SYMPLECTIC INVOLUTIONS OF $K3^{[n]}$ TYPE AND KUMMER n TYPE MANIFOLDS

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Abstract. In this paper we describe the fixed locus of a symplectic involution on a hyperkähler manifold of type $K3^{[n]}$ or of Kummer n type. We prove that the fixed locus consists of finitely many copies of Hilbert schemes of $K3$ surfaces of lower dimensions and isolated fixed points.

1. Introduction

An involution ι on a hyperkähler manifold is symplectic if it preserves the holomorphic symplectic form. Consider a hyperkähler manifold X of $K3^{[n]}$ type, i.e., deformation equivalent to a Hilbert scheme of n points on a $K3$ surface or of Kummer n type, i.e. deformation equivalent to the Albanese fibre of the Hilbert scheme of $n+1$ points on an Abelian surface. We are interested in describing the fixed loci of symplectic involutions on X. In [17] Nikulin proved that the fixed locus of a symplectic involution on a $K3$ surface consists of 8 isolated fixed points. The second named author proved in [9] that the fixed locus of a symplectic involution on a hyperkähler manifold deformation equivalent to a Hilbert square of a $K3$ surface consists of 28 isolated points and one $K3$ surface. Here we generalize these results to any hyperkähler manifold of $K3^{[n]}$ type.

Theorem 1.1. *Let X be a hyperkähler manifold of $K3^{[n]}$ type, and let ι be a symplectic involution on X. Then, up to deformation, the fixed locus F of ι consists of finitely many copies of Hilbert schemes of $K3$ surfaces $S^{[m]}$ (where $m \leq \frac{n}{2}$) and possibly isolated fixed points (only when $n \leq 24$). The fixed locus F is stratified into loci of even dimensions F_{2m}, where $\max(0, \frac{n}{2} - 12) \leq m \leq \frac{n}{2}$. Each fixed locus F_{2m} of dimension $2m$ has*

$$\sum_{2m=n-k-2l} \binom{8}{k}\binom{k}{l}$$

connected components, each one of which is a deformation of a copy of $S^{[m]}$. In particular, the fixed locus Z of largest dimension is the following.

(i) If n is even, then Z consists of one copy of $S^{[\frac{n}{2}]}$;

(ii) If n is odd, then Z consists of 8 copies of $S^{[\frac{n-1}{2}]}$.

A key ingredient in the proof of this theorem is that the moduli space of pairs consisting of a hyperkähler manifold of $K3^{[n]}$ type (or of Kummer n type) together with a symplectic involution is connected. Using the Global Torelli theorem first we prove that (X, ι) is birational to a "standard pair", and then we prove that birational pairs can be deformed one into the other while preserving the group

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action. A "standard pair" for $K3^{[n]}$ type manifolds is a deformation of $(S^{[n]}, G)$, where S is a $K3$ surface and G is a symplectic involution on $S^{[n]}$ induced by a symplectic involution on S.

Theorem 1.2. *Let X be a manifold of $K3^{[n]}$ type or of Kummer n type. Let $G \subset \mathrm{Aut}_s(X)$ be a finite group of numerically standard automorphisms. Then (X, G) is a standard pair.*

As a corollary to this theorem we obtain that the fixed locus of the symplectic involution ι has the form of the fixed locus on $S^{[n]}$ of a symplectic involution coming from the $K3$ surface S. Thus, we can restrict to the case when $X = S^{[n]}$. Furthermore, since the involution ι acts as the identity on the effective classes in $H^2(X, \mathbb{R})$, it preserves the exceptional divisor of $S^{[n]}$. Therefore, it descends to an action on $\mathrm{Sym}^n S$. Depending on the parity of n, we can see that the number of irreducible components of the fixed locus of largest dimension is either one or eight. Using basic combinatorics, one can count the number of irreducible components of the fixed locus in each possible dimension.

In an analogous way, we compute the fixed locus of a symplectic involution on a hyperkähler manifold of Kummer n type, by reducing to the case of an involution coming from the sign change on the abelian surface A:

Theorem 1.3. *Let X be a hyperkähler manifold of Kummer n type, and let ι be a symplectic involution on X. Then, up to deformation, the fixed locus F of ι consists of finitely many copies of Hilbert schemes of $K3$ surfaces $S^{[m]}$ (where $m \leq \frac{n+1}{2}$) and possibly isolated fixed points (only when $n \leq 48$). The fixed locus F is stratified into loci of even dimensions F_{2m}, where $\max(0, \frac{n+1}{2} - 24) \leq m \leq \frac{n+1}{2}$. Each fixed locus F_{2m} of dimension $2m$ has N_m^n (for the precise formula see the last section) connected components, each one of which is a deformation of a copy of $S^{[m]}$.*

2. Preliminaries

Let X be a hyperkähler manifold, i.e., a complex Kähler simply connected manifold such that $H^{2,0}(X) \cong \mathbb{C}$ is generated by a holomorphic symplectic 2-form. If X is deformation equivalent to the Hilbert scheme $S^{[n]}$ of n points of a $K3$ surface S, we say that X is of $K3^{[n]}$ type. If X is deformation equivalent to the generalized Kummer $2n$-fold $K_n(A)$ of an abelian surface A, we say that X is of Kummer n type.

Definition 2.1. Let X be a complex manifold and let G be a subgroup of $\mathrm{Aut}(X)$, the group of automorphisms of X. A deformation of the pair (X, G) consists of the following data:

(i) A flat family $\mathcal{X} \to B$, where B is connected and $\mathcal{X}$ is smooth, and a distinguished point $0 \in B$ such that $\mathcal{X}_0 \cong X$.

(ii) A faithful action of the group G on $\mathcal{X}$ inducing fibrewise faithful actions of G.

Two pairs (X, G) and (Y, H) are deformation equivalent if (Y, H) is a fibre of a deformation of the pair (X, G).

In this paper we are mostly interested in deformations of the pair $(X, \mathbb{Z}_2)$, where X is of $K3^{[n]}$ type and $\mathbb{Z}_2$ is generated by a symplectic involution.

Definition 2.2. Let S be a $K3$ surface and let $G \subset \mathrm{Aut}_s(S)$ be a subgroup of the symplectic automorphisms on S. Then G induces a subgroup of the symplectic morphisms on $S^{[n]}$ which we still denote by G. We call the pair $(S^{[n]}, G)$ a natural pair. The pair (X, H) is standard if it is deformation equivalent to a natural pair. If A is an abelian surface, the same definitions apply to the generalized Kummer $2n$-fold $K_n(A)$ and symplectic automorphisms preserving $0 \in A$, however the reader should notice that the induced action of G on $H^2(K_n(A))$ is not necessarily faithful.

Definition 2.3. Let G be a finite group acting faithfully on a manifold X. Define the invariant locus $T_G(X)$ inside $H^2(X, \mathbb{Z})$ to be the fixed locus of the induced action of G on the cohomology. The co-invariant locus $S_G(X)$ is the orthogonal complement $T_G(X)^\perp$. The fixed locus of G on X is denoted by X^G.

As automorphisms of K3 and abelian surfaces are better known, it is interesting to determine whether an automorphism group on a manifold of $K3^{[n]}$ type (or of Kummer n type) is standard or not, we give the following criterion:

Definition 2.4. Let Y be a manifold of $K3^{[n]}$ type or of Kummer n type. A pair (Y, H) is called numerically standard if the representation of H on $H^2(Y, \mathbb{Z})$ coincides with that of a standard pair (X, H), up to the action of the monodromy group. More specifically, there exists a $K3$ (or abelian) surface S with an H action such that

- $S_H(S) \cong S_H(Y)$,
- $T_H(S) \oplus \mathbb{Z}\delta = T_H(S^{[n]}) \cong T_H(Y)$ (and analogously for the Kummer n case),
- The two isomorphisms above extend to isomorphisms of the Mukai lattices $U^4 \oplus E_8(-1)^2$ (or U^4 in the Kummer case) after taking the canonical choice of an embedding of H^2 into the Mukai lattice described by Markman [8, Section 9] for the $K3^{[n]}$ type case and by Wieneck [20, Theorem 4.1] for the Kummer case.

This definition is slightly stronger than the one given in [10] for manifolds of $K3^{[n]}$ type , but they coincide when $n-1$ is a prime power, which was the case of interest in that paper. Notice that it is relatively easy to check the first two conditions, while the third one is more involved but often unnecessary, see Proposition 4.13.

Let X be a compact complex manifold and $\mathrm{Diff}^0(X)$ a connected component of its diffeomorphism group. Denote by Comp the space of complex structures on X, equipped with the structure of a Fréchet manifold.

Definition 2.5. The Teichmüller space of X is the quotient $\mathrm{Teich} := \mathrm{Comp} / \mathrm{Diff}^0(X)$.

The Teichmüller space is finite-dimensional for a Calabi-Yau manifold X (see [4]). Let $\mathrm{Diff}^+(X)$ be the group of orientable diffeomorphisms of a complex manifold X. The *mapping class group* $\Gamma := \mathrm{Diff}^+(X)/\mathrm{Diff}^0(X)$ acts on Teich.

By Huybrechts's result [7, Theorem 4.3], non-separated points in the moduli space of marked hyperkähler manifolds correspond to birational hyperkähler manifolds. Consider the equivalence relation $\sim$ on Teich identifying non-separated points. Let $\mathrm{Teich}_b = \mathrm{Teich}/_\sim$ be the *birational Teichmüller space.*

Let X be a hyperkähler manifold, and let Teich be its Teichmüller space. Consider the map $\mathcal{P}$: Teich $\to \mathbb{P}H^2(X,\mathbb{C})$, sending a complex structure J to the line $H^{2,0}(X,J) \in \mathbb{P}H^2(X,\mathbb{C})$. The image of $\mathcal{P}$ is the open subset of a quadric, defined by $\mathbb{P}\mathsf{er} := \{l \in \mathbb{P}H^2(X,\mathbb{C}) \mid q(l,l) = 0,\ q(l,\bar{l}) > 0\}$.

Definition 2.6. The map $\mathcal{P}$: Teich $\to \mathbb{P}\mathsf{er}$ is called the period map, and the set $\mathbb{P}\mathsf{er}$ is called the period domain.

The period domain $\mathbb{P}\mathsf{er}$ is identified with the quotient $\frac{SO(3,b_2-3)}{SO(2)\times SO(1,b_2-3)}$.

Theorem 2.7. *(Verbitsky's Global Torelli,* [19]*) The period map $\mathcal{P} : \mathrm{Teich}_b^0 \to \mathbb{P}\mathsf{er}$ is an isomorphism on each connected component of* Teich_b.

It is possible to compute the Kähler cone of a hyperkähler manifold from numerical data on the second cohomology (see [1] and [11] for the general theory), the following will be needed for our main result:

Proposition 2.8. *Let X be a manifold of $K3^{[n]}$ type, n odd. Let $a \in Pic(X)$ be a class of negative square greater than $-6-2n$ and of divisibility two. Then there are no Kähler classes orthogonal to a.*

Proof. There is a canonical choice of an embedding of $H^2(X,\mathbb{Z})$ into the Mukai lattice $U^4 \oplus E_8(-1)^2$ which is described by Markman [8, Section 9]. Let $\mathbb{Z}v := (H^2)^\perp$ in this embedding. The lattice $L := \langle v, a\rangle$ is generated by v and $\frac{v+a}{2}$, whose square is at least -2 by hypothesis and at most $v^2/4$, therefore by [2, Thm 5.7] $a^\perp$ is a wall for the space of positive classes, hence there cannot be a Kähler class orthogonal to it. □

In particular, it follows from the results in [1] and [11] that if there is a Kähler class orthogonal to the Picard lattice, the Kähler cone coincides with the positive cone.

Some lattice theory will be used in the following, the main reference here is [18], where all of the following can be found. For a lattice L we define the discriminant group $A_L := L^\vee/L$. Let $l(A_L)$ denote the length of this group. If the lattice L is even, A_L has a bilinear form with values in $\mathbb{Q}/\mathbb{Z}$ induced from the bilinear form on L. This associated quadratic form is called the discriminant form of L and is denoted by q_{A_L}. If (l_+, l_-) is the signature of L, the integer $l_+ - l_-$ is called signature of q_{A_L} and, modulo 8, it is well defined.

The following concerns primitive embeddings of lattices, i.e., embeddings where the quotient is torsion free:

Lemma 2.9. [18, Proposition 1.15.1] *Let S and N be even lattices of signature (s_+, s_-) resp. (n_+, n_-). Primitive embeddings of S into N are determined by the sets $(H_S, H_N, \gamma, K, \gamma_K)$, where K is an even lattice with signature $(n_+ - s_+, n_- - s_-)$ and discriminant form $-\delta$ where $\delta \cong (q_{A_S} \oplus -q_{A_N})_{|\Gamma_\gamma^\perp/\Gamma_\gamma}$ and $\gamma_K : q_K \to (-\delta)$ is an isometry. Moreover, two such sets $(H_S, H_N, \gamma, K, \gamma_K)$ and $(H'_S, H'_N, \gamma', K', \gamma'_K)$ determine isomorphic sublattices if and only if*

- *$H_S = \lambda H'_S$, $\lambda \in O(q_S)$,*
- *$\exists \epsilon \in O(q_{A_N})$ and $\psi \in Isom(K, K')$ such that $\gamma' = \epsilon \circ \gamma$ and $\overline{\epsilon} \circ \gamma_K = \gamma'_K \circ \overline{\psi}$, where $\overline{\epsilon}$ and $\overline{\psi}$ are the isometries induced among discriminant groups.*

Here Γ_γ is the graph of γ. When a lattice L is a G-representation, we will call the sublattice $T_G(L)$ fixed by L the invariant lattice, and its orthogonal complement $S_G(L)$ - the coinvariant lattice.

3. Fixed loci: local case

The Hilbert scheme $(\mathbb{C}^2)^{[n]}$ represents a local model for $\mathrm{K3}^{[n]}$. Thus to analyze the irreducible components of the fixed locus of $\imath$ on $\mathrm{K3}^{[n]}$ we first need to analyze the local geometry. The surface $\mathbb{C}^2$ has a natural symplectic form $\omega = dx \wedge dy$ and the involution

$$\imath : x \mapsto -x, \quad y \mapsto -y$$

preserves this form. The quotient of $\mathbb{C}^2$ by the involution is a singular surface that admits a symplectic resolution:

$$\widehat{\mathrm{A}}_1 \to \mathrm{A}_1 = \mathbb{C}^2/\imath.$$

It is elementary to see that the $\imath$-fixed locus on $(\mathbb{C}^2)^{[2]}$ is isomorphic to $\widehat{\mathrm{A}}_1$. Below we show a generalization of this statement:

Lemma 3.1. *For any n we have*

$$\left((\mathbb{C}^2)^{[n]}\right)^\imath = (\widehat{A}_1)^{n/2},$$

if n is even and for odd n:

$$\left((\mathbb{C}^2)^{[n]}\right)^\imath = \mathfrak{M}((n-1)\delta/2 + e_1, e_1) \bigcup \mathfrak{M}((n-1)\delta/2 + e_2, e_1)$$

where the $\mathfrak{M}(v, w)$ is the quiver variety for the quiver of the affine Dynkin diagram of type $\tilde{A}_1$ and $\delta = e_1 + e_2$ is the imaginary root of the corresponding root system. The quiver varieties are connected and of dimension:

$$\dim\left(\mathfrak{M}((n-1)\delta/2 + e_1, e_1)\right) = n - 1, \quad \dim\left(\mathfrak{M}((n-1)\delta/2 + e_2, e_1)\right) = n - 3.$$

To explain the statement we need to remind our reader some basics of the theory of the quiver varieties [15]. A quiver is a directed graph Q with a set of vertices I. Given $\alpha \in \mathbb{N}^I$, the set of representations of the quiver is:

$$\mathrm{Rep}(Q, \alpha) = \oplus_{a \in Q} \mathrm{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}),$$

where $h(a)$ and $t(a)$ are the head and the tail of the corresponding arrow. The group

$$G(\alpha) = \left(\prod_{i \in I} \mathrm{GL}(\alpha_i)\right)/\mathbb{C}^*$$

acts on the vector space of representations.

The cotangent bundle of the space of representations is the space of the representations of the *double* of the quiver $\bar{Q}$:

$$\mathrm{T}^*\mathrm{Rep} = \mathrm{Rep}(\bar{Q}, \alpha),$$

where the double is the quiver obtained from Q by adjoining a reverse arrow a^* for every arrow $a \in Q$.

The moment map $\mu : \mathrm{Rep}(\bar{Q}, \alpha) \to \mathrm{Lie}(G(\alpha))$ is given by:

$$\mu(x)_i = \sum_{h(a)=i} x_a x_{a^*} - \sum_{t(a)=i} x_{a^*} x_a.$$

Let Q_0 be a quiver and $u, v \in \mathbb{N}^{I_0}$. We define Q to be the quiver with the set of vertices $I = I_0 \cup \infty$ and the set of arrows is the union of the set of arrows of Q_0 and the arrows v_i from ∞ to $i \in Q_0$. Respectively, we define:

$$\mathbf{M}(v,w) = \mathrm{Rep}(\bar{Q}, \alpha), \quad G_v = G(\alpha),$$

where α is the vector with coordinates: $\alpha_i = u_i$, $i \in I_0$ and $\alpha_\infty = 1$. Nakajima [15] defines the quiver variety as the GIT quotient of the subvariety of $\mathbf{M}(v,w)$:

$$\mathfrak{M}(v,w) = \mu^{-1}(0)//(G_v, \chi),$$

where χ is the character of the group defined by $\chi(g) = \prod_{k\in I} \det(g_k^{-1})$. We indicate the dependence of the quiver variety on the underlying quiver by the subindex: $\mathfrak{M}_{Q_0}(v,w)$, $\mathbf{M}_{Q_0}(v,w)$.

In our study we are most interested in the quiver varieties associated to the following two quivers:

$$\mathbf{Q}_0 = \circlearrowleft \bullet\ , \quad \mathbf{Q}_0' = \bullet \rightrightarrows \bullet$$

these are the quivers of affine Dynkin diagrams of types $\tilde{A}_0$ and $\tilde{A}_1$. Let the dimension vectors be $(v,w) = ((n),(1))$, $(v,w) = ((n_1,n_2),(0,1))$, then the corresponding enhanced quivers are:

$$\mathbf{Q} = \circlearrowleft 1 \longleftarrow \infty\ , \quad \mathbf{Q}' = 1 \rightrightarrows 2 \longleftarrow \infty\ ,$$

in the pictures of the quivers we introduced the labels of the vertices.

The starting point of our proof is the quiver description of the space $\left(\mathbb{C}^2\right)^{[n]}$ [16]:

$$\mathfrak{M}_{\mathbf{Q}_0}((n),(1)) = \left(\mathbb{C}^2\right)^{[n]}.$$

Proof of lemma 3.1. The involution $\imath$ on $\mathbb{C}^2$ induces an action on the $\mathbf{M}_{\mathbf{Q}_0}((n),(1))$, this involution acts trivially on $\mathbb{C}^1$ which corresponds to the vertex ∞, and the space $\mathbb{C}^n$ corresponding to the vertex 1 decomposes into the anti-invariant and invariant parts $\mathbb{C}^n = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$.

The vector space $\mathbb{C}^n$ in the quiver description of the Hilbert scheme corresponds to the quotient space $\mathbb{C}[x,y]/I$ in the interpretation of the natural Hilbert scheme. Moreover, the image of the map corresponding to the arrow from ∞ to 1 is the span of $1 \in \mathbb{C}[x,y]/I$, hence the image is invariant under the involution $\imath$. Thus, the involution invariant part of the quiver variety union of the quiver varieties constructed from the quiver representations is of the form:

$$\mathbb{C}^{n_1} \rightrightarrows \mathbb{C}^{n_2} \longrightarrow \mathbb{C}^1\ .$$

More formally, we conclude that we have an inclusion:

$$\left(\left(\mathbb{C}^2\right)^{[n]}\right)^{\imath} \subset \bigcup_{n_1+n_2=n} \mathfrak{M}_{\mathbf{Q}_0'}((n_1,n_2),(0,1)).$$

Next we recall the result of [5] that concerns with the classification of connected non-empty quiver varieties. The result from the last part of the introduction in [5] states that the quiver variety $\mathfrak{M}_{Q_0}(v,w)$ is non-empty if and only if v is a positive root of the Kac-Moody Lie algebra corresponding to the quiver Q_0 and if it is non-empty, then it is connected.

The Kac-Moody Lie algebra corresponding to the quiver $\mathbf{Q}_0'$ is the Lie algebra of the loop group of SL_2 and the roots of this Lie algebra are:

$$n\delta, \quad e_1 + n\delta, \quad e_2 + n\delta,$$

here $\delta = e_1 + e_2$.

Thus, we can refine our previous inclusion:

$$\left((\mathbb{C}^2)^{[n]}\right)^{\imath} \subset \mathfrak{M}_{\mathbf{Q}_0'}((n/2, n/2), (0,1)), \qquad \text{if } n \text{ is even},$$

$$\left((\mathbb{C}^2)^{[n]}\right)^{\imath} \subset \bigcup_{\epsilon=\pm} \mathfrak{M}_{\mathbf{Q}_0'}(((n+\epsilon 1)/2, (n-\epsilon 1)/2), (0,1)), \qquad \text{if } n \text{ is odd}.$$

In the case of even n we observe that since the $\imath$-invariant locus is not empty, the inclusion is actually an equality. The fact that the corresponding quiver variety is the Hilbert scheme of points on the surface $\hat{A}_1$ is standard (see, for example, Theorem 4.9 in [14]).

In the case of odd n we observe that the involution fixed locus must have at least two connected components. Indeed, if $I \subset \mathbb{C}[x,y]$ is an involution fixed ideal, then the quotient space $\mathbb{C}[x,y]/I$ has an action of the involution and the dimension $d(I) := \dim(\mathbb{C}[x,y]/I)^{\imath}$ is constant along any connected component of $\left((\mathbb{C}^2)^{[n]}\right)^{\imath}$. It is not hard to find two monomial ideals $I^{\pm}$ of codimension n and $d(I^{\pm}) = (n \pm 1)/2$, these two ideals belong to two disjoint connected components. Thus, we conclude that in the case of odd n, the inclusion is also an equality.

The formula for the dimension of the quiver varieties is standard and could be found, for example, in [5]. □

For small value of n, the result above has a more intuitive explanation. Indeed, if $n = 2$, then the fixed locus $\left((\mathbb{C}^2)^{[2]}\right)$ is the closure of the locus consisting of the pairs of points $z, \imath(z)$, $z \neq (0,0)$. On the other hand, if $n = 3$, there are two connected components of the involution: the closure of the locus of triples $(z, (0,0), \imath(z))$, $z \neq (0,0)$ and an isolated point which is the square of the maximal ideal $(x,y)^2$. Then the connected components could be revealed by the analysis of the punctual Hilbert scheme $(\mathbb{C}^2)^{[3]}_{(0,0)}$ which is the cone over the twisted cubic: the involution acts on the rulings of the cone preserving the vertex of the cone and infinite points of the rulings.

Let us fix notations for the two connected components of the involution, we denote the component of smaller dimension and large dimension by:

$$\left((\mathbb{C}^2)^{[n]}\right)^{\imath}_{-}, \quad \left((\mathbb{C}^2)^{[n]}\right)^{\imath}_{+}$$

4. Fixed loci of symplectic involutions

We recall some properties of the irreducible components of the fixed locus of a symplectic involution.

Proposition 4.1. [3, Proposition 3] *Let X be a hyperkähler manifold and ι be a symplectic involution on X. Then the irreducible components of the fixed locus of ι are symplectic subvarieties of X.*

Proof. For completeness we are going to include the proof. Let Z be an irreducible component of the fixed locus of ι. Since ι is a periodic endomorphism, Z is smooth by [6]. After restricting to Z, we have the orthogonal decomposition $TX_{|Z} =$

$TZ \oplus N_Z$, where TZ and N_Z are the eigenspaces corresponding to $+1$ and -1, respectively. Since ι is a symplectic involution, then for any $z \in Z$, T_zZ and $N_{Z,z}$ are orthogonal and symplectic. □

The moduli space of pairs consisting of a hyperkähler manifold of $K3^{[n]}$ type together with a symplectic involution is connected. This follows from the following result of the second named author [10, Theorem 2.5]. Here we include a more general version of the result where we remove the assumption that $n-1$ is a prime power.

Theorem 4.2. *Let X be a manifold of $K3^{[n]}$ type or of Kummer n type. Let $G \subset \mathrm{Aut}_s(X)$ be a finite group of numerically standard automorphisms. Then (X, G) is a standard pair.*

Proof. First, we want to prove that (X, G) is birational to a standard pair by using the Global Torelli theorem, and then we want to prove that birational pairs can be deformed one into the other while preserving the group action. For the first step, up to deforming X, we can suppose that $\mathrm{Pic}(X) := S_G(X) \oplus \mathbb{Z}\delta$, where $\delta \subset T_G(X)$ is as in Definition 2.4. Let S be the $K3$ (resp. Abelian) surface with a G-action such that $NS(S) = S_G(S) = S_G(X)$ and $T_G(S) = T(S) = T(X)$. The $K3$ (resp. Abelian) surface S is uniquely determined if, under the identification $T(S) = T(X)$, we have $\sigma_S = \sigma_X$ in $T(X) \otimes \mathbb{C}$. An easy computation shows that $S^{[n]}$ (resp. $K_n(S)$) and X are Hodge isometric and, by the requirement of Definition 2.4, this Hodge isometry extends to an isometry of the Mukai lattice, so with a suitable choice of markings f, g the pairs (X, f) and $(S^{[n]}, g)$ (resp. $K_n(S)$) are in the same component of the Teichmuller space, thus by Theorem 2.7 they are birational and the birational map commutes with the G-action by our construction of the Hodge isometry.

Now we continue with the second step. Let U be a representative of local deformations of X, with total family $\mathcal{X}$ and let V be a representative of local deformations of Y, with total family $\mathcal{Y}$. By classical results of Huybrechts [7], up to shrinking the family there is an isomorphism $U \cong V$ and a fibrewise birational map $\varphi : \mathcal{X} \dashrightarrow \mathcal{Y}$. Let us restrict to the local deformations of the pairs (X, G) and (Y, G), which are the families over U^G (which coincides with V^G). Let $t \in U^G$ be a point such that $\mathrm{Pic}(\mathcal{X}_t) = S_G(\mathcal{X}_t)$, which is true for very general points. There is a Kähler class orthogonal to $S_G(\mathcal{X}_t)$ and, as G is symplectic, this class lies in the orthogonal complement to $\mathrm{Pic}(\mathcal{X}_t)$. Thus, the Kähler cone is the full positive cone and all manifolds birational to $\mathcal{X}_t$ are isomorphic to it, so $\mathcal{X}_t \cong \mathcal{Y}_t$ (and the isomorphism is compatible with the G-action), so the pairs (X, G) and (Y, G) are both deformation equivalent to $(\mathcal{X}_t, G)$, and our claim holds. □

Corollary 4.3. *Let X be a manifold of $K3^{[n]}$ type and let ι be a symplectic involution. Then the fixed locus of ι has the form of the fixed locus on $S^{[n]}$ of a symplectic involution coming from the $K3$ surface S.*

Proof. By [12, Cor. 37], the coinvariant lattice of a symplectic involution is always isometric to $E_8(-2)$. Thus, by Proposition 4.12, there is an embedding of $E_8(-2)$ which is numerically standard and all others have an element v inside $E_8(-2)$ which has divisibility 2 and is of square at least $-6-2n$. The latter case cannot be induced by an involution on a manifold of $K3^{[n]}$ type, because otherwise an invariant Kähler class would be orthogonal to v and this is in contradiction with Proposition 2.8.

Thus, all pairs (X, ι) are numerically standard, therefore Thm. 4.2 applies and we obtain our claim. $\square$

Corollary 4.4. *Let X be a manifold of Kummer n type and let ι be a symplectic involution. Then the fixed locus of ι has the form of the fixed locus on $K_n(A)$ of the -1 involution coming from the Abelian surface A.*

Proof. All possible symplectic involutions on the second cohomology of a manifold of Kummer n type X have been classified in [13, Section 5 and 6]. Notice that there is only one such involution by [13, Prop. 6.1], which however acts as an order four automorphism on X, therefore a symplectic involution on X must have a trivial action on $H^2(X)$. By Theorem 4.2, the pair (X, ι) is deformation equivalent to the pair $(K_n(A), -1)$ and our claim holds. $\square$

Example 4.5. Consider $\mathrm{Sym}^2(S)$, where S is a $K3$ surface with an involution i. The involution i induces an involution ι on $\mathrm{Sym}^2(S)$. The fixed locus of ι has a component of the form S/i, because locally it consists of unordered pairs of points $\{p, i(p)\}$, where $p \in S$. The rest of the fixed points are isolated unordered pairs of the form $\{p, q\}$, where p and q are fixed points of i on S, with possible repetitions. In [17] Nikulin proved that i has 8 fixed points on S. In [9] the second named author proved that the fixed locus of ι on $S^{[2]}$ consists of $28 = \binom{8}{2}$ isolated points and one copy of S. The eight fixed points of type $\{p, p\}$ are contained in the minimal resolution of S/i which is S.

Example 4.6. Consider $\mathrm{Sym}^3(S)$, where S is a $K3$ surface with an involution i. The involution i induces an involution ι on $\mathrm{Sym}^3(S)$. The fixed locus of largest dimension of ι locally looks like $\{p, i(p), q\}$, where $p \in S$ and q is a fixed point of i on S. There are 8 connected components of fixed loci of the form S/i, because there are 8 possibilities for q by Nikulin's result [17]. The rest of the fixed points are isolated of the form $\{p, q, r\}$, where p, q and r are fixed points of i on S. In total, there are 56 points on $S^{[3]}$ corresponding to triples consisting of three different points $\{p, q, r\}$, and all fixed points of the form $\{p, p, q\}$ with $p \neq q$ are contained in the resolution of S/ι. There are eight more isolated fixed points given by schemes fully supported on one point whose reduced scheme structure encompasses all possible tangent directions.

The two examples above illustrate the difference between the even and the odd cases of n when considering symplectic involutions on $S^{[n]}$ and are indicative of the approach towards the following theorem.

Theorem 4.7. *Let X be a hyperkähler manifold of $K3^{[n]}$ type, and let ι be a symplectic involution on X. Then, up to deformation, the fixed locus F of ι consists of finitely many copies of Hilbert schemes of $K3$ surfaces $S^{[m]}$ (where $m \leq \frac{n}{2}$) and possibly isolated fixed points (only when $n \leq 24$). The fixed locus F is stratified into loci of even dimensions F_{2m}, where $\max(0, \frac{n}{2} - 12) \leq m \leq \frac{n}{2}$. Each fixed locus F_{2m} of dimension $2m$ has*

$$\sum_{2m=n-k-2l} \binom{8}{k}\binom{k}{l}$$

connected components, each one of which is a deformation of a copy of $Y^{[m]}$, where Y is the $K3$ resolution of S/ι. In particular, the fixed locus Z of largest dimension is the following.

(i) If n is even, then Z consists of one copy of $Y^{[\frac{n}{2}]}$;
(ii) If n is odd, then Z consists of 8 copies of $Y^{[\frac{n-1}{2}]}$.

Proof. From Corollary 4.3, we can restrict to the case when $X = S^{[n]}$ and ι comes from a symplectic involution on S. Let us denote by Y the minimal resolution of S/ι. Since the involution ι acts as the identity on the effective classes in $H^2(X,\mathbb{R})$, it preserves the exceptional divisor of $S^{[n]}$. Therefore, it descends to an action on $\mathrm{Sym}^n S$.

By Proposition 4.1, the irreducible components of the fixed locus of ι are symplectic subvarieties of X, and in this case each one of them has even dimension $2m$. Let us label the fixed locus of dimension $2m$ by F_{2m}. By Nikulin's theorem in [17], the symplectic involution on the $K3$ has 8 fixed points $f_1, \cdots, f_8$. Each strata F_{2m} looks like $\mathrm{Sym}^m S/\iota \times \mathrm{Sym}^{n-2m}(f_1 \cup \cdots \cup f_8)$. Let $l_1, \cdots, l_8$ be the degrees of $f_1, \cdots, f_8$.

Let us denote by U_i some small analytic neighborhoods around f_i. Since S is connected, we can connect any involution fixed point point to a point inside $(U_1)^{[n_1]} \times \cdots \times (U_8)^{[n_8]}$ for some $n_i \geq 0$. Respectively, the fixed locus inside these analytic neighborhoods are the products $U_{\vec{n},\vec{s}} = \left((U_1)^{[n_1]}\right)^{\iota}_{s_1} \times \cdots \times \left((U_8)^{[n_8]}\right)^{\iota}_{s_8}$ where s_i is $\pm$ if n_i is odd and $s_i = \emptyset$ if n_i is even. Let $k(\vec{n},\vec{s})$ be a number of odd n_i and $l(\vec{n},\vec{s})$ is the number i such that $s_i = -$ then the dimension of $U_{\vec{n},\vec{s}}$ is $n - k(\vec{n},\vec{s}) - 2l(\vec{n},\vec{s})$.

By moving a pair of points $z, i(z)$ from one neighborhood U_i to another U_j we connect analytic sets $U_{\vec{n},\vec{s}}$ and $U_{\vec{n'},\vec{s'}}$ as long as $k(\vec{n},\vec{s}) = k(\vec{n'},\vec{s'})$ and $l(\vec{n},\vec{s}) = l(\vec{n'},\vec{s'})$. On the other hand, it's also clear that the if we can connect analytic sets $U_{\vec{n},\vec{s}}$ and $U_{\vec{n'},\vec{s'}}$ then $k(\vec{n},\vec{s}) = k(\vec{n'},\vec{s'})$, because we can only move points between the neighborhoods in pairs. Finally, if the invariant $l(\cdot,\cdot)$ changes along a path then the dimension of the connected component would change too. Thus we proved our formula for the number of connected components.

Now we shall describe explicitely the fixed locus $Z = F_{2[\frac{n}{2}]}$ of largest dimension. Let $m = [\frac{n}{2}]$ be the largest integer not greater than $\frac{n}{2}$, i.e., $m = \frac{n}{2}$ if n is even and $m = \frac{n-1}{2}$ if n is odd. Then the fixed locus of largest dimension is of the form $\{x_1, \iota(x_1), x_2, \iota(x_2), \cdots, x_m, \iota(x_m)\}$ if n is even, i.e., one copy of $\mathrm{Sym}^m S/\iota$, where $x_i \in S$ for $1 \leq i \leq m$. In the case when n is odd, the fixed locus of largest dimension is of the form $\{x_1, \iota(x_1), x_2, \iota(x_2), \cdots, x_m, \iota(x_m), x_{m+1}\}$, where $x_i \in S$ for $1 \leq i \leq m$ and x_{m+1} is a fixed point, i.e., Z contains eight copies of $\mathrm{Sym}^m S/\iota$ since there are 8 choices for x_{m+1}. In both cases, the dimension of $\mathrm{Sym}^m S/\iota$ is $2m$. □

Remark 4.8. As a special case of this theorem when $n = 2$ we obtain the description of the fixed locus F of ι on $S^{[2]}$ that the second named author proved in [9], namely that F consists of $28 = \binom{8}{2}$ isolated points and one copy of the minimal resolution of S/ι.

Remark 4.9. The involution fixed locus does not have a zero dimensional component if $n > 24$. On the other hand, $\left(\mathrm{K3}^{[24]}\right)^{\iota}$ has a zero dimensional component which is the product of eight squares of maximal ideals of the involution fixed locus.

Finally, let us conclude with the analogous result in the Kummer case. To state the combinatorial part of the result we need a few extra notations. Given a subset

$I \subset \mathbb{Z}_2^4$, we denote by $|I|$ the size of the set and $||I||$ the element of $\mathbb{Z}_2^4$ which is the total product of the elements in the set. Thus, we define:

$$N_m^n = \sum_{I,||I||=1} \binom{(n-|I|)/2-m}{|I|}.$$

Theorem 4.10. *Let X be a hyperkähler manifold of Kummer n type, and let ι be a symplectic involution on X. Then, up to deformation, the fixed locus F of ι consists of finitely many copies of Hilbert schemes of $K3$ surfaces $S^{[m]}$ (where $m \leq \frac{n+1}{2}$) and possibly isolated fixed points (only when $n \leq 48$). The fixed locus F is stratified into loci of even dimensions F_{2m}, where $\max(0, \frac{n+1}{2} - 24) \leq m \leq \frac{n+1}{2}$. Each fixed locus F_{2m} of dimension $2m$ has N_m^n connected components, each one of which is a deformation of a copy of $S^{[m]}$. In particular, the fixed locus Z of largest dimension is one copy of $S^{[\frac{n+1}{2}]}$.*

Proof. From Corollary 4.4, we can restrict to the case when $X = K_n(A)$ and ι comes from a the -1 involution on A. Since the involution ι acts as the identity on the classes in $H^2(X, \mathbb{R})$, it preserves the exceptional divisor of $K_n(A)$. Therefore, it descends to an action on $\mathrm{Sym}^{n+1}A$ preserving the fibre over 0 of the Albanese map. Let us first look at this action on $\mathrm{Sym}^{n+1}A$, and then let us look at the Albanese fibre of the fixed locus.

By Proposition 4.1, the irreducible components of the fixed locus of ι are symplectic subvarieties of X, and in this case each one of them has even dimension $2m+2$. Let us label the fixed locus of dimension $2m+2$ by F_{2m}. The symplectic involution on the Abelian surface A has 16 fixed points $f_1, \cdots, f_{16}$, and there is a natural identification of $\{f_1, \ldots, f_{16}\}$ with $\mathbb{Z}_2^4$. The same argument as in the previous theorem implies that there is a natural correspondence between the $2m$-dimensional connected components of the involution fixed locus and the set of pairs: $I_- \subset I_{odd} \subset \mathbb{Z}_2^4$ such that $2m = n - |S_{odd}| - 2|S_-|$.

Finally, let us notice that the Albanese map is constant on the connected components and the value of the map on the connected component is 1 iff $||I_{odd}|| = 1$. $\square$

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Appendix: lattice computations

In this appendix we include all the computations needed in the proof of Corollary 4.3. Let E_8 be the unique unimodular even positive definite lattice of rank 8 and let $S := E_8(-2)$ be the same lattice with the quadratic form multiplied by -2. The discriminant group of S is $\mathbb{Z}_2^8$ and its elements are classes $[v/2]$, where $v \in S$. The discriminant form on these elements takes the values 0 or 1 modulo $2\mathbb{Z}$ (i.e., it is 0 if v^2 is divisible by 8 and 1 otherwise). We have the following:

Lemma 4.11. *For every element α in A_S there is an element $v \in S$ such that $[v/2] = \alpha$ and $v^2 \geq -16$.*

Proof. As E_8 is generated by its roots, the discriminant group of S is generated by classes of half-roots, which have square -4. Thus, all elements of A_S can be represented by half the sum of at most eight distinct roots. The sum of two roots is either a root (if they are not orthogonal) or an element of square -8. As there can be at most a set of four orthogonal roots, the claim follows. $\square$

Let $L_n := U^3 \oplus E_8(-1)^2 \oplus (-2n+2)$ be the lattice corresponding to the second cohomology of a $K3^{[n]}$ type manifold. We then have the following:

Proposition 4.12. *Let L_n and S be as above. Then, up to isometry, there is only one primitive embedding of S into L_n such that S contains no element of divisibility two and square bigger than $-6-2n$.*

Proof. The discriminant group of L_n has one generator whose discriminant form is $-\frac{1}{2(n-1)}$. There is only one element in this group whose order is precisely two, and it has discriminant form $-\frac{n-1}{2}$ modulo $2\mathbb{Z}$. As per Lemma 2.9, primitive embedding of S into L_n are determined by a quintuple $(H_S, H_{L_n}, \gamma, K, \gamma_K)$, where the first two are subgroups of A_S and A_{L_n}, respectively, and γ is an anti-isometry between the two. Thus, when n is even, H_S, H_{L_n} and γ are trivial as all elements of A_S have integer square. When n is odd, either we are in the same case as before or we have nontrivial H_S, H_{L_n} and γ. In the latter case, H_{L_n} is unambiguously determined and by Lemma 4.11 the non trivial element of H_S is represented by half an element v of square at least -16 (more specifically, at least -16 if $n-1$ is divisible by four and at least -12 otherwise). Thus, v is an element of square at least $-6-2n$ and of divisibility 2 in L_n, as $[v/2]$ is non trivial in A_{L_n}. $\square$

We conclude this appendix with a criterion to avoid checking the last condition in Definition 2.4:

Proposition 4.13. *Let (X, G) be a pair such that there exists a $K3$ (resp. Abelian) surface S and $G \subset Aut_s(S)$ such that $H^2(S^{[n]})$ (resp. $H^2(K_n(A))$) and $H^2(X)$ are isomorphic G representations. Moreover, suppose that $U \subset T_G(S)$. Then (X, G) is numerically standard.*

Proof. By the hypothesis we have a Hodge isometry $\varphi : H^2(X) \to H^2(S^{[n]})$ (resp. $H^2(K_n(A))$) which might not extend to an isometry of the Mukai lattice $\Lambda := U^4 \oplus E_8(-1)^2$ (resp. $\Lambda := U^4$), that is, it is not compatible with the two embeddings $\psi_1 : H^2(X, \mathbb{Z}) \to \Lambda$ and $\psi_2 : H^2(S^{[n]}, \mathbb{Z}) \to \Lambda$. Let δ be half the class of the exceptional divisor in $S^{[n]}$ (resp. $K_n(A)$) and let $\delta_x := \varphi^{[-1]}(\delta)$. Let v be a generator of $\psi_2(H^2)^\perp$ and v_x be the same for $\psi_1(H^2)^\perp$. As discussed in in [8, Section 9], the fact that φ does not extend to Λ means that it does not respect the two gluing data associated to the pairs (v, δ) and (v_x, δ_x). The gluing data corresponds to a choice of an anti-isometry between the two discriminant groups A_{H^2} and $A_{\langle v \rangle}$. However, we have $U \subset T_G(S)$ and let $L := U \oplus \mathbb{Z}\delta \subset T_G(S^{[n]})$ (resp. $T_G(K_n(A))$), thus, by [18, Thm 1.14.2] applied to L, we can compose φ with an isometry γ of $H^2(S^{[n]})$ (resp. $H^2(K_n(A))$) which is trivial on $L^\perp$ and arbitrarily non trivial on $A_L \cong A_{H^2(S^{[n]})}$, thus $\delta \circ \varphi$ extends to an isometry of the Mukai lattices.

$\square$

In particular, this proposition applies to symplectic involutions on manifolds of $K3^{[n]}$ type .

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