant. This proves the lemma. □

If X and Y are projective schemes over k, then the functor $\underline{\mathrm{Hom}}_k(X, Y)$ parametrizing morphisms $X \to Y$ is representable by a countable disjoint union of quasi-projective schemes over k ([13, Section 4.c, pp. 221-19 – 221-20]). If X is a projective scheme over a field k, then the functor $\mathrm{Aut}_{X/k}$ parametrizing automorphisms of X over k is representable by a locally finite type separated group scheme over k (which we also denote by $\mathrm{Aut}_{X/k}$). If X is a projective groupless scheme over k, then $\mathrm{Aut}_{X/k}$ is a zero-dimensional scheme, as we show now.

Lemma 2.6. *Let X be a proper variety over k. If X is groupless, then $\mathrm{Aut}^0_{X/k}$ is trivial. In particular, the group $\mathrm{Aut}(X)$ is a countable discrete group.*

Proof. Let $G = \mathrm{Aut}^0_{X/k}$. This is a finitely presented connected group algebraic space over k; see [1, Theorem 6.1]. Thus, it is a finite type connected group scheme over k (see [32, Tag 06E9]). Since X is groupless and G is a finite type connected group scheme over k, for x in X, the morphism $G \to X$ defined by $g \mapsto gx$ is constant. In other words, since every g in G acts trivially, we see that G is the trivial group. □

We can combine Lemma 2.6 with a theorem of Hwang-Kebekus-Peternell to get a stronger conclusion. To state it, for Y is a projective scheme over k, recall that the functor $\underline{\mathrm{Sur}}(Y, X)$ parametrizing surjective morphisms from Y to X is representable by a locally finite type separated scheme over k. Indeed, it is an open subscheme of $\underline{\mathrm{Hom}}_k(Y, X)$.

Theorem 2.7 (Hwang-Kebekus-Peternell)**.** *Let X be a projective groupless variety over k. Let Y be a normal projective variety over k. Then $\underline{\mathrm{Sur}}(Y,X)$ is a countable union of zero-dimensional projective schemes over k.*

Proof. Let $f : Y \to X$ be a surjective morphism. Let $\mathrm{Hom}_f(Y,X)$ be the connected component of $\underline{\mathrm{Hom}}(Y,X)$ containing f. By [17, Theorem 1.2], as X does not have any rational curves, there exists a factorization $Y \to Z \to X$ with $Z \to X$ a finite morphism, and a surjective morphism $\mathrm{Aut}^0_{Z/k} \to \underline{\mathrm{Hom}}_f(Y,X)$. Now, since X is groupless, it follows that Z is groupless, and thus $\mathrm{Aut}^0_{Z/k}$ is trivial. This implies that $\underline{\mathrm{Hom}}_f(Y,X)$ is a point. Therefore, since $\underline{\mathrm{Hom}}_k(Y,X)$ is a countable union of finite type schemes over k, this concludes the proof. $\square$

Remark 2.8. The "converse" of the theorem of Hwang-Kebekus-Peternell is not true. Let C be a smooth projective curve of genus two, and let X be the blow-up of $C\times_k C$ in a point. Then X is a smooth projective surface of general type. Note that X is not groupless (as it contains a rational curve). However, for any projective variety Y over k, the set of surjective morphism $Y \to X$ is finite.

3. Projective varieties with no rational curves

It turns out that, if Y is a projective scheme over k and X is a projective scheme over k with no rational curves, then the scheme $\underline{\mathrm{Hom}}_k(Y,X)$ is a countable union of *projective* schemes; this is a well-known ingredient in Mori's "bend-and-break". To state this in an efficient manner, we introduce some terminology.

Definition 3.1. A variety X over k is *pure (over k)* if, for every normal variety T over k and every dense open $U \subset T$ with $\mathrm{codim}(T\setminus U) \geq 2$, we have that every morphism $U \to X$ extends (uniquely) to a morphism $T \to X$.

Lemma 3.2. *Let X be a proper pure variety over k. Let Y be a normal variety over k. Then every rational dominant map from Y to X extends to a morphism $Y \to X$. In particular, if X is normal, every rational dominant self-map $X \dashrightarrow X$ extends to a surjective endomorphism $X \to X$.*

Proof. Since X is proper, every rational dominant map from Y to X can be defined on an open $U \subset Y$ with $\mathrm{codim}(Y \setminus U) \geq 2$. Therefore, the lemma follows from the definition of a pure variety (Definition 3.1). $\square$

Remark 3.3. Let X be a variety over k. Let $k \subset L$ be a field extension with L algebraically closed. Then X is pure over k if and only if X_L is pure over L. This follows from a standard spreading out and specialization argument (similar to the argument in the proof of Lemma 2.2).

Lemma 3.4. *Let $X \to Y$ be an affine morphism of varieties over k. If Y is pure over k, then X is pure over k.*

Proof. This is a consequence of Hartog's lemma. $\square$

Lemma 3.5. *A proper variety X over k is pure if and only if it has no rational curves, i.e., every morphism $\mathbb{P}^1_k \to X$ is constant.*

Proof. Let $0 = (0:0:1) \in \mathbb{P}^2(k)$. Since the projection $\mathbb{P}^2_k \setminus \{0\} \to \mathbb{P}^1$ does not extend to a morphism $\mathbb{P}^2 \to \mathbb{P}^1$, we see that $\mathbb{P}^1$ is not pure. Therefore, as a non-constant morphism $\mathbb{P}^1_k \to X$ is finite (hence affine), a proper variety with a rational curve is not pure by Lemma 3.4. Conversely, suppose that X has no rational curves, let $U \subset T$ be an open of a normal variety T with $\operatorname{codim}(T \setminus U) \geq 2$, and let $\varphi : U \to X$ be a morphism. Note that, as X is proper over k, we may resolve the locus of indeterminacy of φ by blowing up the complement of U in T. However, as the exceptional locus of this blow-up is covered by rational curves and X has no rational curves, we see that $U \to X$ extends to a morphism $T \to X$. ☐

Example 3.6. If X is an affine variety over k, then X is pure. Abelian varieties over k are pure. An algebraic K3 surface over k is not pure (as it contains a rational curve).

The relation between groupless varieties and pure varieties is provided by the following proposition.

Proposition 3.7. *If X is a proper groupless variety over k, then X is pure over k.*

Proof. Since proper groupless varieties have no rational curves, the proposition follows from Lemma 3.5. ☐

Note that a smooth proper genus one curve over k is pure, but not groupless. Thus, there are pure smooth projective varieties over k which are not groupless.

We show now that the rigidity lemma implies that "evaluation maps" restricted to closed subschemes of certain pieces of Hom-schemes are finite.

Lemma 3.8. *Let X be a proper variety over k and let Y be a projective variety over k. Let Z be a locally closed subscheme of $\underline{\mathrm{Hom}}_k(Y, X)$. Assume that Z is proper over k. Then, for any y in $Y(k)$, the evaluation morphism*

$$Z \to X, f \mapsto f(y)$$

is finite.

Proof. This is an application of the rigidity lemma [28, Chapter II] (cf. the argument of the proof of [21, Corollary 5.3.4]). To be more precise, let y be a point in $Y(k)$ and x a point in $X(k)$. Note that the fibre of the evaluation map $\mathrm{eval}_y : Z \to X$ defined by $\mathrm{eval}_y(f) = f(y)$ is the set $\mathrm{Hom}_k((Y,y),(X,x)) \cap Z(k)$ of morphisms $f : Y \to X$ in $Z(k)$ with $f(y) = x$. To show that $\mathrm{Hom}_k((Y,y),(X,x)) \cap Z(k)$ is finite, consider the closed subscheme $\underline{\mathrm{Hom}}_k((Y,y),(X,x)) \subset \underline{\mathrm{Hom}}_k(Y,X)$ parametrizing maps $f : Y \to X$ with $f(y) = x$. Let H be a connected component of $Z \cap \underline{\mathrm{Hom}}_k((Y,y),(X,x))$. Since Z is proper, the scheme H is proper over k. It suffices to show that $H(k)$ is a singleton.

The morphism $\mathrm{eval} : Y \times H \to X$ given by $(y', f) \mapsto f(y')$ has the property that $(y, f) = x$ for all $f \in H$, i.e., it contracts $\{y\} \times H$ to a point. Thus, since H is proper, the rigidity lemma implies that the morphism $\mathrm{eval} : Y \times H \to X$ factors over some morphism $g : Y \to X$, i.e., $\mathrm{eval} = g \circ \mathrm{pr}_Y$. In other words, for any f in in H and any y in Y, we have that $f(y) = g(y)$. Thus, $H(k) = \{g\}$. This concludes the proof. ☐

To apply Lemma 3.8 we now show that finite type (separated) subschemes of Hom-schemes of pure varieties are proper.

Proposition 3.9. *Let X be a projective variety over k which is pure over k. Then, for every normal projective variety Y over k, the locally finite type scheme $\underline{\mathrm{Hom}}_k(Y, X)$ satisfies* *the valuative criterion of properness over k. In particular, for any $P \in \mathbb{Q}[t]$, the scheme $\underline{\mathrm{Hom}}_k^P(Y,X)$ is projective and pure over k.*

Proof. Let S be a smooth affine curve over k and let $K = K(S)$ be its function. Note that $X_S := X \times_k S$ is pure over k (as X and S are pure over k). We claim that the injective map of sets

$$\mathrm{Hom}_S(Y_S, X_S) \to \mathrm{Hom}_K(Y_K, X_K)$$

is surjective. To do this, let $f : Y_K \to X_K$ be a morphism over K. Since X is proper over k, the scheme X_S is proper over S. In particular, by the valuative criterion of properness, there is an open $U \subset Y_S$ with $U_K \cong Y_K$ and a morphism $U \to X_S$ extending $Y_K \to X_K$ with $\mathrm{codim}(Y_S \setminus U) \geq 2$. Since $Y_S = Y \times_k S$ is a normal variety over k, the purity of $X \times_k S$ implies that the morphism $U \to X_S$ extends to a morphism $Y_S \to X_S$. This shows that $\underline{\mathrm{Hom}}_k(Y,X)$ satisfies the valuative criterion of properness over k. In particular, for any P in $\mathbb{Q}[t]$, the quasi-projective scheme $\underline{\mathrm{Hom}}_k^P(Y,X)$ over k is projective over k. Now, since $Z := \underline{\mathrm{Hom}}_k^P(Y,X)$ is proper over k, by Lemma 3.8, for y in $Y(k)$, the evaluation morphism $\mathrm{eval}_y : Z \to X$ is finite. Thus, as X has no rational curves (Lemma 3.5), we conclude that Z has no rational curves (Lemma 3.5). Therefore, $Z = \underline{\mathrm{Hom}}_k^P(Y,X)$ is pure. This concludes the proof. □

The arguments used in the proof of Proposition 3.9 can be used to prove the properness of other Hom-schemes (and Hom-stacks) as we show now. Concerning algebraic stacks, we follow the conventions of the stacks project [32, Tag 026N].

Lemma 3.10. *Let X be a projective pure variety over k, and let $U \to M$ be a smooth proper representable morphism of finite type separated algebraic stacks over k. Let $\mathcal{L}$ be a M-relative ample line bundle on U. Then, the natural (representable) morphism $\underline{\mathrm{Hom}}_M(U, X \times M) \to M$ of algebraic stacks satisfies the valuative criterion of properness over k. Therefore, for any polynomial $P \in \mathbb{Q}[t]$, the morphism $\underline{\mathrm{Hom}}_M^P(U, X \times M) \to M$ is proper.*

Proof. Since $\underline{\mathrm{Hom}}_M^P(U, X \times M) \to M$ is a finite type separated morphism of finite type separated algebraic stacks, it suffices to prove the first statement. Let S be a smooth affine curve over k, and let $S \to M$ be a morphism, and suppose that the morphism $\mathrm{Spec}\, K(S) \to S \to M$ lifts to a morphism $\mathrm{Spec}\, K(S) \to \underline{\mathrm{Hom}}_M(U, X \times M)$. In other words, we are given a smooth finite type morphism $U_S \to S$ of schemes and a morphism $U_{K(S)} \to X_{K(S)}$. By properness of $X \times_k S \to S$, there is a dense open $V \subset U_S$ with $\mathrm{codim}(U_S \setminus V) \geq 2$ and a morphism $V \to X \times_k S$ which extends the morphism $U_{K(S)} \to X_{K(S)}$ over $K(S)$ to a morphism over S. Since S is affine, the curve S is pure over k (Lemma 3.4). Therefore, since X and S are pure over k, the variety $X \times_k S$ is pure over k. Since U_S is a smooth (hence normal) variety over k and $X \times_k S$ is pure over k, the morphism $V \to X \times_k S$ extends to a morphism $U_S \to X \times_k S$. This concludes the proof of the lemma. □

We now combine Lemma 3.8 and Proposition 3.9 to show that "evaluation maps" on finite type closed subschemes of Hom-schemes of pure varieties are finite.

Corollary 3.11. *Let X be a proper pure variety over k and let Y be a normal projective variety over k. Let Z be a finite type locally closed subscheme of $\underline{\mathrm{Hom}}_k(Y,X)$. Then Z is projective and, for any y in $Y(k)$, the evaluation morphism*

$$Z \to X, f \mapsto f(y)$$

is finite.

Proof. We may and do assume that Z is closed in $\underline{\mathrm{Hom}}_k(Y,X)$. Then, by Proposition 3.9, the quasi-projective k-scheme Z is proper over k (hence projective). The result now follows from Lemma 3.8. □

Let $m \geq 1$ be an integer. Let Y and X be proper schemes over k. Let $y_1,\ldots,y_m$ be elements of $Y(k)$, and let $x_1,\ldots,x_m$ be elements of $X(k)$. The functor

$$\underline{\mathrm{Hom}}_k((Y,y_1,\ldots,y_m),(X,x_1,\ldots,x_m))$$

parametrizing morphisms $f : Y \to X$ with $f(y_1)=x_1,\ldots,f(y_m)=x_m$ is representable by a (possibly empty) closed subscheme of $\underline{\mathrm{Hom}}_k(Y,X)$ which we (also) denote by

$$\underline{\mathrm{Hom}}_k((Y,y_1,\ldots,y_m),(X,x_1,\ldots,x_m)).$$

We conclude this section with the following proposition which says that algebraic sets of pointed maps to a pure variety are zero-dimensional.

Proposition 3.12. *Let X be a projective pure variety over k. Let $m \geq 1$ be an integer, let Y be a projective variety over k, let $y_1,\ldots,y_m$ be pairwise distinct points in $Y(k)$, and let $x_1,\ldots,x_m \in X(k)$. Then the locally finite type scheme*

$$\underline{\mathrm{Hom}}_k((Y,y_1,\ldots,y_m),(X,x_1,\ldots,x_m))$$

parametrizing morphisms $f : Y \to X$ with $f(y_1)=x_1,\ldots,f(y_m)=x_m$ is zero-dimensional.

Proof. We show that, for y in $Y(k)$ and x in $X(k)$, the scheme $\underline{\mathrm{Hom}}_k((Y,y),(X,x))$ is zero-dimensional. (This is clearly enough.) Since $\underline{\mathrm{Hom}}(Y,X)$ is a (countable) disjoint union of finite type open subschemes, it suffices to show that, for each finite type open subscheme Z of $\underline{\mathrm{Hom}}_k(Y,X)$ and every y in $Y(k)$, the evaluation morphism $Z \to X$ of k-schemes given by $f \mapsto f(y)$ is quasi-finite. Since the projective variety X is pure over k and Z is a finite type locally closed subscheme of $\mathrm{Hom}_k(Y,X)$, the claim follows from Corollary 3.11. □

4. Bounded varieties: definitions and grouplessness

By definition, a projective algebraically hyperbolic variety satisfies a "strong" form of boundedness with respect to maps from curves. As we explained in the introduction, a lot of properties of algebraically hyperbolic varieties we prove in this paper also hold for varieties satisfying (a priori) "weaker" properties of boundedness with respect to maps from curves. To state and prove our results, we start by defining what we mean by "bounded" and "(n,m)-bounded" projective schemes.

Definition 4.1. Let n be a non-negative integer. A projective scheme X over k is *n-bounded over k* if, for all normal projective integral schemes Y of dimension at most n over k, the scheme $\underline{\mathrm{Hom}}_k(Y,X)$ is of finite type over k. A projective scheme X over k is *bounded* if, for all integers $n \geq 1$, the scheme X is n-bounded.

Definition 4.2. Let n and m be non-negative integers. A projective scheme X over k is *(n,m)-bounded (over k)* if, for all normal projective integral schemes Y of dimension at most n over k, all pairwise distinct points $y_1,\ldots,y_m \in VYk)$, and all $x_1,\ldots,x_m \in X(k)$, the scheme

$$\underline{\mathrm{Hom}}_k((Y,y_1,\ldots,y_m),(X,x_1,\ldots,x_m))$$

is of finite type.

Remark 4.3. Note that a projective variety over k is n-bounded over k if and only if it is $(n,0)$-bounded over k (by definition). Obviously, if X is n-bounded, then X is $(n-1)$-bounded. Moreover, if X is (n,m)-bounded over k, then X is $(n,m+1)$-bounded.

Proposition 4.4. *Let $n \geq 1$ and $m \geq 0$ be integers. If X is a projective (n,m)-bounded scheme over k, then X is groupless and pure over k.*

Proof. Since a projective (a,b)-bounded scheme is $(a,b+1)$-bounded (Remark 4.3), we may and do assume that the integer m is greater or equal to 1. Suppose that X is not groupless over k. Let A be an abelian variety and let $f : A \to X$ be a non-constant morphism. Let $a_1, \ldots, a_m$ be pairwise distinct points in $A[m]$. Let $C \subset A$ be a smooth projective curve containing $a_1, \ldots, a_n$. Let ℓ be a positive integer such that $\ell = 1 \mod m$. Since x_i is m-torsion, we see that $\ell x_i = x_i$ in A. The morphism $f \circ [\ell] : A \to X$ sends $a_1, \ldots, a_n$ to $f(a_1), \ldots, f(a_n)$, respectively. In particular, as the morphisms $f \circ [\ell]$ correspond to k-points of different components of $\underline{\mathrm{Hom}}_k((A, a_1, \ldots, a_n), (X, f(a_1), \ldots, f(a_n))$, we see that X is not $(1,m)$-bounded. It follows that X is not (n,m)-bounded. We conclude that a projective (n,m)-bounded scheme is groupless. Finally, since projective groupless varieties are pure (Proposition 3.7), this concludes the proof. $\square$

Corollary 4.5. *A projective algebraically hyperbolic variety over k is groupless and pure over k.*

Proof. Note that algebraically hyperbolic projective varieties are 1-bounded. Therefore, the result follows from Proposition 4.4. $\square$

Assuming $m \geq 1$ is a positive integer, we now show that an (n,m)-bounded projective variety admits only finitely many pointed maps $(Y, y_1, \ldots, y_m) \to (X, x_1, \ldots, x_m)$. The precise statement reads as follows.

Lemma 4.6. *Let X be a projective variety over k. Let $n \geq 1$ and $m \geq 1$ be integers. The following are equivalent.*

1. *The projective variety X is (n,m)-bounded over k.*
2. *For all projective integral schemes Y of dimension at most n over k, all pairwise distinct points $y_1, \ldots, y_m \in V(k)$, and all $x_1, \ldots, x_m \in X(k)$, the set*
$$\mathrm{Hom}_k((Y, y_1, \ldots, y_m), (X, x_1, \ldots, x_m))$$
is finite.

Proof. Clearly, (2) $\implies$ (1). Let us show that (1) $\implies$ (2). Replacing Y by its normalization, we may and do assume that Y is normal. Now, to show that (1) $\implies$ (2), note that an (n,m)-bounded projective variety is groupless and pure (Proposition 4.4). Therefore, by purity and projectivity of X, the scheme $\underline{\mathrm{Hom}}_k((Y, y_1, \ldots, y_m), (X, x_1, \ldots, x_m))$ is zero-dimensional (Proposition 3.12). By our assumption (1), the latter scheme is of finite type. As a finite type zero-dimensional k-scheme is finite, this concludes the proof. $\square$

Remark 4.7. Note that the assumption that m is positive in Lemma 4.6 is necessary. Indeed, $\mathrm{Hom}(C, X)$ contains all constant maps $C \to X$, and is therefore infinite (even if X is bounded). However, more interestingly, there is a bounded projective surface X over $\mathbb{C}$ and a smooth projective curve C over $\mathbb{C}$ such that there are infinitely many non-constant morphisms $C \to X$ (of bounded degree).

Proposition 4.8. *Let X be a projective variety over k. The following are equivalent.*

(1) The projective variety X is $(1,1)$-bounded.
(2) There is an integer $m \geq 1$ such that X is $(1,m)$-bounded.

Proof. If X is $(1,1)$-bounded, then X is $(1,m)$-bounded (Remark 4.3). Now, assume that X is $(1,m)$-bounded. Let C be a smooth projective curve over k, let $c \in C(k)$, and let $x \in X(k)$. We now show that set $\mathrm{Hom}_k((C,c),(X,x))$ is finite.

Let D be a smooth projective connected curve and let $f : D \to C$ be a finite surjective morphism of degree m which is étale over c. Write $\{d_1,\ldots,d_m\} = f^{-1}\{c\}$, and note that $d_1,\ldots,d_m$ are pairwise distinct points of D. Define $x_1 = \ldots = x_m = x$. Since $D \to C$ is surjective, the map of sets

$$\mathrm{Hom}_k((C,c),(X,x)) \to \mathrm{Hom}_k((D,d_1,\ldots,d_m),(X,x_1,\ldots,x_m)), \quad g \mapsto g \circ f$$

is injective. Since X is $(1,m)$-bounded, the set $\mathrm{Hom}_k((D,d_1,\ldots,d_m),(X,x_1,\ldots,x_m))$ is finite (Lemma 4.6). We conclude that $\mathrm{Hom}_k((C,c),(X,x))$ is finite, so that X is $(1,1)$-bounded, as required. □

We will later show that a projective variety X over k is $(1,1)$-bounded over k if and only if, for every $n \geq 1$ and $m \geq 1$, we have that X is (n,m)-bounded; see Theorem 8.4 for a more precise statement.

5. Hyperbolicity and boundedness along finite maps

In this section we show that the notions of being algebraically hyperbolic, Kobayashi hyperbolic, and (n,m)-bounded (for some fixed n and m) behave in a similar manner along finite maps.

In our proofs below we will use the "slope" of a morphism $f : C \to X$ with respect to a fixed ample line bundle on X.

Definition 5.1. Let $\mathcal{L}$ be an ample line bundle on a projective scheme X over k. Let $C \to X$ be a morphism of projective schemes over k with C a smooth projective connected curve over k. The *slope $s(f)$ of f (with respect to L)* is defined as

$$s(f) = \frac{\deg_C f^*\mathcal{L}}{\max(1, \mathrm{genus}(C))}.$$

Note that a projective scheme X over k is algebraically hyperbolic over k if and only if there is a real number α (depending only on X and some fixed ample line bundle $\mathcal{L}$ on X) such that, for every smooth projective connected curve C over k and every morphism $f : C \to X$, the slope (with respect to the aforementioned fixed ample line bundle on X) satisfies $s(f) \leq \alpha$. On the other hand, a projective scheme X over k is 1-bounded over k if and only if for every smooth projective connected curve C over k there is a real number α_C such that $s(f) \leq \alpha_C$. Thus, one could say that algebraic hyperbolicity is a "uniform" version of 1-boundedness, as a projective variety X is algebraically hyperbolic if and only if the slope of a morphism from a smooth projective curve to X is uniformly bounded.

Proposition 5.2. *Let $f : X \to Y$ be a finite morphism of projective varieties over k. Then the following statements hold.*

(1) If Y is algebraically hyperbolic over k, then X is algebraically hyperbolic over k.
(2) Let $n \geq 1$ and $m \geq 0$ be integers. If Y is (n,m)-bounded over k, then X is (n,m)-bounded over k.

(3) *Assume $k = \mathbb{C}$. If Y is Kobayashi hyperbolic over k, then X is Kobayashi hyperbolic over k.*

Proof. To prove (1), let $\mathcal{L}$ be an ample line bundle on Y. Since $f : X \to Y$ is finite, the line bundle $f^*\mathcal{L}$ is ample on X. Suppose that X is not algebraically hyperbolic over k, so that there is a smooth projective connected curve C over k, and infinitely many morphisms $f_i : C \to X$ such that the slope $s(f_i) = \deg(f_i)/\mathrm{genus}(C)$ (Definition 5.1) tends to infinity as i tends to infinity, where we compute the degree of $f_i : C \to X$ with respect to $f^*\mathcal{L}$. For every i, let $g_i := f \circ f_i$, and note that the slope of the finite morphism $g_i : C \to Y$ equals the slope of f_i, and is therefore unbounded. This shows that Y is algebraically hyperbolic, and proves (1).

To prove (2), suppose that X is not (n, m)-bounded. We now show that Y is not (n, m)-bounded. Let V be a normal projective variety of dimension 1, let $v_1, \ldots, v_m$ be pairwise distinct points in $V(k)$, and let $x_1, \ldots, x_m \in X(k)$ be such that

$$\underline{\mathrm{Hom}}_k((V, v_1, \ldots, v_m), (X, x_1, \ldots, x_m))$$

is not of finite type over k. Let $f_i \in \mathrm{Hom}((V, v_1, \ldots, v_m), (X, x_1, \ldots, x_m))$ be elements with pairwise distinct Hilbert polynomials. For $i \in \{1, \ldots, m\}$, define $y_i := f(x_i)$ and $g_i := f_i \circ f$. Note that the elements

$$g_i \in \mathrm{Hom}((V, v_1, \ldots, v_m), (Y, y_1, \ldots, y_m))$$

have pairwise distinct Hilbert polynomial. This shows that

$$\underline{\mathrm{Hom}}_k((V, v_1, \ldots, v_m), (Y, y_1 \ldots, y_m))$$

is not of finite type, so that Y is not (n, m)-bounded over k.

We note that (3) is due to Kwack [24, Theorem 1]. (One could also use Brody's lemma and the analogous statement for Brody hyperbolicity. One could also appeal to [21, Proposition 3.2.11]) This concludes the proof. $\square$

Corollary 5.3. *Let X be a projective scheme over k. Let Y be a normal projective scheme and let $P \in \mathbb{Q}[t]$ be a non-zero polynomial. Then the following statements hold.*

(1) *If X is algebraically hyperbolic over k, then $\underline{\mathrm{Hom}}_k^P(Y, X)$ is a projective algebraically hyperbolic scheme over k with $\dim \underline{\mathrm{Hom}}_k^P(Y, X) \leq \dim X$.*

(2) *If $n \geq 1$ and $m \geq 0$ are integers and X is (n, m)-bounded over k, then $\underline{\mathrm{Hom}}_k^P(Y, X)$ is a projective (n, m)-bounded scheme over k with $\dim \underline{\mathrm{Hom}}_k^P(Y, X) \leq \dim X$.*

Proof. Let X be as in (1) or (2). First, note that X is groupless by Corollary 4.5) and Proposition 4.4. Therefore, as X is proper and groupless, it follows that X is pure (Proposition 3.7). Thus, since $Z := \underline{\mathrm{Hom}}_k^P(Y, X)$ is a finite type subscheme of $\underline{\mathrm{Hom}}_k(Y, X)$, by Corollary 3.11, the scheme $\underline{\mathrm{Hom}}_k^P(Y, X)$ is projective and, for every y in $Y(k)$, the evaluation morphism $\mathrm{eval}_y \colon \underline{\mathrm{Hom}}_k^P(Y, X) \to X$ is finite. This implies that $\dim \underline{\mathrm{Hom}}_k^P(Y, X) \leq \dim X$. Now, if X is algebraically hyperbolic (resp. (n, m)-bounded), it follows from Proposition 5.2 that $\underline{\mathrm{Hom}}_k^P(Y, X)$ is algebraically hyperbolic (resp. (n, m)-bounded). This concludes the proof. $\square$

Proposition 5.4. *Let $f : X \to Y$ be a finite étale morphism of projective varieties. Then the following statements hold.*

(1) *If X is algebraically hyperbolic over k, then Y is algebraically hyperbolic over k.*

(2) If $n \geq 1$ and $m \geq 0$ are integers and X is (n,m)-bounded over k, then Y is (n,m)-bounded over k.

(3) Assume $k = \mathbb{C}$. If X^{an} is Kobayashi hyperbolic, then Y^{an} is Kobayashi hyperbolic.

Proof. Let $d = \deg(Y/X)$, let $\mathcal{L}$ be an ample line bundle on Y, and note that $f^*\mathcal{L}$ is ample on X.

To prove (1), assume that X is algebraically hyperbolic over k, and let α be a real number (which depends on $\mathcal{L}$ and $f : X \to Y$) such that, for every smooth projective connected curve C' over k and every morphism $f' : C' \to X$ we have $s(f') \leq \alpha$. To show that Y is algebraically hyperbolic over k, let C be a smooth projective curve over k and let $f : C \to Y$ be a morphism. Let $D := C \times_Y X$, and let $g : D \to X$ be the natural morphism. Note that D is a smooth projective curve over k. We now bound the slope $s(f)$ of f (Definition 5.1). Note that $\mathrm{genus}(D) = d\mathrm{genus}(C) > 0$ and that

$$\alpha \geq \quad s(g) = \frac{\deg_D g^* f^* \mathcal{L}}{\mathrm{genus}(D)} = \frac{d \deg_C f^* \mathcal{L}}{d\mathrm{genus}(C)} = s(f).$$

In particular, the slope of f is bounded by α. We conclude that Y is algebraically hyperbolic over k.

To prove (2), assume that Y is not (n,m)-bounded. Let V be a normal projective variety of dimension at most n over k, let $v_1, \ldots, v_m$ be points in $V(k)$, let $y_1, \ldots, y_m$ be points in $Y(k)$, and let $f_i : V \to Y$ be a sequence of morphisms with pairwise distinct Hilbert polynomials and $f(v_i) = y_i$. Since k is an algebraically closed field of characteristic zero, it follows from [15, Exposé II. Theorem 2.3.1] that the set of k-isomorphism classes of (normal) projective varieties W such that there is a finite étale morphism $W \to V$ of degree at most d is finite. Therefore, replacing $(f_i)_{i=1}^{\infty}$ by a subsequence if necessary, we may and do assume that, for all positive integers i, we have $W := V \times_{f_1,Y,f} X \cong V \times_{f_i,Y,f} X$. Let $g_i : W = V_i \times_Y X \to X$ be the natural morphism. For every positive integer i, consider the morphism $W = V_i \times_Y X \to V$ and let w_i be a point lying over v_i. Replacing $(f_i)_{i=1}^{\infty}$ by a subsequence if necessary, we may and do assume that $g_i(w_1), \ldots, g_i(w_m)$ are independent of i. Let $x_1 := g_1(w_1), \ldots, x_m = g_1(w_m)$. Note that g_i is an element of

$$\mathrm{Hom}_k((W, w_1, \ldots, w_m), (X, x_1, \ldots, x_m)).$$

Since the f_i have pairwise distinct Hilbert polynomial, it follows that the g_i have pairwise distinct Hilbert polynomial. This shows that X is not (n,m)-bounded.

Note that (3) follows from [21, Theorem 3.2.8.(2)]. (One can also use Brody's lemma and the easier to establish analogous statement of (3) for Brody hyperbolicity to prove (3).) □

6. Finiteness results for bounded varieties

In this section we prove finiteness results for certain moduli spaces of maps. The main ingredients in this section are the theorem of Hwang–Kebekus–Peternell (Theorem 2.7), and the properties of Hom-schemes of pure varieties established in Section 3.

Our first lemma gives the finiteness of surjective maps from a given projective variety Y to a bounded projective variety X.

Lemma 6.1. *Let $n \geq 1$ be an integer. Let X be a projective n-bounded variety over k. If Y is a reduced projective variety of dimension at most n, then the set of surjective morphisms $Y \to X$ is finite.*

Proof. We may and do assume that Y is integral. Let $Y' \to Y$ be the normalization of Y. Note that the natural map of sets $\mathrm{Sur}_k(Y,X) \to \mathrm{Sur}_k(Y',X)$ is injective. Therefore, replacing Y by its normalization if necessary, we may and do assume that Y is normal. Since X is n-bounded, it follows that X is groupless (Proposition 4.4). Therefore, by Hwang-Kebekus-Peternell's theorem (Theorem 2.7), the scheme $\underline{\mathrm{Sur}}_k(Y,X)$ is zero-dimensional. Since X is n-bounded and Y is a normal projective variety, the scheme $\underline{\mathrm{Hom}}_k(Y,X)$ is of finite type over k. In particular, the zero-dimensional scheme $\underline{\mathrm{Sur}}_k(Y,X)$ is of finite type over k, and is therefore finite over k. This concludes the proof. □

Corollary 6.2. *Let $n \geq 1$ be an integer. Let X be an n-dimensional reduced projective n-bounded variety over k. Then X has only finitely many surjective endomorphisms, and every surjective endomorphism of X is an automorphism of X. In particular,* $\mathrm{Aut}(X)$ *is finite.*

Proof. This follows from Lemma 6.1. □

Proposition 6.3. *Let $n \geq 1$ be an integer. Let X be a projective n-bounded scheme over k. If Y is a normal projective variety of dimension at most n over k, then the set of dominant rational maps $Y \dashrightarrow X$ is finite.*

Proof. Since X is n-bounded, it is pure (Proposition 4.4). Therefore, every dominant rational map $Y \dashrightarrow X$ extends uniquely to a well-defined surjective morphism $Y \to X$ (Lemma 3.2). The result now follows from Lemma 6.1. □

Corollary 6.4. *Let $n \geq 1$ be an integer. Let X be a projective n-bounded scheme over k. Let Y be a projective scheme of dimension at most n over k. Let $A \subset X$ be a non-empty reduced closed subscheme of X, and let $B \subset Y$ be a non-empty reduced closed subscheme of Y. Then the set*

$$\{f \in \mathrm{Hom}_k(Y,X) \mid f(B) = A\}$$

is finite.

Proof. Note that the inclusion $A \to X$ is finite. Therefore, since X is n-bounded over k, it follows that A is n-bounded over k (Proposition 5.2). Thus, as $\dim B \leq \dim Y \leq n$ and B is reduced, the set $\mathrm{Sur}(B,A)$ of surjective morphisms $B \to A$ is finite (Lemma 6.1). Fix b in $B(k)$. Then, the finiteness of $\mathrm{Sur}(B,A)$ implies that the set

$$I := I_b := \{a \in A \mid \text{there is a surjective morphism } f: B \to A \text{ with } f(b) = a\}$$

is finite. Now, it is clear that

$$\mathrm{Hom}_k((Y,B),(X,A)) \subset \cup_{a \in I} \mathrm{Hom}_k((Y,b),(X,a)).$$

Thus, as the set I is finite, it suffices to show that, for every $a \in X(k)$, the set

$$\mathrm{Hom}_k((Y,b),(X,a))$$

is finite. Since an n-bounded variety is $(n,1)$-bounded (Remark 4.3), the latter finiteness follows from Lemma 4.6. □

7. Geometricity theorems

Note that a variety over k is of general type if and only if it remains so after any algebraically closed field extension. In other words, the property that a variety is of general type is "geometric" (in the sense that it persists over any algebraically closed field extension). Similarly, by Lemma 2.2, the property that a variety is groupless is also "geometric", and the property that a variety is pure is also "geometric" (Remark 3.3). Therefore, as the Green–Griffiths–Lang's conjecture says that a projective variety is groupless if and only if it is algebraically hyperbolic, we see that the Green–Griffiths–Lang conjecture predicts that algebraic hyperbolicity is a "geometric" property (i.e., persists over any algebraically closed field extension). We now prove this.

Theorem 7.1 (Algebraic hyperbolicity is a geometric property)**.** *Let X be a projective scheme over k and let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. Then X is algebraically hyperbolic over k if and only if X_L is algebraically hyperbolic over L.*

Proof. Since X is algebraically hyperbolic over k, it is groupless over k (Corollary 4.5). In particular, X_L is groupless over L (by Lemma 2.3). In particular, the variety X_L admits no maps from a smooth projective curve of genus at most one.

Let α be a real number such that, for any $g \geq 2$, any $C \in \mathcal{M}_g(k)$, and any non-constant morphism $f : C \to X$, we have that the slope $s(f)$, as defined in Definition5.1, satisfies $s(f) \leq \alpha$. Such a real number α exists, as X is algebraically hyperbolic over k.

Let C be a smooth projective curve of genus g (at least two) over L and let $g : C \to X_L$ be a non-constant morphism. Choose a smooth affine variety U over $\operatorname{Spec} k$, choose a smooth proper geometrically connected genus g curve $\mathcal{C} \to U$ over U with $\mathcal{C}_L \cong C$, and choose a morphism $\mathcal{C} \to X \times U$ of U-schemes which equals $C \to X_L$ after pull-back along $\operatorname{Spec} L \to U$. Let $u \in U(k)$ and consider the induced morphism $f : \mathcal{C}_u \to X \times \{u\} \cong X$. Note that the slope of the morphism $f : \mathcal{C}_u \to X$ equals the slope of the morphism $g : C \to X_L$, i.e., $s(g) = s(f)$. Therefore, since $\mathcal{C}_u$ is in $\mathcal{M}_g(k)$ and $\mathcal{C}_u \to X$ is non-constant, we have that

$$s(g) = s(f) \leq \alpha.$$

This implies that X_L is algebraically hyperbolic over L, and concludes the proof. $\square$

Motivated by Green–Griffiths–Lang conjecture, and the similarities between the notions of boundedness and algebraic hyperbolicity established in Sections 4 and 5, we now establish the geometricity of boundedness.

To do so, for $g \geq 2$ an integer, let $\mathcal{M}_g$ be the stack of smooth proper curves of genus g over $\mathbb{Z}$, and let $\mathcal{U}_g \to \mathcal{M}_g$ be the universal smooth proper geometrically connected curve of genus g over $\mathcal{M}_g$. Recall that $\mathcal{M}_g$ is a smooth finite type separated Deligne-Mumford algebraic stack over $\mathbb{Z}$. More generally, for $g \geq 2$ and $m \geq 1$ an integer, let $\mathcal{M}_{g,m}$ be the stack of m-pointed smooth proper geometrically connected curves of genus g, and let $\mathcal{U}_{g,m} \to \mathcal{M}_{g,m}$ be the universal family.

Theorem 7.2 (1-boundedness is a geometric property)**.** *Let X be a projective scheme over k. Then X is 1-bounded over k if and only if X_L is 1-bounded over L.*

Proof. Clearly, if X_L is 1-bounded over L, then X is 1-bounded over k. To prove the converse, assume that X_L is not 1-bounded over L. Then, there is an integer $g \geq 2$, a smooth projective curve C over L of genus g, and an increasing sequence of integers $d_1 < d_2 < \ldots$ such that $\underline{\mathrm{Hom}}_L^{d_i}(C, X_L)$ has an L-point.

Since X is 1-bounded over k, it follows that X is pure and groupless (Proposition 4.4). Define $M := \mathcal{M}_g \otimes_{\mathbb{Z}} k$ and $U := \mathcal{U}_g \otimes_{\mathbb{Z}} k$. Note that $U \to M$ is a smooth proper representable morphism of finite type separated algebraic stacks over k. Moreover, the relative dualizing sheaf $\omega_{U/M}$ is an M-relative ample line bundle on U (as $g \geq 2$).

Note that, as X is projective and pure over k, the natural morphism $\underline{\mathrm{Hom}}_M(U, X \times M) \to M$ satisfies the valuative criterion of properness over k (Lemma 3.10). In particular, for any integer d, the finite type separated morphism $\phi_d : \underline{\mathrm{Hom}}_M^d(U, X \times M) \to M$ is proper. Let $Z_d \subset M$ be the stack-theoretic image of ϕ_d, and note that Z_d is a closed substack of M.

Now, since $\underline{\mathrm{Hom}}_L^{d_i}(C, X_L)$ has an L-point for all $i \in \{1, 2, \ldots\}$, the algebraic stack Z_{d_i} (over k) has an L-object (corresponding to the curve C) for all $i \in \{1, 2, \ldots\}$. Define $Z := \cap_{i=1}^{\infty} Z_{d_i}$, and note that Z is a closed substack of $\mathcal{M}_g$ with an L-point. Since Z is a finite type separated algebraic stack over k with an L-point, we conclude that $Z(k)$ is non-empty. This means that $Z_{d_i}(k) \neq \emptyset$ for all $i = 1, 2, \ldots$. Thus, there is a smooth projective curve C' of genus g and a sequence of morphisms $g_i : C' \to X$ of increasing degree. This shows that X is not 1-bounded over k, and concludes the proof. $\square$

The argument to prove Theorem 7.2 can be used to show that $(1, m)$-boundedness is a geometric property, as we show now.

Theorem 7.3 ($(1, m)$-boundedness is a geometric property)**.** *Let X be a projective scheme over k, and let $m \geq 1$. Then X is $(1, m)$-bounded over k if and only if X_L is $(1, m)$-bounded over L.*

Proof. We follow the proof of Theorem 7.2 with only minor modifications. Assume that X_L is not $(1, m)$-bounded over L. Then, there is an integer $g \geq 2$, a smooth proper connected curve C over L of genus g, pairwise distinct points $c_1, \ldots, c_m \in C(L)$, and points $x_1, \ldots, x_m \in X(L)$ such that the scheme

$$\underline{\mathrm{Hom}}_L((C, c_1, \ldots, c_m), (X_L, x_1, \ldots, x_m))$$

is not of finite type over L. We fix C, $c_1, \ldots, c_m$, and $x_1, \ldots, x_m$ with this property.

Define $M := \mathcal{M}_{g,m} \otimes_{\mathbb{Z}} k$ and $U := \mathcal{U}_{g,m} \otimes_{\mathbb{Z}} k$. Note that $U \to M$ is a smooth proper morphism of finite type separated algebraic stacks over k which is representable by schemes. Moreover, there is an M-relative ample line bundle on U.

For any integer d, let

$$\phi_d : \underline{\mathrm{Hom}}_M^d(U, X \times M) \to M \times X^m$$

be the morphism defined by

$$((D, d_1, \ldots, d_m), f : D \to X) \mapsto ((D, d_1, \ldots, d_m), (f(d_1), \ldots, f(d_m)).$$

Since X is $(1, m)$-bounded over k, it follows that X is pure (Proposition 4.4). Therefore, as X is projective and pure over k, the natural morphism

$$\underline{\mathrm{Hom}}_M^d(U, X \times M) \to M$$

is proper (Lemma 3.10). As X^m is separated over k and the composed morphism $\underline{\mathrm{Hom}}_M^d(U, X \times M) \to M \times X^m \to M$ is proper, the morphism

$$\phi_d : \underline{\mathrm{Hom}}_M^d(U, X \times M) \to M \times X^m$$

is proper [26].

Let Z_d be the image of ϕ_d in $M \times X^m$, and note that Z_d is a closed substack of $M \times X^m$. Since $\underline{\mathrm{Hom}}_L((C, c_1, \ldots, c_m), (X_L, x_1, \ldots, x_m))$ is not of finite type over L (by assumption), there is a sequence of integers $d_1 < d_2 < \ldots$ such that

$$\mathrm{Hom}_L((C, c_1, \ldots, c_m), (X_L, x_1, \ldots, x_m)) = \underline{\mathrm{Hom}}_L((C, c_1, \ldots, c_m), (X_L, x_1, \ldots, x_m))(L)$$

is non-empty. In particular, $Z_{d_i}(L)$ is non-empty. Define $Z := \cap_{i=1}^{\infty} Z_{d_i}$. Then Z is a closed substack of $M \times X^m$ (over k) with an L-point. Thus, it follows that $Z(k) \neq \emptyset$. This means precisely that there is a smooth projective connected curve C' of genus g over k , pairwise distinct points $c'_1, \ldots, c'_m \in C(k)$, and points $x'_1, \ldots, x'_m \in X(k)$ such that $\underline{\mathrm{Hom}}_k((C', c'_1, \ldots, c'_m), (X, x'_1, \ldots, x'_m))$ is not of finite type over k. This shows that X is not $(1, m)$-bounded over k, and concludes the proof of the theorem. ☐

8. Relating $(1, m)$-boundedness and (n, m)-boundedness

We use the geometricity theorems in the previous section, and a specialization argument, to prove that $(1, m)$-bounded varieties are $(n, 1)$-bounded. To prove our result we use the following application of Bertini's theorem.

Lemma 8.1. *Let X be a variety over an uncountable algebraically closed field k. Let I be a countable set and let $(Z_i)_{i\in I}$ be a collection of proper closed subsets of X. Let $S \subset \cap_{i\in I} Z_i$ be a finite (possibly empty) closed subset. Then there is a smooth irreducible curve $C \subset X$ such that, for all i in I, the set $C \cap Z_i$ is finite and contains S.*

Proof. Since k is uncountable and $Z_i \neq X$ for all i in I, we have that $X(k) \neq \cup_{i\in I} Z_i(k)$. Therefore, there is a k-point Q in X not contained in any of the Z_i. By Bertini's theorem, a general complete intersection curve C containing the set S and the point Q is smooth and irreducible. For every i in I, the intersection $Z_i \cap C$ does not contain the specified point Q. Therefore, the intersection is a proper closed subset of the irreducible curve C. We conclude that, for all i in I, the intersection of C and Z_i is a finite subset of C containing S. ☐

Proposition 8.2. *Let $m \geq 1$ be an integer. Let X be a $(1, m)$-bounded projective variety. Then, for every integer $n \geq 1$, the projective variety X is (n, m)-bounded.*

Proof. By the geometricity of $(1, m)$-boundedness (Theorem 7.3), for any algebraically closed field extension $k \subset L$, the projective scheme X_L is $(1, m)$-bounded over L (Theorem 7.2). Therefore, to prove that X is (n, m)-bounded, we may and do assume that k is uncountable.

Now, assume that X is not (n, m)-bounded over k. Let Y be a projective variety of dimension at most n over k, let $y_1, \ldots, y_m \in Y(k)$ be pairwise distinct points, let $x_1, \ldots, x_m \in X(k)$, and let $f_1, \ldots$ be pairwise distinct morphisms $Y \to X$ such that

$$f_i \in \mathrm{Hom}_k((Y, y_1, \ldots, y_m), (X, x_1, \ldots, x_m)).$$

(We will show that this leads to a contradiction.)

For any pair of positive integers, define $Y^{n,m} := \{y \in Y \mid f_n(y) = f_m(y)\}$. Note that $Y^{n,m}$ is a proper closed subset of Y which contains the points $y_1, \ldots, y_m$. (The fact that $Y^{n,m} \neq Y$ is equivalent to the fact that $f_n \neq f_m$.)

Let $I = \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1} \setminus \Delta$ be the set of pairs of distinct positive integers. For i in I (corresponding to (n, m)), define $Z_i := Y^{n,m}$. As the collection of proper closed subsets $(Z_i)_{i\in I}$ is countable and contains $\{y_1, \ldots, y_m\}$, there is a smooth projective connected curve C in X such that the intersection of C with any Z_i is finite and contains $\{y_1, \ldots, y_m\}$. This means that the morphisms f_i restricted to C are all still pairwise distinct. Thus, their restrictions

$f_i|_C$ give rise to pairwise distinct elements of $\mathrm{Hom}_k((C, y_1, \ldots, y_m), (X, x_1, \ldots, x_m))$. This implies that

$$\mathrm{Hom}_k((C, y_1, \ldots, y_m), (X, x_1, \ldots, x_m))$$

is infinite. By Lemma 4.6, we see that $\underline{\mathrm{Hom}}_k((C, c_1, \ldots, c_m), (X, x_1, \ldots, x_m))$ is not of finite type. In particular, X is not $(1, m)$-bounded over k. This contradicts our hypothesis. □

Corollary 8.3. *Let X be a projective variety over k. Assume that there is an integer $m \geq 1$ such that X is $(1, m)$-bounded. Then, for every $n \geq 1$, the projective variety X is $(n, 1)$-bounded over k.*

Proof. Since X is $(1, m)$-bounded, it is $(1, 1)$-bounded (Proposition 4.8). Therefore, it is $(n, 1)$-bounded (Proposition 8.2). □

Theorem 8.4. *Let X be a projective variety over k. Then the following are equivalent.*

(1) There exist $n \geq 1$ and $m \geq 1$ such that X is (n, m)-bounded.
(2) For every $n \geq 1$ and $m \geq 1$, we have that X is (n, m)-bounded over k.

Proof. This follows from Corollary 8.3. □

9. Relating 1-boundedness, boundedness, and algebraic hyperbolicity

The property of being bounded has to be (by definition) "tested" on maps from all projective varieties. In this section, we prove that a 1-bounded variety is in fact bounded, i.e., one can "test" boundedness of a variety on maps from curves. This result is an algebraic analogue of the complex-analytic fact that one can "test" the *Borel hyperbolicity* of a variety on holomorphic maps from a curve [19, Theorem. 1.5]. To prove the main result of this section, we start with a simple intersection-theoretic lemma.

Lemma 9.1. *Let D be a very ample divisor on a reduced projective scheme Y over k of dimension at least two. Let κ be a positive real number. Then, the set of numerical equivalence classes of big base-point free divisors L with intersection number $L \cdot D^{\dim Y - 1} \leq \kappa$ is finite.*

Proof. Let $f : \widetilde{Y} \to Y$ be a projective birational surjective morphism with W a smooth projective variety over k. By the projection formula, we have that $(f^*L) \cdot (f^*D)^{\dim \widetilde{Y} - 1} = L \cdot D^{\dim Y - 1} \leq \kappa$. Therefore, since $f^* : \mathrm{NS}(Y) \to \mathrm{NS}(\widetilde{Y})$ is injective, replacing Y by $\widetilde{Y}$ if necessary, we may and do assume that Y is a smooth projective variety over k. Moreover, we may and do assume that Y is connected.

Suppose that $\dim Y = 2$. Define $e := L \cdot L$, $g := L \cdot D$, and $h := D \cdot D$. Then $(hL - gD) \cdot D = 0$. Also, by the Hodge index theorem, the inequality $(hL - gD)^2 \leq 0$ holds. Therefore, since h is fixed and $g \leq \kappa$, we conclude that

$$L \cdot L = e = \frac{g^2}{h} \leq \frac{\kappa^2}{h}$$

is bounded from above. Now, since L is big and base-point free, the general member C of the linear system defined by L is a smooth projective connected curve. Let $\mathrm{genus}(L)$ be the genus of C. Since $2\mathrm{genus}(L) - 2 = L \cdot (K_Y + L)$, we see that $2\mathrm{genus}(L) - 2$ is also bounded. Thus, the lemma holds when $\dim Y \leq 2$.

Therefore, to prove the lemma, we may and do assume that $\dim Y \geq 3$. In this case, by the Lefschetz hyperplane theorem, the induced map from the Picard group of Y to the Picard group of a smooth hyperplane section is injective. Therefore, the lemma follows from induction on $\dim Y$. □

Theorem 9.2 (1-bounded implies bounded)**.** *Let X be a 1-bounded projective scheme over k. Then X is bounded over k.*

Proof. We show by induction on $n \geq 1$ that X is n-bounded. By assumption, the projective scheme X is 1-bounded. Thus, let $n \geq 2$ and assume that X is $(n-1)$-bounded. Assume that X is not n-bounded over k. Let Y be a projective n-dimensional normal variety over k, and let $f_1, \ldots$ be morphisms $Y \to X$ with pairwise distinct Hilbert polynomials. The Hilbert polynomial of a morphism $f : Y \to X$ is uniquely determined by the numerical equivalence class of $f^*\mathcal{O}(1)$. Indeed, by Hirzebruch-Riemann-Roch, $\chi(f^*(\mathcal{O}(d)) = \deg \mathrm{ch}(f^*(\mathcal{O}(d)) \cdot \tau_Y$, where τ_Y is the refined Todd class (as in Fulton [11]), and the Chern character depends only on the first Chern class, which is determined by the numerical equivalence class of $f^*\mathcal{O}(1)$.

The infinitude of the Hilbert polynomials of $f_1, \ldots$ is equivalent to the infinitude of the numerical equivalence classes of $f_1^*\mathcal{O}(1), \ldots$. Fix a very ample divisor class D on Y. From Lemma 9.1 it follows that if there is a collection $f_1, \ldots$ with infinitely many distinct Hilbert polynomials, then for this collection $f_i^*(\mathcal{O}(1)) \cdot D^{\dim V - 1} \to \infty$. Take the restrictions to D, whose dimension is $\dim D = \dim Y - 1$ and for which the induction hypothesis holds. We have that $f_i^*(\mathcal{O}(1))|_D \cdot D|_D^{\dim Y - 2} \to \infty$, because $f_i|_D : D \to X$ have the same properties as $f_i : Y \to X$. This contradicts the induction hypothesis applied to D. □

As an application of our results, we obtain that (n, m)-boundedness (and thus boundedness) is a geometric property.

Corollary 9.3 (Boundedness is a geometric property)**.** *Let $n \geq 1$ and $m \geq 0$ be integers. Let $k \subset L$ be an extension of algebraically closed fields of characteristic zero. A projective variety X over k is (n, m)-bounded over k if and only if X_L is (n, m)-bounded over L.*

Proof. Combine Theorem 7.2, Theorem 7.3, Theorem 8.4, and Theorem 9.2. □

The relation between algebraic hyperbolicity and bounded varieties is provided by the following theorem.

Theorem 9.4. *A projective algebraically hyperbolic scheme over k is bounded over k.*

Proof. (This follows from [22, Theorem 1.7]. We give a self-contained proof using the results of this paper.) Let X be a projective algebraically hyperbolic variety over k. Then, for every projective normal (hence smooth) curve C over k, the degree of any morphism $C \to X$ is bounded linearly in the genus of C. In particular, the scheme $\underline{\mathrm{Hom}}_k(C, X)$ is of finite type over k. This implies (clearly) that X is 1-bounded over k. Therefore, X is bounded over k (Theorem 9.2). □

It seems reasonable to suspect that (n, m)-bounded varieties are in fact bounded. Indeed, as we explain in Section 11, the Green–Griffiths–Lang conjecture implies that a $(1, m)$-bounded projective variety is 1-bounded, and hence bounded (Theorem 9.2). In the direction of this "reasonable" expectation, we prove the following result.

Proposition 9.5. *Let $m \geq 1$ be an integer, and let X be a $(1, m)$-bounded projective scheme over k. Let Y be a projective variety over k. Then, almost all (non-empty) connected components of the scheme $\underline{\mathrm{Hom}}_k(Y, X)$ have dimension $< \dim X$.*

Proof. Note that X is $(n, 1)$-bounded for all $n \geq 1$ (Corollary 8.3). Let $y \in Y(k)$. Consider the evaluation map $\mathrm{eval}_y : \underline{\mathrm{Hom}}_k(Y, X) \to X$ defined by $f \mapsto f(y)$. Suppose that $\underline{\mathrm{Hom}}_k(Y, X)$ has infinitely many pairwise distinct connected components $H_1, \ldots$ of dimension at least

$\dim X$. Then, as the restriction $\mathrm{eval}_y : H_i \to X$ of eval_y to H_i is finite, it is surjective. Let x be any point in X. Then, for every i, the fibre of $H_i \to X$ over x is non-empty. Thus, for every i, the set $\mathrm{Hom}((Y, y), (X, x))$ contains a point from H_i, and is therefore infinite. This contradicts the fact that X is $(\dim Y, 1)$-bounded. We conclude that almost all components of $\underline{\mathrm{Hom}}_k(Y, X)$ have dimension $< \dim X$. ☐

Remark 9.6. Let $n \geq 1$ and $m \geq 1$ be positive integers. Let X be a projective scheme over k. We have shown the following statements.

- If X is algebraically hyperbolic over k, then X is bounded over k.
- The scheme X is bounded over k if and only if X is 1-bounded over k.
- The scheme X is (n, m)-bounded over k if and only if X is $(1, 1)$-bounded over k.
- If X is 1-bounded over k, then X is (n, m)-bounded over k.
- If X is (n, m)-bounded over k, then X is groupless over k.

10. Proofs of main results

In this section we prove the results on algebraic hyperbolicity and 1-bounded varieties.

10.1. **Algebraically hyperbolic varieties.**

We prove all the results on algebraic hyperbolicity stated in Section 1.1. The proofs are applications and combinations of all our results.

Proof of Theorem 1.5. This is Theorem 7.1. ☐

Proof of Theorem 1.6. Since algebraically hyperbolic projective varieties are bounded (Theorem 9.4), there are only finitely many surjective rational maps $Y \dashrightarrow X$ by Proposition 6.3. The rest of the theorem follows from Corollary 6.2. ☐

Proof of Theorem 1.8. Since algebraically hyperbolic projective varieties are bounded (Theorem 9.4), the theorem follows from Corollary 6.4. ☐

Proof of Theorem 1.9. Let X be a projective algebraically hyperbolic scheme over k and let Y be a projective scheme over k. Since X is bounded (Theorem 9.4), the scheme $\underline{\mathrm{Hom}}_k(Y, X)$ is an algebraically hyperbolic projective scheme over k with $\dim \underline{\mathrm{Hom}}_k(Y, X) \leq \dim X$ (Corollary 5.3).

To see that $\dim \underline{\mathrm{Hom}}_k^{nc}(Y, X) < \dim X$, let $Z \subset \underline{\mathrm{Hom}}_k^{nc}(Y, X)$ be a reduced irreducible component with $\dim Z = \dim X$. For all y in $Y(k)$, consider the evaluation map $\mathrm{eval}_y \colon Z \to X$, and note that it is finite (as shown in the proof of Corollary 5.3). Since $\dim Z = \dim X$, for all y in $Y(k)$, the finite morphism eval_y is surjective. Thus, as $\mathrm{Sur}_k(Z, X)$ is finite (Theorem 1.6), there exist an integer $n \geq 1$ and points $y_1, \ldots, y_n \in Y(k)$ such that, for all y in $Y(k)$, we have that $\mathrm{eval}_y \in \{\mathrm{eval}_{y_1}, \ldots, \mathrm{eval}_{y_n}\}$. In other words, every morphism $f \colon Y \to X$ in Z takes on only finitely many values (namely $f(y_1), \ldots, f(y_n)$). In particular, since Z is irreducible, we conclude that every f in Z takes on precisely one value, i.e., f is constant. This contradicts the fact that $Z \subset \underline{\mathrm{Hom}}^{nc}(Y, X)$. ☐

10.2. **Bounded varieties.**

We prove all the results on 1-bounded varieties stated in Section 1.2.

Proof of Theorem 1.11. This is Theorem 7.2. ☐

Proof of Theorem 1.12. Since 1-bounded projective projective varieties are bounded (Theorem 9.2), the theorem follows from Corollary 6.4. ☐

Proof of Theorem 1.13. (We follow very closely the proof of Theorem 1.9.) Let X be a projective 1-bounded scheme over k and let Y be a projective scheme over k. Since X is bounded (Theorem 9.2), the scheme $\underline{\mathrm{Hom}}_k(Y,X)$ is a bounded projective scheme over k with $\dim \underline{\mathrm{Hom}}_k(Y,X) \leq \dim X$ (Corollary 5.3).

To see that $\dim \underline{\mathrm{Hom}}_k^{nc}(Y,X) < \dim X$, let $Z \subset \underline{\mathrm{Hom}}_k^{nc}(Y,X)$ be a reduced irreducible component with $\dim Z = \dim X$. For all y in $Y(k)$, consider the evaluation map $\mathrm{eval}_y \colon Z \to X$, and note that it is finite (as shown in the proof of Corollary 5.3). Since $\dim Z = \dim X$, for all y in $Y(k)$, the finite morphism eval_y is surjective. Thus, as $\mathrm{Sur}_k(Z,X)$ is finite (Theorem 1.12), there exist an integer $n \geq 1$ and points $y_1,\ldots,y_n \in Y(k)$ such that, for all y in $Y(k)$, we have that $\mathrm{eval}_y \in \{\mathrm{eval}_{y_1},\ldots,\mathrm{eval}_{y_n}\}$. In other words, every morphism $f \colon Y \to X$ in Z takes on only finitely many values (namely $f(y_1),\ldots,f(y_n)$). In particular, since Z is irreducible, we conclude that every f in Z takes on precisely one value, i.e., f is constant. This contradicts the fact that $Z \subset \underline{\mathrm{Hom}}^{nc}(Y,X)$. $\square$

11. Conjectures related to Demailly's and Green–Griffiths–Lang's conjecture

The following conjecture is a consequence of Demailly's conjecture (Conjecture 1.4), and thus also Green–Griffiths–Lang's conjecture [25]. The conjecture says that the total space of a family of projective algebraically hyperbolic varieties over a projective algebraically hyperbolic base variety is algebraically hyperbolic.

Conjecture 11.1 (Fibration property)**.** *Let k be an algebraically closed field of characteristic zero. Let $f : X \to Y$ be a morphism of projective varieties over k. If Y is algebraically hyperbolic over k, and, for all y in $Y(k)$, the projective scheme X_y is algebraically hyperbolic over k, then X is algebraically hyperbolic over k.*

The analogue of this conjecture for projective families of Kobayashi hyperbolic varieties is known and follows from [21, Corollary 3.11.2]. We now explain how Conjecture 11.1 follows from Demailly's conjecture (Conjecture 1.4).

Remark 11.2 (Demailly's conjecture implies Conjecture 11.1)**.** To see that Conjecture 11.1 is a consequence of Demailly's conjecture (Conjecture 1.4), we may and do assume that $k \subset \mathbb{C}$. Then, with the notation as in Conjecture 11.1, by the geometricity of algebraic hyperbolicity (Theorem 1.5), the fibers of the morphism $X_{\mathbb{C}} \to Y_{\mathbb{C}}$ are algebraically hyperbolic over $\mathbb{C}$ and $Y_{\mathbb{C}}$ is algebraically hyperbolic over $\mathbb{C}$. By Demailly's conjecture, for every t in $Y(\mathbb{C})$, the fiber X_y is Kobayashi hyperbolic (as a complex analytic space) and the projective variety $Y_{\mathbb{C}}$ is Kobayashi hyperbolic. Therefore, $X_{\mathbb{C}}$ is Kobayashi hyperbolic [21, Corollary 3.11.2]. Since Kobayashi hyperbolic projective varieties over $\mathbb{C}$ are algebraically hyperbolic (Theorem 1.2), this shows that $X_{\mathbb{C}}$ is algebraically hyperbolic over $\mathbb{C}$. We conclude that X is algebraically hyperbolic over k.

The following conjecture relates all notions of "boundedness" introduced in this paper (see Section 4).

Conjecture 11.3. *Let k be an algebraically closed field of characteristic zero and let X be a projective variety over k. Then the following are equivalent.*

(1) The projective variety X is algebraically hyperbolic over k.
(2) The projective variety X is bounded over k.
(3) For all $n \geq 1$, the projective variety X is $(n,1)$-bounded.

Note that (1) $\implies$ (2) is Theorem 9.4 and that (2) $\implies$ (3) is Remark 4.3. Other relations between the three notions in Conjecture 11.3 are summarized in Remark 9.6. The implication (3) $\implies$ (2) is currently not known and neither is the implication (2) $\implies$ (1). To show that (3) $\implies$ (2), it suffices to show that, if X is $(n,1)$-bounded for all $n \geq 1$, then X is 1-bounded.

We conclude by noting that the implication (3) $\implies$ (1) in Conjecture 11.3 is a consequence of Green–Griffiths–Lang's conjecture in [25]. Indeed, $(1,m)$-bounded varieties are groupless by Proposition 4.4, and it follows from Green–Griffiths–Lang's conjectures that projective groupless varieties are algebraically hyperbolic.

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