MAT 536 SPRING 2020 HOMEWORK 3

Problem 1. Suppose that the function $f(z)$ is holomorphic in a domain D and satisfies
$$|f(z)^2 - 1| < 1$$
for all $z \in D$. Prove that either $\mathrm{Re} f(z) > 0$ or $\mathrm{Re} f(z) < 0$ throughout D.

Problem 2. Recall that a *Möbius transformation* of the complex line $\mathbb{CP}^1$ is a linear transformation with non-zero determinant. The Möbius transformations form a group, called the *Möbius group*. Prove that the action of the Möbius group on $\mathbb{CP}^1$ is *three-transitive*, i.e. for any two triples $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$ of distinct points in $\mathbb{CP}^1$, there is a Möbius transformation T such that $Tz_k = w_k$, $k = 1, 2, 3$.

Problem 3. Prove that for any two quadruples $\{z_1, z_2, z_3, z_4\}$ and $\{w_1, w_2, w_3, w_4\}$ of distinct points in $\mathbb{CP}^1$ there is a Möbius transformation T such that $Tz_k = w_k$, $k = 1, 2, 3, 4$, if and only if their cross-ratios are equal.

Problem 4. Consider the so-called *anharmonic group* – a subgroup of the Möbius group, generated by the transformations $z \to 1/z$ and $z \to 1 - z$. Show that its action on $1, 0, \infty$ gives an isomorphism with S_3, the symmetric group on 3 elements.

Problem 5. Map the domain between the two circles $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ conformally onto the upper half-plane.