MAT544 HW5

Due 11/1 in class. Each problem is worth 10 points

Problem 1.
Given a family $\{A_\alpha\}$ of chain complexes of R-modules, prove that direct sums and direct products commute with homology, i.e., $\oplus_\alpha H_n(A_\alpha) \cong H_n(\oplus_\alpha A_\alpha)$ and $\Pi_\alpha H_n(A_\alpha) \cong H_n(\Pi_\alpha A_\alpha)$ for every n.

Problem 2. Let C be a chain complex of R-modules with boundaries B_n and cycles Z_n in C_n. Remember that C is called *split* if there are maps $s_n : C_n \to C_{n+1}$ such that $d_n \circ s_{n-1} \circ d_n = d_n$. Show that C is split if and only if there are R-module decompositions $C_n \cong Z_n \oplus B'_n$ and $Z_n = B_n \oplus H'_n$. Show that C is split exact if and only if in addition to the above $H'_n = 0$.

Problem 3. Let C and D be chain complexes of R-modules. Recall that a chain map $f : C \to D$ is *null homotopic* if there are maps $s_n : C_n \to D_{n+1}$ such that $f_n = s_{n-1}d_n + d'_{n+1}s_n$. Show that C is a split exact chain complex if and only if the identity map on C is null homotopic.

Problem 4. Let $F : \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. Assume $\mathcal{A}$ has enough projectives, and consider the left derived functors $L_iF, i \geq 0$. If $U : \mathcal{B} \to \mathcal{C}$ is an exact functor, show that $U(L_iF) \cong L_i(UF)$.