Series Logo
Volume 00, Number 00, Xxxx 19xx

COMPLEX STRUCTURES ON RULED SURFACES

LJUDMILA KAMENOVA KAMENOVA

Abstract. We study the twistor space Z of $\mathbf{D} \times \mathbb{CP}^1$. If the scalar curvature of $\mathbf{D} \times \mathbb{CP}^1$ is zero, then it is known that Z is a complex manifold, so every almost complex structure of $\mathbf{D} \times \mathbb{CP}^1$, compatible with the metric is integrable. Our main result is that the set of all integrable structures of $\mathbf{D} \times \mathbb{CP}^1$ is a real quadric, which we describe explicitly. As a corollary we get the same result for ruled surfaces of genus $g \geq 2$ and of an even degree.

1. Preliminaries

Here we introduce the twistor space of an oriented even dimensional Riemannian manifold M, following the notations of [2], [3] and [5].

Let (M, g) be an oriented connected Riemannian manifold of real dimension $2n$. Let P be the $SO(2n)$-principle bundle of oriented g-orthonormal frames on M. Denote by $\pi : P \to M$ the canonical projection.Then $SO(2n)$ acts on the right on P.

Consider local coordinates $\{x_1, .., x_{2n}\}$ in a nieghborhood U of $x \in M$, and let $\{\theta_1, ..., \theta_{2n}\}$ be a local oriented g-orthonormal frame.

The fibre $\pi^{-1}(x)$ is diffeomorphic to $SO(2n)$.Let us denote by $i : \pi^{-1}(x) \to P$ the fibre's inclusion, then

$$\tilde{V}_a = i_{*|a}(T_a \pi^{-1}(x))$$

is called the vertical tangent space at the point a.

Let $\{\theta_1^*, ..., \theta_{2n}^*\}$ be the local coframe dual to $\{\theta_1, ..., \theta_{2n}\}$, then the covariant derivative ∇ on M defined by the Levi-Civita connection of g is locally expressed by: $\nabla \theta_j = \Gamma_{ij}^k \theta_i^* \otimes \theta_k$, and the Christoffel's symbols satisfy $\Gamma_{ij}^k = -\Gamma_{ik}^j$, so the matrix $(\Gamma_{i\cdot}^{\cdot}) \in so(2n)$. Hence (according to [4]), the Riemannian connection on P at the point $a = (x; \tilde{X}) = (x; (X_j^i))$, induced by g, can be expressed by:

$$\omega_j^i(x; \tilde{X}) = \frac{1}{2}(X_i^r dX_j^r - X_j^r dX_i^r) + X_i^r \Gamma_{ml}^r(x) X_j^l \theta_m^*(x) \tag{1}$$

The connection $\omega = (\omega^i_j)$ on P induces a splitting of the tangent bundle in horizontal and vertical subbundles, at the point a:

$$T_aP = \tilde{H}_a \oplus \tilde{V}_a,$$

where

$$\tilde{H}_a := \{Y \in T_aP \mid \omega(Y) = 0\}$$

is the horizontal tangent space at the point a.

$SO(2n)$ acts transitively on $\frac{SO(2n)}{U(n)}$. Then we have an action of $SO(2n)$ on $P \times \frac{SO(2n)}{U(n)}$, defined by:

$$(2) \qquad \begin{aligned} SO(2n) \times P \times \frac{SO(2n)}{U(n)} &\to P \times \frac{SO(2n)}{U(n)}, \\ (A, a, XU(n)) &\to (aA, A^{-1}XU(n)). \end{aligned}$$

Definition 1. *The twistor space of (M, g) is the associated bundle to P defined as the quotient of $P \times \frac{SO(2n)}{U(n)}$ with respect to the action (2). It will be denoted by $Z(M, g)$, or simply by Z, when the manifold (M, g) is understood.*

Z is a bundle over M with fibre $\frac{SO(2n)}{U(n)}$ and structure group $SO(2n)$. Denote by $\Pi : P \to Z$ and by $r : Z \to M$ the bundle projections, and by $Z_x := r^{-1}(x)$ the fibre of Z at the point $x \in M$. The geometric meaning of Z is clear from the following:

Theorem 1.1. *Z_x parametrizes the complex structures on T_xM compatible with the metric g and the orientation.*

The splitting $T_aP = \tilde{H}_a \oplus \tilde{V}_a$, via Π induces a corresponding splitting of T_pZ for $p \in Z$. Let $a \in \Pi^{-1}(p)$. Then

$$T_pZ = (\Pi)_{*|a}(\tilde{H}_a) \oplus (\Pi)_{*|a}(\tilde{V}_a) := H_p \oplus V_p$$

We have immediately that $r_{*|p}(H_p) = T_{r(p)}M$. Then for every $p \in Z$ and for every tangent vector $X \in T_pZ$, we have a decomposition of X in horizontal and vertical component:

$$X = X_h + X_v, X_h \in H_p, X_v \in V_p.$$

Let us define an almost complex structure J on Z: for $X = X_h + X_v \in T_pZ$, we set:

$$J(X) = r^{-1}_{*|r(p)} \circ p \circ r_{*|p}(X_h) + J_V(X_v),$$

where p acts on $T_{r(p)}M$ according to Theorem 1.1, and J_V is the almost complex structure of the symmetric space $Z_{r(p)} \cong \frac{SO(2n)}{U(n)}$.

2. Main Results

We explore the complex inclusions of $\mathbf{D}\times\mathbb{CP}^1$ in its twistor space and in consequence we get the explicit record of its complex structures.

We use the notations of the previous section. Let $\mathbf{D}$ be the one-dimensional disc equipped with Poincare metric with scalar curvature $-1-\epsilon$, $(\epsilon>-1)$, and $\mathbb{CP}^1$ denotes the projective 1-dimensional space with Fubini-Study metric with scalar curvature $+1$. We denote the real local coordinates on $\mathbf{D}$ and $\mathbb{CP}^1$ with x_1, y_1 and x_2, y_2, respectively. According to [8] their metrics are :

$$g_{\mathbf{D}}=\frac{dx_1\otimes dx_1+dy_1\otimes dy_1}{1-\frac{1+\epsilon}{4}(x_1^2+y_1^2)}, g_{\mathbb{CP}^1}=\frac{dx_2\otimes dx_2+dy_2\otimes dy_2}{1+\frac{1}{4}(x_2^2+y_2^2)}.$$

We set:

$$A:=\frac{1}{1-\frac{1+\epsilon}{4}(x_1^2+y_1^2)}, B:=\frac{1}{1+\frac{1}{4}(x_2^2+y_2^2)}.$$

We choose $x=(x_1,y_1,x_2,y_2)\in\mathbf{D}\times\mathbb{CP}^1$. An orthonormal frame for $T_x(\mathbf{D}\times\mathbb{CP}^1)$ is

$$\left\{\theta_1=\frac{1}{A}\frac{\partial}{\partial x_1},\theta_2=\frac{1}{A}\frac{\partial}{\partial y_1},\theta_3=\frac{1}{B}\frac{\partial}{\partial x_2},\theta_4=\frac{1}{B}\frac{\partial}{\partial y_2}\right\}$$

and an orthonormal frame for $T_x^*(\mathbf{D}\times\mathbb{CP}^1)$ is

$$\{\theta_1^*=Adx_1,\theta_2^*=Ady_1,\theta_3^*=Bdx_2,\theta_4^*=Bdy_2\}.$$

From the condition $\nabla\theta_j=\Gamma_{ij}^k\theta_i^*\otimes\theta_k$ for the metric of $\mathbf{D}\times\mathbb{CP}^1$, we compute its Christoffel's symbols and therefore we get the following:

Lemma 2.1. *The curvature components of the metric of* $\mathbf{D}\times\mathbb{CP}^1$ *defined above are:*

$$R_{212}^1=\epsilon A^2, R_{434}^3=\epsilon B^2,$$
$$R_{313}^1=R_{414}^1=R_{323}^2=R_{424}^2=\epsilon AB,$$

$R_{jkl}^i=0$, *else, where* $i<j, k<l$.

In order to parametrize the fibre in more convinient way, we use the following:

Lemma 2.2.

$$\frac{SO(4)}{U(2)}\cong\mathbb{CP}^1.$$

Proof. Let $\mathbb{H}\cong\mathbb{R}^4$ be the space of quaternions with a basis $\{\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k}\}$ over $\mathbb{R}$ with the relations:

$$\mathbf{i^2}=\mathbf{j^2}=\mathbf{k^2}=-\mathbf{1},\mathbf{ij}=\mathbf{k}=-\mathbf{ji},\mathbf{jk}=\mathbf{i}=-\mathbf{kj},\mathbf{ki}=\mathbf{j}=-\mathbf{ik}.$$

The standart complex structure J_2 of $\mathbb{R}^4$ is identified with the left multiplication with $\mathbf{i}$. Let S^3 be the unit sphere in $\mathbb{R}^4$, i.e. the space of unit quaternions. Let $q \in S^3, q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d = z_1 + z_2\mathbf{j}$, where $z_1 = a + \mathbf{i}b, z_2 = c + \mathbf{i}d$. We define the following maps:

$$c_1 : \mathbb{C}P^1 \to S^3, [z_1 : z_2] \mapsto q; c_2 : S^3 \to \mathbb{C}P^1, q \mapsto [z_1 : z_2].$$

For every $q \in S^3$ we define a matrix $A_q \in SO(4)$, which corresponds to a left multiplication with the conjugate of q, in other words, $A_q x = \bar{q} \cdot x$, where $x \in \mathbb{H} = \mathbb{R}^4$ and " $\cdot$ " is the quaternionic multiplication. Reversly, for every matrix $A \in SO(4)$, we define a unit quaternion q with the same equation. We obtain the following maps:

$$s_1 : S^3 \to SO(4), q \mapsto A_q; s_2 : SO(4) \to S^3, A_q \mapsto q.$$

Note that for $q_1, q_2 \in S^3$, $A_{q_1} = A_{q_2}$ if and only if $\bar{q}_1^{-1}\bar{q}_2 \in U(2)$

Now we can define the maps:

$$p_1 : \mathbb{C}P^1 \to \frac{SO(4)}{U(2)}, [z_1 : z_2] \mapsto A_q U(2), p_1 = pr \circ s_1 \circ c_1$$

and

$$p_2 : \frac{SO(4)}{U(2)} \to \mathbb{C}P^1, A_q U(2) \mapsto [z_1 : z_2], p_2 = c_2 \circ s_2 \circ i,$$

where $pr : SO(4) \to \frac{SO(4)}{U(2)}$ and $i : \frac{SO(4)}{U(2)} \to SO(4)$.

It is easily seen that the maps p_1 and p_2 are correctly defined and they are isomorphisms between $\mathbb{C}P^1$ and $\frac{SO(4)}{U(2)}$.

$\square$

Remark 1. *We associate to A_q the almost complex structure $J_q = A_q^{-1} J_2 A_q$, comming from Theorem 1.1. After some computations we get:*

$$J_q = \begin{pmatrix} 0 & -S & 2(ad+bc) & 2(bd-ac) \\ S & 0 & 2(bd-ac) & -2(ad+bc) \\ -2(ad+bc) & 2(ac-bd) & 0 & -S \\ 2(ac-bd) & 2(ad+bc) & S & 0 \end{pmatrix},$$

where $S = a^2 + b^2 - c^2 - d^2$.

In order to compute a local frame for $\tilde{H}_a, a \in \pi^{-1}(U)$, it suffices to compute horizontal lifts $\tilde{\theta}_j$ of θ_j for $1 \leq j \leq 4$. In the notations of the previous paragraph, $\tilde{\theta}_j$ is uniquely determined by: $\pi_{*|a}(\tilde{\theta}_j) = \theta_j(\pi(a))$ and $\omega(a)(\tilde{\theta}_j) = 0$ for all $a \in \pi^{-1}(U)$, where ω is defined by (1). The local frames $\hat{\theta}_j, 1 \leq j \leq 4$, for $H_p, p \in Z$ are the projections of $\tilde{\theta}_j$:

$$\hat{\theta}_j = \Pi_{*|a}(\tilde{\theta}_j).$$

We consider local coordinates $\{u_1, u_2\}$ on the fibre $\mathbb{C}P^1$, where $u := \frac{z_2}{z_1}, u = u_1 + \mathbf{i}u_2$. So the local coordinates on the twistor space Z are $\{x_1, y_1, x_2, y_2, u_1, u_2\}$. If we choose local coordinates $\{v_1, v_2\}$ on the fibre $\mathbb{C}P^1$, where $v := \frac{z_1}{z_2}, v = v_1 + \mathbf{i}v_2$, then the computations will be analogous. Let $p = (x_1, y_1, x_2, y_2, u_1, u_2) \in Z$. According to Remark 1,we can compute J in the chosen local coordinates,as well as $J\hat{\theta}_j, 1 \leq j \leq 4$.

Now we can compute explicitely all terms of the Njienhuis tensor

$$N(J)(p)(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

For convenience we will write $N(X,Y)$ instead of $N(J)(p)(X,Y)$.

Lemma 2.3.

$$N(\hat{\theta}_1, \hat{\theta}_2) = N(\hat{\theta}_3, \hat{\theta}_4) = \frac{8\epsilon(u_2\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_2})((A^2+B^2)(u_1^2+u_2^2) - 2AB)}{(1+u_1^2+u_2^2)^2},$$

$$N(\hat{\theta}_1, \hat{\theta}_3) = -N(\hat{\theta}_2, \hat{\theta}_4) = -\frac{4\epsilon(1+u_1^2-u_2^2)((A^2+B^2)(u_1^2+u_2^2) - 2AB)}{(1+u_1^2+u_2^2)^2}\frac{\partial}{\partial u_1} - $$
$$-\frac{8\epsilon u_1 u_2((A^2+B^2)(u_1^2+u_2^2) - 2AB)}{(1+u_1^2+u_2^2)^2}\frac{\partial}{\partial u_2},$$

$$N(\hat{\theta}_1, \hat{\theta}_4) = N(\hat{\theta}_2, \hat{\theta}_3) = -\frac{8\epsilon u_1 u_2((A^2+B^2)(u_1^2+u_2^2) - 2AB)}{(1+u_1^2+u_2^2)^2}\frac{\partial}{\partial u_1} - $$
$$-\frac{4\epsilon(1-u_1^2+u_2^2)((A^2+B^2)(u_1^2+u_2^2) - 2AB)}{(1+u_1^2+u_2^2)^2}\frac{\partial}{\partial u_2}.$$

According to the Newlander-Nirenberg Theorem, an almost complex structure is integrable if and only if its Nijenhuis tensor vanishes. We apply it for the horizontal subbundle of Z and get the following result:

Theorem 2.1. *If $\epsilon = 0$, then Z is a complex manifold, else $\mathbf{D} \times \mathbb{CP}^1$ embeds in Z as a complex submanifold with the help of the section*

$$\phi : \mathbf{D} \times \mathbb{CP}^1 \to Z,$$
$$(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, x_2, y_2, u_1, u_2)$$

if and only if

$$u_1^2 + u_2^2 = \frac{2AB}{A^2 + B^2},$$

where A and B are defined above.

Corollary 2.1.1. *If $\epsilon = 0$, then all almost complex structures of $\mathbf{D} \times \mathbb{CP}^1$, compatible with the metric, are integrable, else the set of integrable complex structures is a quadric.*

Remark 2. *In the case $\epsilon \neq 0$, all integrable almost complex structures, compatible with the metric are in the equivalent class with representative of the form:* $J = \begin{pmatrix} J_H & 0 \\ 0 & J_V \end{pmatrix}$, *where* $J_V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$J_H = \frac{1}{1+u_1^2+u_2^2} \begin{pmatrix} 0 & 1-u_1^2-u_2^2 & 2u_2 & -2u_1 \\ u_1^2+u_2^2-1 & 0 & -2u_1 & 2u_2 \\ -2u_2 & 2u_1 & 0 & 1-u_1^2-u_2^2 \\ 2u_1 & 2u_2 & u_1^2+u_2^2-1 & 0 \end{pmatrix},$$

and moreover $u_1^2 + u_2^2 = \frac{2AB}{A^2+B^2}$.

Remark 3. *It is interesting that the above properties we got do not depend on the exact value of ϵ, but only on the condition if it is equel or not to zero.*

Remark 4. *Although the submanifolds $\phi(\mathbf{D} \times \mathbb{CP}^1)$ of Z are the same as smooth manifolds for different values of ϵ, their induced metrics are different.*

Remark 5. *Since $\mathbf{D}^1$ is the universal cover of Riemmanian surfaces of genus $g \geq 2$, then Theorem 2.1 is true for every ruled surface of genus $g \geq 2$ and of an even degree (see* [6] *) .*

References

[1] M.F. Atiyah,N. Hitchin, Singer, *Self-duality in four-dimensional Riemannian Geometry* , Proc.R.Soc.London, Ser.A **362** (1978) 425-461

[2] A. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin,1987

[3] P.de Bartolomais, A. Nannicini, *Introduction to Differential Geometry of Twistor Spaces*, Proc.of the workshop, Cortona, Italy, May 15-19,1995, Cambridge Univ. Press,Symp.Math. **38** (1998) 91-160

[4] S. Kobayashi,K. Nomizu, *Foundations of Differential Geometry,* Intersc.Publ., New York, 1963,1969

[5] N.R. O'Brian, J.H. Rawnsley, *Twistor Spaces*, Ann.Global Anal.Geom. **3** (1985) 29-58

[6] M.S.Narasimhan, C.S. Seshadri, *Stable and Unitary Vector Bundles on a Compact Riemann Surface,* Ann.Math., II. Ser. **82** (1965) 540-567

[7] Séminaire Arthur Besse 1978/79, *Géometrie Riemannienne en dimension* 4, Cedic/Fernand Nathan, Paris,1981

[8] J. Wolf, *Spaces of Constant Curvature* , Univ. of California, Berkley, 1972

Faculty of Mathematisc and Informatics, Sofia University, 5 James Bourchier, Sofia 1164, Bulgaria

E-mail address: `kamenova@fmi.uni-sofia.bg`