Inversions. Draft

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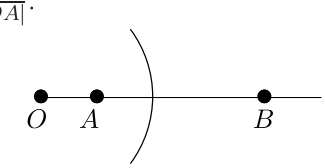
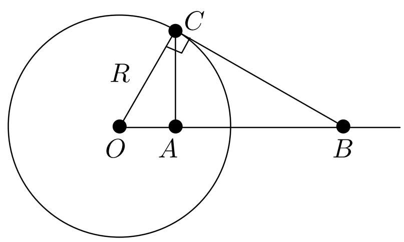
1. Definitions of inversion

Inversion is a sort of symmetry in a circle. It is defined as follows. The *inversion of degree R^2 centered at a point O* maps a point $A \neq O$ to the point B on the ray OA such that R is the geometric mean of OA and OB, that is $|OA||OB| = R^2$, or $|OB| = \frac{R^2}{|OA|}$.



A geometric construction relating points O, A and B looks as follows.



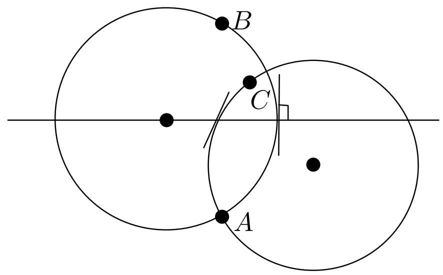
On O the inversion is not defined, and O is not the image of any point under the inversion. Thus, the inversion is a mapping of the plane punctured at O to itself.

Observe that points of the circle centered at O of radius R are fixed under the inversion, points of the disk bounded by this circle are mapped to points outside the disk and vice versa. This circle is called the *circle of the inversion* and the inversion is referred to as the *inversion in this circle.*

The square of an inversion, that is an inversion composed with itself, is the identity map. In other words, an inversion is invertible map and the map inverse to an inversion is the same inversion.

The definitions of reflection in a line and inversion does not look similar. However these two transformations admit similar definitions.

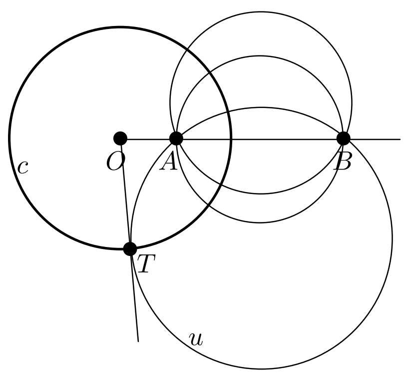
Theorem A. *Any circle passing through points symmetric with respect to a line is orthogonal to the line. For any two points non-symmetric with respect to a* *line l there exists a circle passing through the points which is not orthogonal to l.*



Exercise. Prove Theorem A. $\square$

This theorem allows to define reflection in a line l as a map which maps a point A to a point B such that any circle passing through A and B is orthogonal to l.

Theorem B. *Any circle passing through a point A and its image B under the inversion in a circle c is orthogonal to c. If B is not the image of A under the inversion in a circle c, then there exists a circle passing through the points which is not orthogonal to c.*



Proof. For any circle u passing through points A and B the degree of the center of inversion O with respect to u is $|OA||OB|$. Therefore $|OT|^2 = |OA||OB|$, where OT is the segment of the line tangent to u between the center of inversion and the point of tangency.

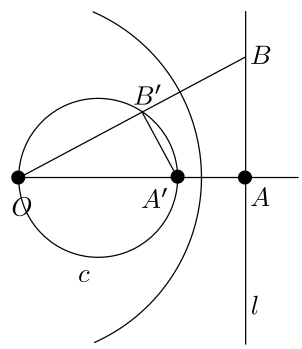
On the other hand, $|OA||OB|$ is equal to the degree R^2 of the inversion, that is to the square of the radius of circle c. Therefore $T \in c$. Hence, OT is a radius of c. As a radius of c, it is perpendicular to the tangent of c. Thus at T the lines to u and c are perpendicular to each other.

A proof of the second statement is an exercise. $\square$

2. Images of lines and circles

Obviously, a line passing through the center of an inversion is mapped by the inversion to itself.

Theorem C. *The image under an inversion of a line l not passing through the center O of the inversion is a circle c passing through O and having at O a tangent line parallel to l.*



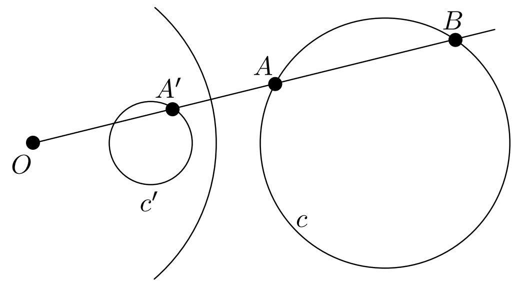
Proof. Drop the perpendicular OA to l from O. Let A be its intersection with l. Let A' be the image of A under the inversion. Take arbitrary point $B \in l$. Denote by B' its image under the inversion. By the definition of inversion $|OA||OA'| = |OB||OB'|$. Therefore $\frac{OA}{OB} = \frac{OB'}{OA'}$. By SAS-test for similar triangles, $\triangle OAB$ is similar to $\triangle OB'A'$. Therefore $\angle A'B'O = \angle OAB$. The latter angle is right, because $OA \perp l$. Hence B' belongs to the circle with diameter OA'.

Vice versa, let us take any point B' of the circle with diameter OA'. Draw a ray OB' and denote the intersection of this ray with l by B. Triangles $\triangle OB'A'$ and $\triangle OAB$ similar by the AA-test. Hence $\frac{OB}{OA} = \frac{OA'}{OB'}$ and $|OB||OB'| = |OA||OA'|$. Therefore, B' is the image of B under the inversion. □

Corollary D. *The image under an inversion of a circle c passing through the center O of the inversion is a line which is parallel to the line tangent to c at O.*

Proof. It follows from Theorem C, because an inversion is inverse to itself. □

Theorem E. *The image under an inversion of a circle c that does not pass through the center O of the inversion is a circle c' that is the image of c under a homothety centered at O.*



Proof. Let A be a point of circle c, and A' be the image of A under the inversion. Denote by B the second intersection point of the ray OA with c. By definition of inversion, $|OA'| = \frac{R^2}{|OA|}$, where R^2 is the degree of inversion. On the other hand, $|OA| = \frac{d^2}{|OB|}$, where d^2 is the degree of O with respect to the circle c. Recall that d does not depend on the points A and B, this is the length of segment of a tangent line from O to c between O and the point of tangency.

Substituting this formula to the formula for $|OA'|$, we get

$$|OA'| = \frac{R^2}{d^2}|OB|.$$

This means that A' is the image of B under the homothety with center O and ratio $\frac{R^2}{d^2}$. Hence, the image of c under the inversion is the image of c under this homothety. □

Theorem F. *A composition of two inversions with the same center is a homothety centered at the same center. The ratio of this homothety is the ratio of the degrees of the inversions.*

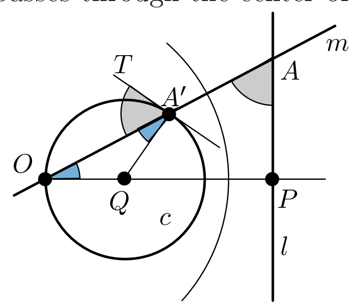
Proof. Let O be the center of inversions I_1, I_2 and R_1^2, R_2^2 be their degrees. Take an arbitrary point A. Then $A_1 = I_1(A)$ is a point on the ray OA, and $|OA_1| = \frac{R_1^2}{|OA|}$. Further, $A_2 = I_2 \circ I_1(A)$ is also a point on the ray OA and

$$|OA_2| = \frac{R_2^2}{|OA_1|} = \frac{R_2^2}{\frac{R_1^2}{|OA|}} = \frac{R_2^2}{R_1^2}|OA|.$$

□

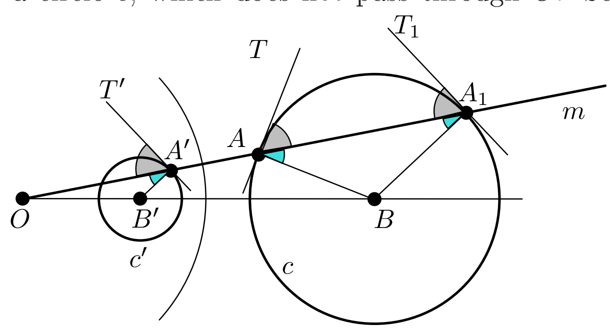
Theorem G. *An inversion preserves angles between lines and circles.*

Proof. Let us begin with special cases. Consider, first, the angle between two lines, one of which passes through the center of inversion.



Let the lines be l and m, the center of inversion be $O \in m$. Then the image of m under the inversion is m, while the image of l is a circle c passing through O. The center Q of c lies on the perpendicular OP dropped from O to l. The angles $\angle AOP$ and $\angle OAP$ are complementary. The angles $\angle QA'O$ and $\angle AOQ$ are equal as angles in an isosceles triangle $\triangle OQA'$. The angle $\angle TA'Q$ is right as an angle between a radius QA' and tangent line $A'T$. Therefore angles $\angle TA'O$ and $\angle QA'O$ are complementary. Consequently, $\angle OAP = \angle TA'O$. □

Consider now the angle between a line m passing through the center O of inversion and a circle c, which does not pass through O. See the picture:



The prove in this case is similar to the preceding one, and it is left as an exercise. □

An angle between arbitrary two lines or circles can be presented as the sum or difference of angles between the same lines or circles and a line passing through the center of inversion. Thus, the general case reduces to the special ones above. □

3. The Apollonius problem

A general formulation: find a circle tangent to three given circles.

The problem is invariant under inversions, and therefore circles should be understood in wide sense, invariant under inversions: both lines and circles.

Furthermore, one may consider degenerations of the Apollonius problem, a circle may shrink to a point. A line tangent to a shrinking circle turns in the limit to a line passing through the point to which the circle has shrunk.

We start from degenerate cases since they are easier.

Problem 1. *Find a circle passing through two given points and tangent to a given line.*

Denote the points by A and B and the line by l. Draw a line through A and B and denote its intersection point with l by I. Let c_1 and c_2 be the circles we want to find. Denote by T_i the point of tangency of c_i and l. The segments IT_1, IT_2 are geometric mean of IA and IB:

$$|IT_1|^2 = |IA||IB| = |IT_2|^2.$$

This suggests a construction of IT_1 and IT_2.

Problem 2. *Find a circle passing through two given points and tangent to a given circle.*

This problem can be obtained from Problem 1 by an inversion at a point. A special choice of the center of inversion converts this problem to a simpler one. For instance, if we choose the center of inversion at one of the given points, the desired circle would turn into a line passing through the image of another given point and tangent to the image of the given circle. So, we reduce the problem to the following easy one:

Problem 3. *Find a line tangent to a given circle and passing through a given point.*

Problem 4. *Find a circle passing through a given point and tangent to two lines.*

This problem was solved using homothety. It can be also reduced to Problem 1 by reflecting the given point in the bisector of the angle formed by the given lines.

If, instead, we would apply an inversion centered at the given point, then Problem 4 would turn into the following problem

Problem 5. *Find a common tangent line of two given circles.*

A general Apollonius problem can be reduced to Problem 4 by simultaneous replacing of the circles by circles with radii differing from the original ones by the same number such that one of the circles shrinks to a point.

4. Pairs of circles

Any two circles can be transformed to each other by homothety or by translation. Therefore any two circles can be transformed to each other by an inversion. What about pairs of circles? Certainly, there are two pairs of circles which cannot be transformed by an inversion to each other. For example, an inversion preserves the angle between circles, therefore circles intersecting at one angle cannot be transformed to circles intersecting at a different angle. Disjoint circles cannot be made intersecting by an inversion.

However a pair of circles can be made quite special by an inversion.

Theorem H. *Any two circles meeting at non-zero angle can be transformed by an inversion into a pair of lines intersecting at the same angle.*

Proof. Any inversion centered at any of the two intersection point of the circles does the job. □

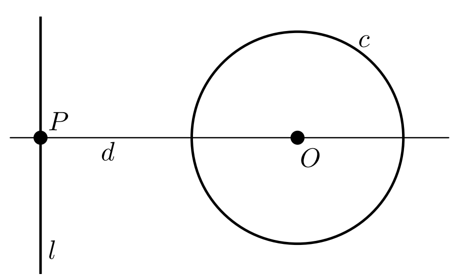
Theorem I. *Any two circles tangent to each other can be transformed by an inversion into a pair of parallel lines.* □

Proof. An inversion centered at the point of tangency of the circles maps the circles to lines, and the lines do not intersect, because otherwise the preimage of the intersection point of the lines was an intersection point of the circles different from the center of inversion. □

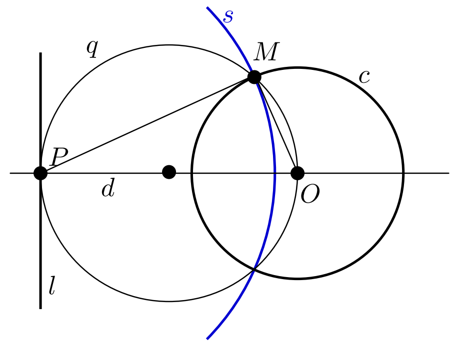
Theorem J. *For any disjoint line l and circle c, there exists an inversion that map them to a pair of concentric circles.*

Lemma K. *For any disjoint line l and circle c, there exist a line d and a circle s perpendicular to each other and to l and c.*

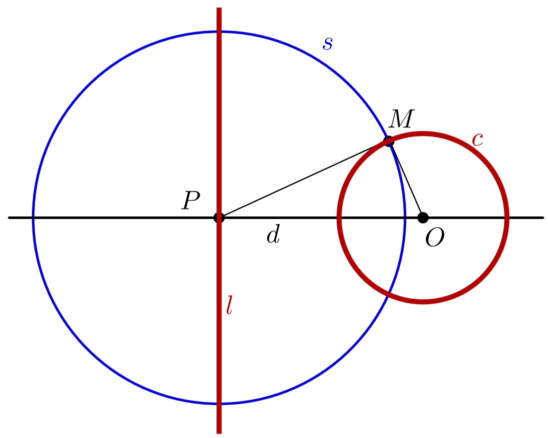
Proof. Drop from the center O of circle c a perpendicular d to the line l. Let P be the intersection point of d and l.



Draw a circle s centered at P perpendicular to the circle c. For this draw a circle q with diameter PO, find an intersection point M of q and c and draw the circle with center P through M. This is s.



After removing auxiliary figures, the result looks as follows:



□

Proof of Theorem J. An inversion centered at an intersection point of d and s maps d to d and the circle s to a line e. Since s is perpendicular to d, its image e is perpendicular to d. The inversion maps l and c to circles perpendicular to the lines d and e. A line perpendicular to a circle passes through the center of the circle. Therefore the circles perpendicular to lines d and e have the center in the intersection point of d and e. □

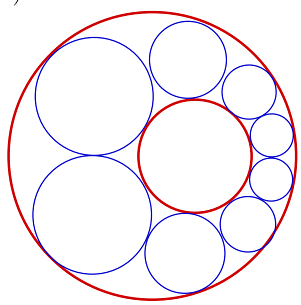
Theorem L. *Any pair of disjoint circles can be transformed by an inversion into a pair of concentric circles.*

Proof. By an inversion centered at some point of one of the given disjoint circles a and b, the circles turn into disjoint line l and circle c. By Lemma K, there exist line d and circle s perpendicular to each other and to l and c. The same inversion returns l and c back to a and b and transform line d and circle s into a pair of circles, say, p and q, perpendicular to each other and to a and b. An inversion centered at an intersection point of p and q transforms p and q into a pair of lines perpendicular to each other and transforms a and

b into a pair of circles a' and b' perpendicular to these lines. These a' and b' are concentric, since they are perpendicular to a pair of intersecting lines. Cf. the proof of Theorem J. □

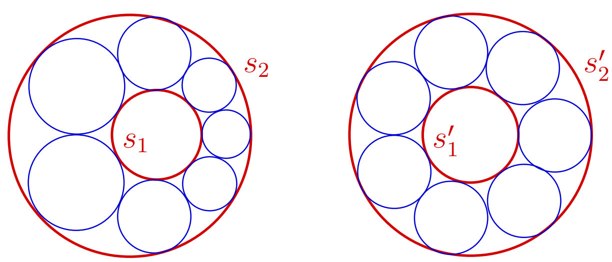
5. Necklaces of circles. Steiner porism

By a necklace of circles we will call a finite sequence of circles c_1, c_2, $\ldots$, c_n each of which is tangent to two fixed disjoint circles s_1 and s_2 (called the base circles of the necklace) and two other circles from the sequence.



Theorem M. *If circles s_1 and s_2 form a base of a necklace, then any circle c which is tangent to s_1 and s_2 can be included into a necklace of the same number of circles. In particular, s_1 and s_2 form a base for infinitely many necklaces consisting of the same number of circles.*

Proof. By Theorem L, circles s_1 and s_2 can be mapped by an inversion to concentric circles s_1' and s_2'. This inversion maps the circles of a necklace with base s_1, s_2 into circles of the same radius forming a necklace with base s_1', s_2'.



Such a necklace can be freely rotated between the base concentric circles, and any circle from the necklace can be identified by a rotation with any circle tangent to s_1' and s_2'. The inversion can be used to transform the necklaces between s_1' and s_2' into necklaces between the original circles s_1 and s_2. □