MAT544 HW3

Due 10/4 in class. Each problem is worth 10 points

Problem 1. Let A be a UFD. Prove that A is normal. In other words, let $K = \mathrm{Frac}(A)$ and $f \in A[X]$ be a monic polynomial with a root $\alpha \in K$, then in fact $\alpha \in A$. [Hint: write $\alpha = \frac{p}{q}$, where p and q have no common factors, and consider the equation $f(\frac{p}{q}) = 0$.]

Problem 2. Let k be an infinite field, $A = k[y_1, \cdots, y_n]$ is a finitely generated k-algebra and $0 \neq F \in k[Y_1, \cdots, Y_n]$ is such that $F(y_1, \cdots, y_n) = 0$. Then there exist $y_1^*, \cdots, y_{n-1}^* \in A$ such that y_n is integral over $A^* = A[y_1^*, \cdots, y_{n-1}^*]$ and $A = A^*[y_n]$. [Hint. Set $y_i^* = y_i - \alpha_i y_n$ for some $\alpha_1, \cdots, \alpha_{n-1} \in k$ to be chosen later. Define the polynomial G by the relation $G(y_1^*, \cdots, y_{n-1}^*, y_n) = F(y_1, \cdots, y_n) = 0$. Then, if F has degree d, the coefficient of y_n^d in G is $F_d(\alpha_1, \cdots, \alpha_{n-1}, 1)$, where F_d is the homogeneous piece of F of degree d. Prove that for an infinite field k and any homogeneous $F_d \neq 0$, there are elements $\alpha_1, \cdots, \alpha_{n-1} \in k$ such that $F_d(\alpha_1, \cdots, \alpha_{n-1}, 1) \neq 0$. Then, for this choice, the relation $G(y_1^*, \cdots, y_{n-1}^*, y_n) = 0$ is an integral dependence relation for y_n over $k[y_1^*, \cdots, y_{n-1}^*]$.]

Problem 3. Show that:

(a) If X is an affine variety, then $X = V(I(X))$.

(b) If $J \subset k[x_1, \ldots, x_n]$ is any ideal, then $V(J) = V(\sqrt{J})$ and $\sqrt{J} \subset I(V(J))$.

Problem 4.

(a) Show that the set $X = \{(t, t^2, t^3) \in \mathbb{A}^3 | t \in k\}$ is closed in $\mathbb{A}^3$ and find $I(X)$.

(b) Same for the subset $Y = \{(t^3, t^4, t^5) \in \mathbb{A}^3 | t \in k\}$ of $\mathbb{A}^3$.

(c) Show that $I(Y)$ can't be generated by less than three polynomials.
Hint: Is $I(Y)$ a graded ideal ?