SHINTANI ZETA FUNCTION ASSOCIATED TO PAIRS OF QUATERNARY ALTERNATING 2-FORMS

JUN WEN

Abstract. We investigate the Shintani zeta functions associated to the prehomogeneous spaces, the example under consideration is the set of pairs of quaternary alternating 2-forms. We show that there are three relative invariants under a certain parabolic group action, and they completely determine the integer orbits. We compute the local density of the stabilizer group and use reduction theory to count the integer orbits with given relative invariants. The associated Shintani zeta function coincides with the $A_3$-Weyl group multiple Dirichlet series with certain generating function for the coefficients that are powers of a single prime $p$.

1. Introduction

1.1. Prehomogeneous vector space and its relative invariants. This is a second paper in a series of exploring the relation between Weyl group multiple Dirichlet series (WMDS) and the Shintani zeta associated with prehomogeneous vector space (PHVS). We continue to identify examples of multiple Dirichlet series which come from Bhargava’s paper [Bha04]. In the paper just mentioned, the author generalizes the Gauss’s law of composition of binary quadratic forms to the composition of projective integer orbits in certain PHVS, and shows that they are equipped with the structure of finitely abelian groups. We note that the PHVS over algebraically closed field are classified by Sato-Kimura in [SK77]. For example, the $D_4$ case refers to the $2 \times 2 \times 2$ integer cubes equipped with the natural group action of $\text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})$. It has a single relative invariant which is the discriminant. One of the main results of Bhargava is to establish the isomorphism of the group law of composition of projective $2 \times 2 \times 2$ integer cubes with the group $\text{CL}^+(S) \times \text{CL}^+(S)$, i.e., the square products of narrow class group of the quadratic order.

In our previous paper, we consider the set of $2 \times 2 \times 2$ integer cubes equipped with a suitable parabolic group action and show that the associated Shintani zeta function coincides with the $A_3$-Weyl Group multiple Dirichlet series. This paper will focuses on the $D_5$ case, which is the set of pairs of integer quaternary alternating 2-forms equipped with the natural $\text{SL}_2(\mathbb{Z}) \times \text{SL}_4(\mathbb{Z})$ action. It also has a single relative invariant which is called the discriminant. Instead we consider the parabolic subgroup action of $\text{SL}_2(\mathbb{Z}) \times P_4(\mathbb{Z})$ on the certain subset $V_2$. It appears that the pair $(\text{GL}_2(\mathbb{C}), P_4(\mathbb{C}))$ is a PHVS and admits three relative invariants. However, the stabilizer group of each semi-stable integer orbits is not finite, which is different from our previous $D_4$ case. We need to consider the local density, which is the volume of the fundamental domain in stabilizer group under suitable measure. We note that

Date: August 5, 2013.
the Shintani zeta function with coefficients the local density was first considered in [Shi75], where the mean value estimate was obtained, strengthening the earlier result of Siegel [Sie44]. We also note that in [WY92] the mean value of the volume of stabilizer group is expected, in particular for D5 case it is predicted that the mean value theorem gives rises to the average density of Res_{z=1} \zeta_{k'}(z)\zeta_{k'}(2) for quadratic extensions k' of k.

We begin with some definitions. Let \( G_C \) be a connected complex Lie group, usually we assume that \( G_C \) is the complexification of a real Lie group \( G_R \). A PHVS \((G_C, V_C)\) is a complex finite dimensional vector space \( V_C \) together with a holomorphic representation of \( G_C \) such that \( G_C \) has an open orbit in \( V_C \). One of the important properties is that if \( V_C \) is a PHVS for \( G_C \) then there is just one open orbit, and that orbit is dense (see [Kna02, Chapter X]). Let \( P \) be a complex polynomial function on \( V_C \). We call it a relative invariant polynomial if \( P(gv) = \chi(g)P(v) \) for some rational character \( \chi \) of \( G_C \) and all \( g \in G_C \), \( v \in V_C \). We say the PHVS has \( n \) relative invariants if \( n \) algebraically independent relative invariant polynomials generate the invariant ring. We define the set of semi-stable points \( V_C^{ss} \) to be the subset on which no relative invariant polynomial vanishes.

Let \( \mathbb{Z}^2 \times \wedge^2 \mathbb{Z}^4 \) be the set of integer quaternary alternating 2-forms. Denote \( V_\mathbb{Z} \) to be the subset as follows

\[
\left\{ \begin{pmatrix} 0 & r_1 & a & b \\ -r_1 & 0 & c & d \\ -a & -c & 0 & l_1 \\ -b & -d & -l_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_2 & e & f \\ -r_2 & 0 & g & h \\ -e & -g & 0 & l_2 \\ -f & -h & -l_2 & 0 \end{pmatrix} \right\} \mid r_1 = r_2 = 0 \}.
\]

We consider the parabolic subgroup \( G_R = GL_2^+(\mathbb{R}) \times P_4^+(\mathbb{R}) \) of \( GL_2(\mathbb{R}) \times GL_4(\mathbb{R}) \) which is given by

\[
\left\{ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \begin{pmatrix} t_{2,1} & 0 & 0 & 0 \\ u_2 & t_{2,2} & 0 & 0 \\ s & t & t_{3,1} & 0 \\ u & v & u_3 & t_{3,2} \end{pmatrix} \right\} \mid g_{11}g_{22} - g_{12}g_{21} > 0; t_{i,1}t_{i,2} > 0 \}.
\]

Let \( G_\mathbb{Z} = G_R \cap \text{SL}_2(\mathbb{Z}) \times \text{SL}_4(\mathbb{Z}) \). Then we can show that \((G_C, V_C)\) is a prehomogeneous vector space with three relative invariants. Denote by \( A(F) \) to be the embedded \( 2 \times 2 \times 2 \) cube

\[
\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right),
\]

then the relative invariants are \( \text{Disc}(A(F)); m(A(F)) = ag - ce; n(A(F)) = af - be \). For any \( x \in V_\mathbb{Z}^{ss} \) and any bounded domain \( U_x \) such that \( x \in U_x \subseteq \tilde{U}_x \subseteq V_\mathbb{R}^{ss} \). Let \( W_x = \{ g \in G_\mathbb{R} : g \cdot x \in U_x \} \) and let \((W_x)_0\) ne a fundamental domain of \( W_x \) with respect to \( G_\mathbb{R} \). Then the ratio

\[
\mu(x) = \int_{(W_x)_0} dg' / (16\pi) \int_{U_x} |\text{Disc}(x)|^{-1} |m(x)|^{-1} |n(x)|^{-1} dx,
\]

where \( dx \) is the standard Euclidean measure on \( V_\mathbb{R} \), \( \tilde{T} = \ker(G_\mathbb{R} \rightarrow \text{GL}(V_\mathbb{R})) \) and \( dg' \) is the invariant measure on the quotient group, does not depend on the choice of \( U_x \). The Shintani zeta function is defined to be

\[
Z_{\text{Shintani}}(s_1, s_2, w) = \sum_{F \in V_\mathbb{Z}^{ss}/G_\mathbb{Z}} \frac{\mu(x)}{|\text{Disc}(A(F))|^w |m(A(F))|^{s_1} |n(A(F))|^{s_2}}.
\]
Our main result is the following:

**Theorem 1.1.** Take the series with sum over odd discriminants, then the Shintani zeta function \( Z_{\text{Shintani}}(s_1, s_2, w) \) can be written as

\[
Z_{\text{Shintani}}(s_1, s_2, w) = 2^{-1}(1 - 2^{-s_1})(2 - 2^{-s_1})(1 - 2^{-s_2})(2 - 2^{-s_2})\zeta(s_1)\zeta(s_2) \\
\cdot Z_{\text{WMD}}^{(1)}(s_1, s_2, w) \\
+ 2^{-1}(1 - 2^{-s_1})(1 + 2^{-s_1})(1 - 2^{-s_2})(1 + 2^{-s_2})\zeta(s_1)\zeta(s_2) \\
\cdot Z_{\text{WMD}}^{(2)}(s_1, s_2, w),
\]

where

\[
Z_{\text{WMD}}^{(1)}(s_1, s_2, w) = \sum_{D \equiv 1 \pmod{8}} \sum_{m, n \geq 1} \frac{\chi_D(m)\chi_D(n)}{m^{s_1}n^{s_2}D^w} b(D, m, n),
\]

\[
Z_{\text{WMD}}^{(2)}(s_1, s_2, w) = \sum_{D \equiv 5 \pmod{8}} \sum_{m, n \geq 1} \frac{\chi_D(m)\chi_D(n)}{m^{s_1}n^{s_2}D^w} b(D, m, n).
\]

The generating function of coefficients \( b(p^k, p^l, p^t) \) satisfies

\[
H(x, y, x) = f_{A3}(x, y, z) + \frac{(1 - p^{-2})}{4} \left( f_{A3}(x, y, z) \frac{pxy^2z}{1 - pxy^2z} - f_{A3}(-x, y, z) \frac{pxy^2z}{1 + pxy^2z} \right) \\
+ \frac{p(1 - p^{-2})}{4} \left( f_{A3}(x, y, z) \frac{pxy^2z}{1 - pxy^2z} - f_{A3}(-x, y, z) \frac{pxy^2z}{1 + pxy^2z} \right).
\]

The organization of this paper is as follows. In section 2, we review the Shintani zeta function associated to the \( 2 \times 2 \times 2 \) integer cubes. In section 3, using reduction theory, we will count the integer orbits of pairs of quaternary alternating 2-forms with given relative invariants. Finally we will relate the Shintani zeta function to a \( A_3 \)-WMDS. The idea is to compute the generating function of \( b(p^k, p^l, p^t) \) with the single prime \( p \).

## 2. Shintani zeta function associated to the Bhargava integer cubes

In this section we briefly recall the main results in our previous paper. Let \( V_{\text{cube}}^2 \) be the space of \( 2 \times 2 \times 2 \) integer matrices. For the element in \( V_{\text{cube}}^2 \), it is denoted by

\[
A = (a_{ijk}) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} e & f \\ g & h \end{array} \right).
\]

There are three pairs of opposite sides denoted by

\[
(M_1, N_1) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} e & f \\ g & h \end{array} \right),
\]

\[
(M_2, N_2) = \left( \begin{array}{cc} a & c \\ e & g \end{array} \right) \left( \begin{array}{cc} b & d \\ f & h \end{array} \right),
\]

\[
(M_3, N_3) = \left( \begin{array}{cc} a & c \\ b & f \end{array} \right) \left( \begin{array}{cc} c & g \\ d & h \end{array} \right).
\]
There is a natural group $GL_2(\mathbb{Z}) \times GL_2(\mathbb{Z}) \times GL_2(\mathbb{Z})$ action which is given as follows: for $g = (g_1, g_2, g_3)$ with $g_i = \begin{pmatrix} g_{i1} & g_{i2} \\ g_{i3} & g_{i4} \end{pmatrix}$, it acts on the $A$ by

$$g \cdot A = \begin{pmatrix} g_{i1} & g_{i2} \\ g_{i3} & g_{i4} \end{pmatrix} \cdot \begin{pmatrix} M_i \\ N_i \end{pmatrix}.$$  

We consider the parabolic subgroup $G_2^{\text{cube}} = \text{SL}_2(\mathbb{Z}) \times B_2(\mathbb{Z}) \times B_2(\mathbb{Z})$, where $B_2(\mathbb{Z}) = B_2^+(\mathbb{R}) \cap \text{SL}_2(\mathbb{Z})$ denotes the lower triangular subgroup with determinant 1. We are interested in the binary quadratic forms associated to the three opposite sides, they are

$$Q_{1,A}(u, v) = (ad - be)u^2 + (-ah + bg + cf - de)uv + (eh - fg)v^2,$$

$$Q_{2,A}(u, v) = (ag - ce)u^2 + (-ah - bg + cf + de)uv + (bh - df)v^2,$$

$$Q_{3,A}(u, v) = (af - be)u^2 + (-ah + bg - cf + de)uv + (ch - dg)v^2.$$  

There are three relative invariants under the given parabolic group $G_2^{\text{cube}}$ action, they are

$$\text{Disc}(A) = \text{Disc}(Q_{i,A}),$$

$$m(A) = ag - ce,$$

$$n(A) = af - be.$$  

We proved that $(G_2^{\text{cube}}, V_2^{\text{cube}})$ is a prehomogeneous vector space and further showed that the shintani zeta function can be written as

$$Z_{\text{shintani}}(s_1, s_2, w) = 4 \sum_{D > 0} \frac{1}{|D|^w} \sum_{m, n > 0} \frac{B(D, m, n)}{m^{s_1} n^{s_2}} + 4 \sum_{D < 0} \frac{1}{|D|^w} \sum_{m, n > 0} \frac{B(D, m, n)}{m^{s_1} n^{s_2}},$$

where $B(D, m, n)$ is the counting function:

$$B(D, m, n) = \# \{ A \in V_2^{\text{cube}} / |\text{Disc}(A) = D, m(A) = m, n(A) = n \}.$$  

Set

$$\xi_1^{\text{cube}} = 4 \sum_{D > 0} \frac{1}{|D|^w} \sum_{m, n > 0} \frac{B(D, m, n)}{m^{s_1} n^{s_2}},$$

$$\xi_2^{\text{cube}} = 4 \sum_{D < 0} \frac{1}{|D|^w} \sum_{m, n > 0} \frac{B(D, m, n)}{m^{s_1} n^{s_2}}.$$  

are the zeta functions taking the sum over positive or negative discriminants respectively. Note that $G_2^{\text{cube}}$ acts on $V_2^{\text{cube}}$ as the linear representation, and we have

$$\text{Disc}(g \cdot A) = \chi_0(g)\text{Disc}(A), \text{ where } \chi_0(g) = \det(g)^2,$$

$$m(g \cdot A) = \chi_1(g)m(A), \text{ where } \chi_1(g) = \det(g_1) t_{2,1}^2 \det(g_3),$$

$$n(g \cdot A) = \chi_2(g)n(A), \text{ where } \chi_2(g) = \det(g_1) \det(g_2) t_{3,1}^2.$$  

for $g = (g_1, g_2, g_3) = \begin{pmatrix} g_{11} & t_{2,1} & 0 \\ u_2 & t_{2,2} & 0 \\ u_3 & t_{3,2} & t_{3,1} \end{pmatrix}$.  

Let $G_2^R$ be the real group $GL_2^+(\mathbb{R}) \times B_2^+(\mathbb{R}) \times B_2^+(\mathbb{R})$ such that for $g = (g_1, g_2, g_3) \in G_2^R$, it satisfies $\det(g_2), \det(g_3) > 0$. Further, let $dg$ denote the
right invariant measure on $G^{\text{cube}}_\mathbb{R}$ defined by

$$dg = \left( (\det g_1)^{-2} \prod_{1 \leq i,j \leq 2} dg_{1,ij} \right) \cdot \left( \prod_{i=2,3} t_{i,1}^{-2} t_{i,2}^{-1} dt_{i,1} dt_{i,2} du_i \right)$$

where $g_i = \begin{pmatrix} t_{i,1} & 0 \\ u_i & t_{i,2} \end{pmatrix}$ for $i = 2, 3$. Set

$$V^{\text{cube}}_\mathbb{R} = \{ A \in V^{\text{cube}}_\mathbb{R}; \text{Disc}(A) m(A) n(A) \neq 0 \}$$

$$V_1 = \{ A \in V^{\text{cube}}_\mathbb{R}; \text{Disc}(A) > 0 \}, V_2 = \{ A \in V^{\text{cube}}_\mathbb{R}; \text{Disc}(A) < 0 \}.$$  

It is easy to see that $V^{\text{cube}}_\mathbb{R} = (\bigcup_{j=1}^4 V_{1j}) \cup (\bigcup_{j=1}^4 V_{2j})$, where $V_1 = \bigcup_{j=1}^4 V_{1j}$ and $V_2 = \bigcup_{j=1}^4 V_{2j}$ are the disjoint unions according to the cases $(m(A) > 0, n(A) > 0), (m(A) > 0, n(A) < 0), (m(A) < 0, n(A) > 0), (m(A) < 0, n(A) < 0)$.

**Proposition 2.1.**

1. The group $G^{\text{cube}}_\mathbb{R}$ acts on $V_{1j}$ and $V_{2j}$ are all transitive;

2. The stabilizer subgroup of $G^{\text{cube}}_\mathbb{R}$ is isomorphic to $\text{GL}_1(\mathbb{R}) \times \text{GL}_1(\mathbb{R})$.

**Proof.** For any real cube $A \in V^{\text{cube}}_\mathbb{R}$, it is easy to see $G^{\text{cube}}_\mathbb{R}$-equivalent to the element

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ g & 0 \end{pmatrix}.$$  

Then we can further choose the group action of

$$\begin{pmatrix} \sqrt{sg} & 0 \\ 0 & \pm g^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{sg} \end{pmatrix}, \begin{pmatrix} \sqrt{sg} & 0 \\ 0 & 1 \end{pmatrix},$$

where the ‘−’ sign is chosen if $g < 0$. Therefore if $D > 0$, then

$$V_1 = G^{\text{cube}}_\mathbb{R} \cdot A_1, A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

while if $D < 0$, then

$$V_2 = G^{\text{cube}}_\mathbb{R} \cdot A_2, A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

We denote by $G^{\text{cube}}_A$ the isotropy subgroup of $A_1$ in $G^{\text{cube}}_\mathbb{R}$. It is easy to see that if $g = (g_1, g_2, g_3) \in G^{\text{cube}}_A$, then $g_1$ is upper triangular. As $g$ preserves two quadratic forms $Q_{2, A}(u, v)$ and $Q_{3, A}(u, v)$, it follows that $g_2$ and $g_3$ are both diagonal matrices, which in turn requires that $g_1$ is also diagonal. Write

$$g = (g_1, g_2, g_3) = \begin{pmatrix} \lambda_{1,1} & 0 \\ 0 & \lambda_{1,2} \end{pmatrix}, \begin{pmatrix} \lambda_{2,1} & 0 \\ 0 & \lambda_{2,2} \end{pmatrix}, \begin{pmatrix} \lambda_{3,1} & 0 \\ 0 & \lambda_{3,2} \end{pmatrix},$$

then as $g$ preserves $A$, we have

$$\lambda_{3,1} \lambda_{2,1} \lambda_{1,1} = \lambda_{3,1} \lambda_{2,2} \lambda_{1,2} = \lambda_{3,2} \lambda_{1,1} \lambda_{2,2} = \lambda_{3,2} \lambda_{2,1} \lambda_{1,2} = 1.$$

It follows that

$$g = \begin{pmatrix} \lambda^{-1} \mu^{-1} & 0 \\ 0 & \lambda^{-1} \mu^{-1} \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}.$$  

This finishes the proof. □
Let $\tilde{T}_x = \ker(G^x \to \GL_{\mathbb{R}}(V^x))$. Then it is easy to see that

$\tilde{T}_x = \{(\lambda^{-1} \mu^{-1} I_2, \lambda I_2, \mu I_2) | \lambda, \mu \in \GL_1(\mathbb{R})\}$. 

We fix the right invariant measure $dg'$ on $G^x / \tilde{T}_x$ so that

$$dg = dg' \cdot d^x \lambda \cdot d^x \mu.$$ 

Let $V^x = \{ A \in V_{\mathbb{R}}^x ; \text{Disc}(A) m(A)n(A) \neq 0 \}$. Define the global zeta function $Z^x(f, s_1, s_2, w)$, where $f \in S(V_{\mathbb{R}})$, as follows:

$$Z^x(f, s_1, s_2, w) = \int_{G^x / \tilde{T}_x} \chi_1(g)^{s_1} \chi_2(g)^{s_2} \chi_0(g)^w \sum_{A \in V^x} f(g \cdot A) dg'.$$

With the chosen invariant measure, we have

**Proposition 2.2.** Take an $f \in S(V^x_{\mathbb{R}})$, then we have

$$\int_{G^x / \tilde{T}_x} f(g \cdot x) dg' = \frac{1}{16\pi} \int_{V_{ij}} |m(x)|^{-1} |n(x)|^{-1} |\text{Disc}|^{-1} (x) f(x) dx$$

for any $x \in V_{ij}$. Here on each $V_{ij}$, we choose the standard Euclidean measure.

**Proof.** First we observe that

$$\tilde{T}_x = G^x$$

for any $x \in V_i$. And it is easy to see that

$$G^x / \tilde{T}_x \cong \text{SL}_2(\mathbb{R}) \times \begin{pmatrix} t_{2,1} & 0 \\ t_{2,1} \cdot u & 1 \end{pmatrix} | t_{2,1} \in \mathbb{R}^* \times \begin{pmatrix} t_{3,1} & 0 \\ t_{3,2} \cdot u & t_{3,2} \end{pmatrix} | t_{3,1}, t_{3,2} \in \mathbb{R}^*.$$ 

As for the given right invariant measure $dg$ on $G^x_{\mathbb{R}}$, under the left transformation we have

$$d(h \cdot g) = (t_{2,1}^{-1} t_{2,2}) (t_{3,1}^{-1} t_{3,2}) dg$$

for $h = \begin{pmatrix} h_1 & t_{2,1} \\ u & t_{2,2} \end{pmatrix}, \begin{pmatrix} t_{3,1} & 0 \\ v & t_{3,2} \end{pmatrix}$. On the other hand, it is straightforward to check that the volume form

$$m(x)^{-1} n(x)^{-1} \text{Disc}^{-1} (x) dx$$

admits the same rule of transformation under the $G^x_{\mathbb{R}}$-action. So we conclude that

$$dg' = c \cdot m(x)^{-1} n(x)^{-1} \text{Disc}^{-1} (x) dx$$

for a $c \in \mathbb{R}^*$. As $V_i = G^x_{\mathbb{R}} \cdot A_i$, it sufficient to prove the equality for $A_1$ and $A_2$ respectively. Now we choose the coordinates for $G^x_{\mathbb{R}}$ as follows: Put

$$k_{\theta} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} (\theta \in \mathbb{R}),$$

$$a_t = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} (t \in \mathbb{R}^*),$$

$$n(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} (x \in \mathbb{R}),$$

$$\nu(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} (y \in \mathbb{R}).$$
Then under these coordinates, the measure $dg'$ can be written as
\[ dg' = \frac{1}{2\pi} dx \cdot dy \cdot d^x t \cdot (t_{x,1} t_{y,1}) \cdot du \cdot (t_{x,1} t_{y,1}^{-1}) \cdot dv. \]
By computer checking, the Jacobians under the mapping $g \rightarrow g \cdot A_i$ are both equal to\[ \frac{8 t_{x,1}^3 t_{y,1}^3}{t}. \]
It follows that\[ |c| = \frac{1}{8}, \]
which finishes the proof. \[ \square \]

Define\[ \Phi^\text{cube}_i(f, s_1, s_2, w) = \int_{V_i} f(x) |m(x)|^{s_1} |n(x)|^{s_2} |\text{Disc}(x)|^w dx. \]for $f \in S(V^\text{cube}_R)$, $i = 1, 2$. Now we are ready to show the following results.

**Proposition 2.3.** The global zeta function $Z^\text{cube}(f, s_1, s_2, w)$ is absolutely convergent for $\Re s_1 > 1$, $\Re w > 1$. Moreover, the following equality hold
\[ Z^\text{cube}(f, s_1, s_2, w) = \frac{1}{16\pi} \sum_{i=1}^{2} \xi^\text{cube}_i(s_1, s_2, w) \Phi^\text{cube}_i(f, s_1 - 1, s_2 - 1, w - 1). \]

**Proof.** We have the equality
\[ \int_{G^\text{cube}_R/G^\text{cube}_Z} \chi_1(g) g^{s_1} \chi_2(g) g^{s_2} \chi_0(g)^w \sum_{A \in V^\text{cube}_R} f(g \cdot A) dg' \]
\[ = \sum_{A \in G^\text{cube}_Z \backslash V^\text{cube}_Z} \int_{G^\text{cube}_R/G^\text{cube}_Z} \chi_1(g) g^{s_1} \chi_2(g) g^{s_2} \chi_0(g)^w \sum_{\gamma \in G^\text{cube}_Z} f(g \cdot \gamma \cdot A) dg' \]
\[ = \sum_{A \in G^\text{cube}_Z \backslash V^\text{cube}_Z} \int_{G^\text{cube}_R/G^\text{cube}_Z} \chi_1(g) g^{s_1} \chi_2(g) g^{s_2} \chi_0(g)^w f(g \cdot A) dg' \]
\[ = \sum_{A \in G^\text{cube}_Z \backslash V^\text{cube}_Z} \frac{1}{|m(A)|^{s_1} |n(A)|^{s_2} |\text{Disc}(A)|^w} \]
\[ \cdot \int_{G^\text{cube}_R/G^\text{cube}_Z} |m(g \cdot A)|^{s_1} |n(g \cdot A)|^{s_2} |\text{Disc}(g \cdot A)|^w f(g \cdot A) dg' \]
\[ = \frac{1}{16\pi} \sum_{i=1}^{2} \xi^\text{cube}_i(s_1, s_2, w) \Phi^\text{cube}_i(f, s_1 - 1, s_2 - 1, w - 1). \]
\[ \square \]

3. **Shintani zeta function of pairs of quaternary alternating 2-forms.** Let $\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4$ be the free $\mathbb{Z}$-module of pairs of quaternary alternating 2-forms. Bharagava [Bha04] shows the $\text{SL}_2(\mathbb{Z}) \times \text{SL}_4(\mathbb{Z})$-orbits are one to one correspondent to the isomorphism classes of pairs $(R, (I, J))$, where $R$ is a nondegenerate oriented quadratic ring and $(I, J)$ is an equivalence class of balanced pairs of the oriented ideals of $R$ having ranks 1 and 2 respectively.
We recall the construction of Bhargava. Given \( F = (F_1, F_2) \in \mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4 \), with \( F_1 = \{a_{jk}^{(1)}\} \) and \( F_2 = \{a_{jk}^{(2)}\} \), has the form
\[
\begin{pmatrix}
0 & r_1 & a & b \\
r_1 & 0 & c & d \\
-a & -c & 0 & l_1 \\
-b & -d & -l_1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & r_2 & e & f \\
r_2 & 0 & g & h \\
e & -g & 0 & l_2 \\
-f & -h & -l_2 & 0
\end{pmatrix}
\]
Let \( R \) be the unique quadratic ring determined by the discriminant of \( D = \text{Disc}(F) \), and \((1, \tau)\) be its \( \mathbb{Z} \) basis, where
\[
\tau^2 = \frac{D}{4} \text{ or } \tau^2 = \tau + \frac{D-1}{4}
\]
according to whether \( D \equiv 0 \) or 1 (mod 4). By Bhargava [Bha04], we know that \( F \) corresponds to the pair \((R, (I, J))\), where \( R \)-module \( I \) is the rank 2 \( \mathbb{Z} \)-module in \( K = R \otimes \mathbb{Q} \), \( R \)-module \( J \) is the rank 4 \( \mathbb{Z} \)-module in \( K^2 \). By the formula of Bhargava
\[
\alpha_1 : \alpha_2 = c_{jk}^{(1)} + a_{jk}^{(1)} \tau : c_{jk}^{(2)} + a_{jk}^{(2)} \tau,
\]
which is independent of the index \( \{j, k\} \), and \( \{c_{jk}^{(i)}\} \) is determined by
\[
c_{jk}^{(i)} = (i - i') [a_{jk}^{(i)} \text{Pfaff}(F_i) - \frac{1}{2} a_{jk}^{(i)} \{ \text{Pfaff}(F_1 + F_2) - \text{Pfaff}(F_1) - \text{Pfaff}(F_2) \}] - \frac{1}{2} a_{jk}^{(i)} \epsilon,
\]
where \( \{i, i'\} = \{1, 2\} \) and \( \epsilon = 0 \) or 1 according to whether \( D \equiv 0 \) or 1 (mod 4). It may simply set \( \alpha_i = c_{12}^{(i)} + a_{12}^{(i)} \tau \), then the values of \( \det(\beta_i, \beta_j) \) are completely determined by
\[
\alpha_i \det(\beta_i, \beta_k) = c_{jk}^{(i)} + a_{jk}^{(i)} \tau.
\]
As these values satisfy the Plücker relations
\[
\alpha_i \det(\beta_k, \beta_m) \cdot \alpha_j \det(\beta_l, \beta_n) = \alpha_i \det(\beta_k, \beta_l) \cdot \alpha_j \det(\beta_m, \beta_n)
+ \alpha_i \det(\beta_k, \beta_n) \cdot \alpha_j \det(\beta_l, \beta_m),
\]
the values of \( \beta_1, \beta_2, \beta_3, \beta_4 \) are uniquely determined up to a transform in \( \text{SL}_4(K) \).

Conversely, let \((1, \tau)\) be a \( \mathbb{Z} \)-basis for \( R \), and suppose \((\alpha_1, \alpha_2)\) and \((\beta_1, \beta_2, \beta_3, \beta_4)\) are oriented \( \mathbb{Z} \)-basis for \( I \) and \( J \) respectively. The associated pair of quaternary alternating 2-forms \( \{a_{jk}^{(i)}\} \) is given by
\[
\alpha_i \det(\beta_j, \beta_k) = c_{jk}^{(i)} + a_{jk}^{(i)} \tau.
\]
If we define the action of \( g \in \text{SL}_4(\mathbb{Z}) \) on the rank 4 \( \mathbb{Z} \)-module \( J \) to be
\[
(\beta_1', \beta_2', \beta_3', \beta_4')^t = g \cdot (\beta_1, \beta_2, \beta_3, \beta_4)^t,
\]
then it is easy to check that
\[
\{a_{jk}^{(i)}\}' = g \cdot \{a_{jk}^{(i)}\} \cdot g^t.
\]
Explicitly the action of \( G_{\mathbb{Z}} = \text{SL}_2(\mathbb{Z}) \times \text{SL}_4(\mathbb{Z}) \) is given as follows: an element \((g_1, g) \in G_{\mathbb{Z}} \) acts by sending the pair \((F_1, F_2)\) to:
\[
(g_1, g) \cdot (F_1, F_2) = (r \cdot \tilde{g} F_1 \tilde{g} + s \cdot \tilde{g} F_2 \tilde{g}, t \cdot \tilde{g} F_1 \tilde{g} + u \cdot \tilde{g} F_2 \tilde{g}),
\]
where \( g_1 = \left( \begin{array}{cc} r & s \\ t & u \end{array} \right) \).
There is a natural $\mathbb{Z}$-linear mapping
$$\mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2 \rightarrow \mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4$$
taking $2 \times 2 \times 2$ integer cubes to the pairs of quaternary alternating 2-forms. Explicitly, it is given by
$$
\begin{pmatrix}
(a & b \\
(c & d)
\end{pmatrix},
\begin{pmatrix}
e & f \\
g & h
\end{pmatrix}) \mapsto
\begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & c & d
\end{pmatrix},
\begin{pmatrix}
0 & 0 & e & f \\
0 & 0 & g & h
\end{pmatrix},
\begin{pmatrix}
-a & -c & 0 & 0 \\
-b & -d & 0 & 0
\end{pmatrix},
\begin{pmatrix}
-e & -g & 0 & 0 \\
-f & -h & 0 & 0
\end{pmatrix}.
$$

It was shown in [Bha04] that the mapping is surjective on the level of equivalence classes.

For each $F = (F_1, F_2)$, we can associate to it a binary quadratic form $Q$ given by:
$$Q_F(u, v) = \text{Pfaff}(F_1 u - F_2 v) = \sqrt{\text{Det}(F_1 u - F_2 v)},$$
where the sign of the Pfaff is chosen to be
$$\text{Pfaff}\left(\begin{pmatrix} -I \\ F \end{pmatrix}\right) = 1,$$
or alternatively,
$$\text{Pfaff}\begin{pmatrix}
0 & r & a & b \\
-r & 0 & c & d \\
-a & -c & 0 & l \\
-b & -d & -l & 0
\end{pmatrix} = ad - bc - rl.$$

We consider the parabolic group $G_{\mathbb{Z}} = \text{SL}_2(\mathbb{Z}) \times P_4(\mathbb{Z})$, where $P_4(\mathbb{Z})$ is the lower triangular minimal parabolic subgroup of $\text{SL}_4(\mathbb{Z})$ with upper left $2 \times 2$ matrix determinant 1. Let $V_{\mathbb{Z}}$ be the subset of $\mathbb{Z}^2 \otimes \wedge^2 \mathbb{Z}^4$ with $r_1 = r_2 = 0$. There are three relative invariants, they are
$$\text{Disc} F = \text{Disc}(\text{Pfaff}(F)),
\quad m(F) = |ag - ce|,
\quad n(F) = |af - be|.$$

We can show that the group $G_{\mathbb{Z}}$ acts on $V_{\mathbb{Z}}$ such that

**Proposition 3.1.** The pair $(G_{\mathbb{C}}, V_{\mathbb{C}})$ is a prehomogeneous vector space.

**Proof.** Let $\mathcal{H}$ be the hypersurface in $V_{\mathbb{C}}$ defined by the zero loci of the single equation
$$\text{Disc}(F) \cdot m(F) \cdot n(F) = 0.$$

For any $F \in V_{\mathbb{C}} \setminus \mathcal{H}$, under the group action, it is equivalent to some element with the form
$$
\begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & c & d \\
-a & -c & 0 & l_1 \\
-b & -d & -l_1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & f \\
0 & 0 & g & h \\
-e & -g & 0 & l_2 \\
-f & -h & -l_2 & 0
\end{pmatrix}.$$
with \(a, f, g \neq 0\). Next, with the action of \((1, g) \in G_{\mathbb{C}}\), here

\[
\tilde{g} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
s & t & 1 & 0 \\
u & v & 0 & 1
\end{pmatrix},
\]

it gives

\[
l'_1 = sb + td - ua - vc + l_1, \\
l'_2 = af + th - ue - vg + l_2.
\]

So we can find \(s, t, u, v\) such that \(l'_1 = l'_2 = 0\). As we already shown that \((B'_{2}(\mathbb{C}) \times B'_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}), V_{\text{cube}}')\) is a prehomogeneous vector space. This finishes the proof as \(B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})\) is replaced by the embedding of \(\text{GL}_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times B'_2(\mathbb{C})\) into \(G_{\mathbb{C}}\) and the element is in the image of \(V_{\text{cube}}'\) under the mapping

\[
\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^4.
\]

\[\square\]

3.2. Shintani zeta function of quaternary alternating 2-forms. Set \(V'_{\mathbb{C}} = \{F \in V_{\mathbb{C}} : \text{Disc}(F)m(F)n(F) \neq 0\}\). As \((G_{\mathbb{C}}, V_{\mathbb{C}})\) is a prehomogeneous vector space with three relative invariants, set

\[
\begin{align*}
\chi_0(g) &= \det(g_1)^2 \det(\tilde{g})^2, \\
\chi_1(g) &= \det(g_1) \det(g_3)t_{2,1}^2, \\
\chi_2(g) &= \det(g_1) \det(g_3)t_{3,1}^2,
\end{align*}
\]

where \(g = (g_1, \tilde{g})\) and

\[
\tilde{g} = \begin{pmatrix}
g_2 & 0 \\
t & g_3
\end{pmatrix}
\]

\(g_2, g_3\) are the lower triangular \(2 \times 2\) matrix such that \(g_2 = \begin{pmatrix} t_{2,1} & 0 \\ u_2 & t_{2,2} \end{pmatrix}\) and \(g_3 = \begin{pmatrix} t_{3,1} & 0 \\ u_3 & t_{3,2} \end{pmatrix}\), then we have

\[
\begin{align*}
\text{Disc}(g \cdot F) &= \chi_0(g)\text{Disc}(F), \\
m(g \cdot F) &= \chi_1(g)m(F), \\
n(g \cdot F) &= \chi_1(g)n(F).
\end{align*}
\]

The semi-stable points of \(V_{\mathbb{R}}\) under the action of \(G_{\mathbb{R}}\) is denoted by \(V'_{\mathbb{R}}\),

\[V'_{\mathbb{R}} = \{F \in V_{\mathbb{R}}|\text{Disc}(F)m(F)n(F) \neq 0\}.\]

Let \(G_{\mathbb{R}}\) be the real group \(\text{GL}_2^+(\mathbb{R}) \times P_4^+(\mathbb{R})\), where \(P_4^+(\mathbb{R})\) is the real parabolic subgroup of \(\text{GL}_4^+(\mathbb{R})\) with upper left \(2 \times 2\) triangular matrix positive determinant, \(P_4^+(\mathbb{R}) \cong N \times B_2^+(\mathbb{R}) \times B_2^+(\mathbb{R})\), write \(h \in P_4^+(\mathbb{R})\) as

\[
h = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
s & t & 1 & 0 \\
u & v & 0 & 1
\end{pmatrix} \times \begin{pmatrix}
t_{2,1} & 0 \\
u_2 & t_{2,2}
\end{pmatrix} \times \begin{pmatrix}
t_{3,1} & 0 \\
u_3 & t_{3,2}
\end{pmatrix}.
\]
Furthermore, let $dg$ be the right invariant measure on $G_R$ defined by
\[ dg = dg_1 \cdot dn \cdot (t_{2,1}^{2t_{2,1}^{-1}} dt_{2,1} dt_{2,2} du_2) \cdot (t_{3,1}^{2t_{3,1}^{-1}} dt_{2,1} dt_{2,2} du_3) . \]
Denote $\tilde{T}$ to be $\ker(G_R \to \text{GL}(V_R))$. Then it is easy to see that

**Proposition 3.2.** The subgroup $\tilde{T}$ is isomorphic to the embedding of $\tilde{T}^\text{cube}$ into $G_R$, and therefore
\[ \tilde{T} \cong \text{GL}_1(\mathbb{R}) \times \text{GL}_1(\mathbb{R}) . \]
Therefore
\[ G_R/\tilde{T} \cong N \times G_R^\text{cube}/\tilde{T}^\text{cube} . \]
The right invariant quotient measure $dg'$ on $G_R/\tilde{T}$ is defined to be
\[ dg = d\tilde{t} \cdot dg' . \]
For any $x \in V_R$, denoted by $G_x$ the isotropy subgroup of $G_R$. Then we have
\[ G_x \cong N_x \times \tilde{T} , \]
where $N_x$ is the isotropy subgroup inside $N$. Furthermore, we have
\[ G_x/\tilde{T}G_x \cap G_z \cong N_x/I_x , \]
where $I_x = N_x \cap \text{SL}_4(\mathbb{Z})$.

Next we want to define the Haar measure on $G_x/\tilde{T}$. For every $x \in V^s_R$ and any bounded domain $U_x$ such that $x \in U_x \subset \tilde{U}_x \subset V^s_R$. Let $W_x = \{ g \in G_R/\tilde{T} : g \cdot x \in U_x \}$ and let $(W_x)_0$ be a fundamental domain of $W_x$ with respect to $G_x \cap G_z$. Then the ratio
\[ \mu(x) = \int_{(W_x)_0} \frac{dg'}{(16\pi)^3} \int_{U_x} |\text{Disc}(x)|^{-1} |m(x)|^{-1} |n(x)|^{-1} dx , \]
where $dx$ is the standard Euclidean measure on $V_R$, does not depend on the choice of $U_x$. Note that as
\[ V_R \cong V_R^\text{cube} \times \mathbb{R}^2 , \]
the measure $dx$ on $V_R$ can be written as $dx = dx' \times dx''$. Set
\[ \xi_1(s_1, s_2, w) = \sum_{F \in G \setminus V_R^s, \text{Disc}(F) > 0, \text{Disc}(F) \neq 0} \frac{\mu(F)}{|\text{Disc}(F)|^w |m(F)|^{s_1} |n(F)|^{s_2}} , \]
\[ \xi_2(s_1, s_2, w) = \sum_{F \in G \setminus V_R^s, \text{Disc}(F) < 0, \text{Disc}(F) 
eq 0} \frac{\mu(F)}{|\text{Disc}(F)|^w |m(F)|^{s_1} |n(F)|^{s_2}} . \]

Define the global zeta function $Z(f, s_1, s_2, w)$, where $f \in \mathcal{S}(V_R^s)$, as follows:
\[ Z(f, s_1, s_2, w) = \int_{G_R/\tilde{T}G_x} \chi_1(g)^{s_1} \chi_2(g)^{s_2} \chi_0(g)^w \sum_{F \in V_R^s} f(g \cdot F) dg' . \]
Then we have
Proposition 3.3. Given an $f \in \mathcal{S}(V_2')$, if we set

$$f'(x') = \int_{\mathbb{R}^2} f(x)dx'',$$

then $f' \in \mathcal{S}(V_{\text{cube}}')$. Furthermore, we have

$$Z(f, s_1, s_2, w) = \frac{1}{16\pi} \sum_{i=1}^{2} \xi_i(s_1, s_2, w)\Phi_{i'}^{\text{cube}}(f, s_1 - 1, s_2 - 1, w - 1).$$

Proof.

$$\int_{G_2/\tilde{T}G_2} \chi_1(g)^{s_1} \chi_2(g)^{s_2} \chi_0(g)^w \sum_{F \in V_2'} f(g \cdot F)dg'$$

$$= \sum_{F \in G_2 \setminus V_2'} \int_{G_2/\tilde{T}G_2} \chi_1(g)^{s_1} \chi_2(g)^{s_2} \chi_0(g)^w \sum_{\gamma \in G_2/G_2 \cap G_2} f(g\gamma \cdot F)dg'$$

$$= \sum_{F \in G_2 \setminus V_2'} \int_{G_2/\tilde{T}(G_2 \cap G_2)} \chi_1(g)^{s_1} \chi_2(g)^{s_2} \chi_0(g)^w f(g \cdot F)dg'$$

$$= \frac{1}{16\pi} \int_{V_{\text{cube}}} |m(x')|^{s_1-1}|n(x')|^{s_2-1}|\text{Disc}(x')|^{w-1} f'(x')dx'$$

$$= \frac{1}{16\pi} \sum_{i=1}^{2} \xi_i(s_1, s_2, w)\Phi_{i'}^{\text{cube}}(f, s_1 - 1, s_2 - 1, w - 1).$$

We can further show that

Proposition 3.4. $\mu(x) = 1$ for any $x \in V_2'$.

Proof. Given an $x \in V_2'$ and arbitrary small $\epsilon$, choose the tube domain which is formed by taking the product of intervals of $(x_i - \epsilon, x_i + \epsilon)$, where $x_i$ are the coordinates of $F$.

First we identify the quotient group

$$G_2/\tilde{T} \cong N \times \text{GL}_2^+(\mathbb{R}) \times B_2^0(\mathbb{R}) \times B_2^1(\mathbb{R}),$$

where $B_2^1(\mathbb{R}) = B_2(\mathbb{R}) \cap \text{SL}_2(\mathbb{R})$. Then the measure $dg'$ can be written as

$$dg' = dn \cdot dg''$$

and we have shown that

$$dg'' = \frac{1}{16\pi} dx'.$$
Further, by definition of $\mu(x)$,
\[
\mu(x) = \int_{W_{x'}} dg'' \int_{(V_{g',x'})_0} dn / (16\pi) \int_{U_x'} dx' \cdot \int_{l_1-\epsilon < x_1 < l_1+\epsilon} \int_{l_2-\epsilon < x_2 < l_2+\epsilon} dx_1 dx_2,
\]
where $U_x = U_{x'} \times \{ l_1 - \epsilon < x_1 < l_1 + \epsilon \} \times \{ l_2 - \epsilon < x_2 < l_2 + \epsilon \}$, $W_{x'}$ is the set of $G_{\text{cube}}^x$ such that $W_{x'} \cdot x' \subset U_{x'}$, and for $g' \in W_{x'}$,
\[
g' \cdot x' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}, \begin{pmatrix} e' & f' \end{pmatrix},
\]
accordingly,
\[
V_{g',x'} = \{ |sb' + td' - ua' - cv'| < c; |sf' + th' - ue' - vg'| < \epsilon \}
\]
for $n = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s & t & 1 & 0 \\ u & v & 0 & 1 \end{pmatrix}$. Set
\[
\Gamma_{x'} = \{ n \in N_2 | sb + td - ua - va = 0; sf + th - ue - vg = 0 \},
\]
then
\[
(V_{g',x'})_0 = V_{g',x'/g' \Gamma_{x'}(g')}^{-1}.
\]
Write
\[
(l'_1, l'_2) = (s, t, u, v) \cdot \begin{pmatrix} b & f \\ d & h \\ -a & -e \\ -c & -g \end{pmatrix},
\]
then the transform $g' \cdot x'$ for $g' = (g_1, g_2, g_3)$ gives rise to the change of variables, as $(g_2, g_3) \in B_2^x(\mathbb{R}) \times B_2^x(\mathbb{R})$, the absolute value of Jacobian equals to 1, we conclude that it is sufficient to prove
\[
\int_{(V_{g',x'})_0} dn / \int_{l_1-\epsilon < x_1 < l_1+\epsilon} \int_{l_2-\epsilon < x_2 < l_2+\epsilon} dx_1 dx_2 = \int_{(V_{g',x'})_0} dn / 4\epsilon^2 = 1.
\]
Under the change of variables, we can assume that
\[
x' = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & f \\ g & h \end{pmatrix},
\]
scaling the back side, we further assume $g = 1$. Then the volume
\[
\int_{(V_{g',x'})_0} dn = \int_{t=0}^{t=1} \left( \int_{u=dt-\epsilon}^{u=dt+\epsilon} du \right. \left( \int_{s=0}^{s=1} \left( \int_{v=sf+th-\epsilon}^{v=sf+th+\epsilon} dv \right) ds \right) dt
\]
\[
= 2\epsilon \int_{t=0}^{t=1} \left( \int_{u=dt-\epsilon}^{u=dt+\epsilon} du \right) dt
\]
\[
= 4\epsilon^2.
\]
\[\square\]
3.3. Reduction theory. In our previous paper, we have shown that the Shintani zeta function associated to the $2 \times 2 \times 2$ integer cubes is given explicitly by the following formula

$$Z_{\text{Shintani}}(s_1, s_2, w) = \xi_1^{\text{cube}}(s_1, s_2, w) + \xi_2^{\text{cube}}(s_1, s_2, w)$$

$$= \sum_{D = D_0D_1^2} \frac{1}{|D|^w} \sum_{m,n > 0} \sum_{d|m,d|n} \frac{A(D/d^2, 4m/d) \cdot A(D/d^2, 4n/d)}{m^{s_1}n^{s_2}}$$

where the counting function $A(D, m)$ means the number of solutions to the quadratic congruence $x^2 = D \pmod{m}$.

In this section, we are going to derive the analogous formula for the Shintani zeta function $Z_{\text{Shintani}}(s_1, s_2, w) = \xi_1(s_1, s_2, w) + \xi_2(s_1, s_2, w)$ associated to the pairs of quaternary alternating 2-forms. This reduces to the orbits counting with given relative invariants, as we have shown that density $\mu(F) = 1$ for each orbit. Let set the counting function

$$B(D, m, n) = \sharp\{F = (M, N) \in V_2^4/\sim | \text{Disc}(F) = D, m(F) = m, n(F) = n\}.$$

Proposition 3.5. Suppose that the discriminant of $F$ is square free, then we have

$$B(D, m, n) = \frac{1}{4} A(D, 4m)A(D, 4n).$$

Proof. If $D = \text{Disc}(F)$ is square free, then consider the embedded $2 \times 2 \times 2$ integer cube $A(F)$ denoted by

$$A(F) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$ 

It must satisfies that $\text{g.c.d.}(e, f, g, h) = 1$ which means that

$$sf + th - ue - vg = 1 - l_2,$$

therefore under the group $N_2$-action, we can make $l_2 = 1$, and further transformation can make $l_1 = 0$ then $l_2 = 0$. As the three relative invariants of $F$ are characterized by those of embedded $A(F)$, from

$$B^{\text{cube}}(D, m, n) = \frac{1}{4} A(D, 4m)A(D, 4n),$$

the result follows. \qed

Now we turn to the general case without assuming discriminant square free. First we consider the case when $D = D_0p^2$, where $D_0$ is square free.

Lemma 3.6. Given the nonzero integers $m, n, D = D_0p^2$, where $D_0$ is square free and $p$ is a prime integer, then we have

$$B(D, m, n) = \frac{1}{4} \left( A(D, 4m)A(D, 4n) + p^2A(D_0, 4m/p)A(D_0, 4n/p) \right),$$

where for $p$ not a divisor of $m$ or $n$, then the second term is interpreted as 0.

Proof. Given the solution $(x, y)$ to the congruence equations, we know that there is an integer cube $A$ with the prescribed binary quadratic forms:

$$Q_{2, A} = (ag - ce)u^2 + (-ah - bg + cf + de)uv + (bh - df)v^2,$$

$$Q_{3, A} = (af - be)u^2 + (-ah + bg - cf + de)uv + (ch - dg)v^2.$$
Suppose that \( g.c.d.(e, f, d, g) = 1 \), then from the last proposition we know that the given integer cube \( A \) uniquely determines the orbit of \( F \) with \( A = A(F) \).

Now suppose that \( g.c.d.(e, f, d, g) \neq 1 \), as \( g.c.d.(e, f, d, g)^2|D \) and \( D = D_0p^2 \), it implies that \( g.c.d.(e, f, d, g) = p \) and \( g.c.d.(a, b, c, d) = 1 \). Write \( e = e_0p, f = f_0p, g = g_0p, h = h_0p \), then there are \( p + 1 \) \( G_p^{\text{cube}} \)-orbits of integer cubes. They are

\[
\left( \begin{array}{cc}
ap & bp \\
ep & dp
\end{array} \right), \left( \begin{array}{cc}
e_0 & f_0 \\
g_0 & h_0
\end{array} \right),
\]

and

\[
\left( \begin{array}{cc}
a + j e_0 & b + j f_0 \\
c + j g_0 & d + j h_0
\end{array} \right), \left( \begin{array}{cc}
eg & g \\
h & d
\end{array} \right)
\]

for \( 0 \leq j \leq p - 1 \). Note that for the last \( p \) orbits of integer cubes, each of them has \( p \) ways to embed into an \( F \), they are

\[
\left( \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right), \left( \begin{array}{cc}
a + j e_0 & b + j f_0 \\
c + j g_0 & d + j h_0
\end{array} \right), \left( \begin{array}{cc}
e & g \\
h & d
\end{array} \right)
\]

for \( 0 \leq j, d \leq p - 1 \). Therefore the case \( g.c.d.(e, f, g, h) \neq 1 \) contributes \( p^2 \) orbits of pairs of quaternary alternating 2-forms.

\[\square\]

**Lemma 3.7.** Given the nonzero integers \( m, n, D = D_0p^4 \), where \( D_0 \) is square free and \( p \) is a prime integer, then we have

\[
B(D, m, n) - B^{\text{cube}}(D, m, n) = \frac{1}{4}(p^2 - 1)A(D_0, \frac{4m}{p^2})A(D_0, \frac{4n}{p^2}),
\]

where for \( p \) not a divisor of \( m \) or \( n \) the last term are interpreted as 0.

**Proof.** Consider the embedded integer cube \( A \). Suppose that \( g.c.d.(e, f, g, h) = 1 \), then under the certain group action one can make both \( l_1 = l_2 = 0 \), the orbit of \( F(A) \) is the determined by that of \( A \). This is the contribution of the term \( A(D, 4m)A(D, 4n) \).

Suppose that \( g.c.d.(e, f, g, h) = p \), examine the orbits of integer cubes of

\[
\left( \begin{array}{cc}
a + j e_0 & b + j f_0 \\
c + j g_0 & d + j h_0
\end{array} \right), \left( \begin{array}{cc}
e_0 & f_0 \\
g_0 & h_0p
\end{array} \right)
\]

for \( 0 \leq j \leq p - 1 \). If \( g.c.d.(a + j e_0, b + j f_0, c + j g_0, d + j h_0) = 1 \) for all \( j \), we can make \( l_1 = l_2 = 0 \), therefore the embedded integer cubes determine the orbits of \( F \).

The number of such orbits of integer cubes is counted by

\[
\frac{1}{4}p \left( A(D, \frac{4m}{p^2})A(D, \frac{4n}{p^2}) - A(D, \frac{4m}{p^2})A(D, \frac{4n}{p^2}) \right).
\]

If for some \( j \) there is \( g.c.d.(a + j e_0, b + j f_0, c + j g_0, d + j h_0) = p \), for this \( j \) the integer cube determines \( p^2 \) orbits of \( F \) corresponding to \( 0 \leq l_1, l_2 \leq p \); while for other \( j \), they uniquely determine the orbits of \( F \) by making \( l_1 = l_2 = 0 \). The number of orbits is counted by

\[
\frac{1}{4}p^2A(D, \frac{4m}{p^2})A(D, \frac{4n}{p^2}) + (p - 1)A(D, \frac{4m}{p^2})A(D, \frac{4n}{p^2}).
\]
Proposition 3.9. Given nonzero integers \(m,n\), we have
\[
B(D,m,n) = B_{\text{cube}}(D,m,n) = (p^2 - 1)A_2 + (p^3 - p)A_3 + 2(p^4 - p^2)A_4 + 2(p^5 - p^3)A_5 + \cdots
\]
\[
+ l(p^{2l} - p^{2l-2})A_{2l} + l(p^{2l+1} - p^{2l-1})A_{2l+1} + \cdots
\]
where for \(p\) not a divisor of \(m\) or \(n\) the last term are interpreted as 0.

\(\square\)

Proof. We also start for the case of \(g.c.d.(e,f,g,h) = p\) and consider
\[
\left( \begin{array}{cccc}
  a & j & 0 & 0 \\
  c & j & 0 & 0 \\
  e & j & 0 & 0 \\
  g & j & 0 & 0 \\
\end{array} \right)
\]

Suppose for some \(j\) there is \(g.c.d.(a+j,0,b+jf,0,c+jg,0,d+jh) = p\), as discussed above the number of orbits of \(F\) is
\[
\frac{1}{4} p^2 \left( A(D,\frac{4m}{p^4},\frac{4n}{p^2})A(D,\frac{4m}{p^4},\frac{4n}{p^2}) - A(D,\frac{4m}{p^4},\frac{4n}{p^2})A(D,\frac{4m}{p^4},\frac{4n}{p^2}) \right)
\]
\[
+ \frac{1}{4} (p-1) \left( A(D,\frac{4m}{p^4},\frac{4n}{p^2})A(D,\frac{4m}{p^4},\frac{4n}{p^2}) - A(D,\frac{4m}{p^4},\frac{4n}{p^2})A(D,\frac{4m}{p^4},\frac{4n}{p^2}) \right).
\]

Suppose for some \(j\) there is \(g.c.d.(a+j,0,b+jf,0,c+jg,0,d+jh) = p\), for this \(j\) there are \(p^2\) orbits of \(F\) corresponding to \(0 \leq l_1, l_2 \leq p\); while for other \(j\) they uniquely determine the orbits of \(F\). Therefore the number of orbits of \(F\) is
\[
\frac{1}{4} p^2 \left( A(D,\frac{4m}{p^3},\frac{4n}{p})A(D,\frac{4m}{p^3},\frac{4n}{p}) \right) + \frac{1}{4} (p-1) \left( A(D,\frac{4m}{p^3},\frac{4n}{p})A(D,\frac{4m}{p^3},\frac{4n}{p}) \right).
\]

Next for the case of \(g.c.d.(e,f,g,h) = p^2\), if for some \(j\) there is \(g.c.d.(a+j,0,b+jf,0,c+jg,0,d+jh) = p^2\) it corresponds to \(p^2\) different orbits of \(F\); otherwise they uniquely determine the orbits of \(F\). The number of orbits of \(F\) is
\[
\frac{1}{4} p^2 \cdot p \left( A(D,\frac{4m}{p^3},\frac{4n}{p})A(D,\frac{4m}{p^3},\frac{4n}{p}) \right) + \frac{1}{4} (p^2 - p) \left( A(D,\frac{4m}{p^3},\frac{4n}{p})A(D,\frac{4m}{p^3},\frac{4n}{p}) \right).
\]

\(\square\)

As a direct generalization we have

Proposition 3.9. Given nonzero integers \(m,n\), we have
\[
A_1 = \frac{1}{4} A(D,\frac{4m}{p^2},\frac{4n}{p^2})A(D,\frac{4m}{p^2},\frac{4n}{p^2})
\]
We have
\[
B(D,m,n) - B_{\text{cube}}(D,m,n) = (p^2 - 1)A_2 + (p^3 - p)A_3 + 2(p^4 - p^2)A_4 + 2(p^5 - p^3)A_5 + \cdots
\]
\[
+ l(p^{2l} - p^{2l-2})A_{2l} + l(p^{2l+1} - p^{2l-1})A_{2l+1} + \cdots
\]
where for \( p \) not a divisor of \( m \) or \( n \), \( A(D, \frac{4m}{p^2q}) \) or \( A(D, \frac{4m}{p^2q}, \frac{4m}{pq}) \) is interpreted as 0.

Proof. Without loss of generality we assume that \( k \) is an even integer and \( l = \frac{t}{2} \).

From integer cubes

\[
\left( \begin{array}{ccc}
(a + je) & (b + jf) & (c + jg) \\
(c + jg) & (d + jh) & (e + jg) \\
(e + jg) & (f + jg) & (g + jh)
\end{array} \right)
\]

for \( 0 \leq j \leq p^l - 1 \), if \( \text{g.c.d.}(a + je, b + jf, c + jg, d + jh) = 1 \) for some \( j \), the number of different orbits of \( F \) is

\[
(A_{k-1} - A_k)(1 - p) + p^2 \cdot p^2(p-l-2) + \ldots + p^{l-1}p^{l-1} + (p^l - p^{l-1})).
\]

Suppose that \( \text{g.c.d.}(a + je, b + jf, c + jg, d + jh) = p^l \) for some \( j \), the number of different orbits of \( F \) is

\[
(A_{k} - A_{k+1})(1 - p) + p^2 \cdot p^2(p-l-2) + \ldots + p^{l+1}p^l + (p^l - p^{l-1}) + (p^l - p^{l+1})
\]

Adding them contributes \( (p^k - p^{k-2})A_k \).

Next consider the case of \( \text{g.c.d.}(e, f, g, h) = p^{l+t} \) for \( 1 \leq t \leq l - 1 \). From integer cubes

\[
\left( \begin{array}{ccc}
(a + je) & (b + jf) & (c + jg) \\
(c + jg) & (d + jh) & (e + jg) \\
(e + jg) & (f + jg) & (g + jh)
\end{array} \right)
\]

for \( 0 \leq j \leq p^{l+t} - 1 \), if \( \text{g.c.d.}(a + je, b + jf, c + jg, d + jh) = p^{l-t} \) for some \( j \), the number of different orbits of \( F \) is

\[
(A_{k-1} - A_k)(1 - p) + p^2 \cdot p^2(p-l-2) + \ldots + p^{l-t-1}p^{l-t}p^{l-t} + (p^{l-t} - p^{l-1})
\]

Suppose that \( \text{g.c.d.}(a + je, b + jf, c + jg, d + jh) = p^{l-t} \) for some \( j \), the number of different orbits of \( F \) is

\[
(A_{k} - A_{k+1})(1 - p) + p^2 \cdot p^2(p-l-2) + \ldots + p^{l-t+1}p^{l-t}p^{l-t} + (p^{l-t} - p^{l-1})
\]

Adding them also contributes \( (p^k - p^{k-2})A_k \). Therefore the coefficient of \( A_k \) in

\[
B(D, m, n) - B^{\text{cube}}(D, m, n) = l(p^k - p^{k-2}).
\]

We can also show that

**Lemma 3.10.** Given nonzero integers \( m, n, D = D_0p^2q^2 \), where \( D_0 \) is square free and \( p, q \) are coprime, we have

\[
B(D, m, n) = B^{\text{cube}} = \frac{1}{4} \left(A(D, 4m)A(D, 4n) + pA(D, 4m)A(D, 4n)\right) - \frac{4m}{p^2q^2}A(D, 4m)A(D, 4n) - \frac{4m}{pq}A(D, 4m)A(D, 4n),
\]

where for \( pq \) not a divisor of \( m \) or \( n \), \( A(D, \frac{4m}{pq}) \) or \( A(D, \frac{4m}{pq}) \) is interpreted as 0.
Lemma 3.11. Given nonzero integers $m, n, D = D_0p^i q^j$, where $D_0$ is square free and $p, q$ are coprime, we have

$$B(D, m, n) - B^\text{cub}(D, m, n) = p(q^2 - 1) \frac{1}{4} A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$$

$$+ q(p^2 - 1) \frac{1}{4} A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$$

$$+ p^2(q^2 - 1) \frac{1}{4} A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$$

$$+ q^2(q^2 - 1) \frac{1}{4} A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$$

$$+(p^2 - 1)(q^2 - 1) \frac{1}{4} A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$$

where for $pq$ not a divisor of $m$ or $n$, $A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$ or $A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$ is interpreted as 0.

Proof. We will compute the coefficient of the term $A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$.

In the case of $g.c.d.(e, f, g, h) = pq$, the number of orbits of $F$ is given by

$$\frac{1}{4} pq \left( A(D, \frac{m}{pq}, \frac{n}{pq}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) - A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) \right)$$

$$+ \frac{1}{4} p^2 q \left( A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) - A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) \right)$$

$$+ \frac{1}{4} (pq - q) \left( A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) - A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) \right)$$

$$+ \frac{1}{4} pq \left( A(D, \frac{m}{pq}, \frac{n}{pq}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) - A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) \right)$$

$$+ \frac{1}{4} p^2 q \left( A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) - A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) \right)$$

$$+ \frac{1}{4} (pq - p) \left( A(D, \frac{m}{pq}, \frac{n}{pq}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) - A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2}) \right)$$

$$+ \frac{1}{4} (pq - p + 1) A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$$

$$+ \frac{1}{4} (p^2 + q^2) A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$$

$$+ \frac{1}{4} (p^2 q^2) A(D, \frac{m}{pq^2}, \frac{n}{pq^2}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$$

where the last term comes from eliminating the double counting of the first line and fourth line: the first line tries to count the orbits which satisfies $g.c.d.(a + j e_0, b + j f_0, c + j g_0, d + j h_0) \neq p$ for $0 \leq j \leq pq - 1$; while the fourth line tries to count the orbits which satisfies $g.c.d.(a + j e_0, b + j f_0, c + j g_0, d + j h_0) \neq q$ for $0 \leq j \leq pq - 1$, but they subtract more orbits which satisfies $g.c.d.(a + j e_0, b + j f_0, c + j g_0, d + j h_0) = pq$ for some $0 \leq j \leq pq - 1$. So the coefficient of $\frac{1}{4} A(D, \frac{m}{pq}, \frac{n}{pq}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$ is

$$(p^2 - 1)(q^2 - 1).$$

There are two more cases in which the orbits counting contributes to the coefficient of $\frac{1}{4} A(D, \frac{m}{pq}, \frac{n}{pq}) A(D, \frac{4m}{pq^2}, \frac{4n}{pq^2})$. They are $g.c.d.(e, f, g, h) = p^2 q$ and $g.c.d.(e, f, g, h) = \ldots$
\[ \text{pq}^2. \] In the former case, the orbits counting provides
\[ p^2(q^2 - 1) \]
for the coefficient of \( \frac{1}{4} A(D_p, m_p) A(D_q, m_q) \); while in the latter case it is
\[ q^2(p^2 - 1). \]

\[ \square \]

If we examine the relation among \( B(p^k, p^l, p^t) \), \( B(q^k, q^l, q^t) \), \( B(p^k q^k, p^l q^l, p^t q^t) \) then the above lemma suggests the multiplicativity of them, however, the coefficients of the Shintani zeta function \( Z_{\text{Shintani}}(s_1, s_2, w) \) do not satisfy the multiplicativity property. In order the overcome this difficulty we refer to the \( A_3 \)-Weyl group multiple Dirichlet series in the next section.

### 3.4. Relation to \( A_3 \)-Weyl group multiple Dirichlet series

We focus on the type \( A_r \) multiple Dirichlet series. These series have the form
\[
\sum_{m_1, m_2, \ldots, m_r} a(m_1, m_2, \ldots, m_r) m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r},
\]
where the sum is over certain positive integers. They satisfy, for the square free integers \( m_1, m_2, \ldots, m_r \),
\[
a(m_1, m_2, \ldots, m_r) = \left( \frac{m_1}{m_2} \right) \left( \frac{m_2}{m_3} \right) \cdots \left( \frac{m_{r-1}}{m_r} \right),
\]
and the following twisted multiplicativity property:
\[
a(m_1 m_1', \ldots, m_r m_r') = a(m_1, \ldots, m_r) a(m_1', \ldots, m_r') \prod_{j=1}^{r-1} \left( \frac{m_j}{m_j + 1} \right) \left( \frac{m_j'}{m_j' + 1} \right).
\]

In particular, for lower rank \( r \), we mention two quadratic multiple Dirichlet series which we will use. The first one is found by Siegel [Sie56].
\[
Z(s, w) = \sum_{d, m \geq 1, d, m \text{ odd}} \frac{\chi_d(\hat{m})}{m^s d^w} a(d, m),
\]
where \( \hat{m} \) denotes the factor of \( m \) which is prime to the square free part of \( d \). The factor \( a(d, m) \) is defined by
\[
a(d, m) = \prod_{p \text{ prime}} a(p^k, p^l),
\]
where
\[
a(p^k, p^l) = \begin{cases} \min(p^{k/2}, p^{l/2}) & \text{if } \min(k, l) \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}
\]
The second example is the quadratic \( A_3 \)-Weyl group Dirichlet series, which has the form
\[
Z(s_1, s_2, w) = \sum_{d, m, n \geq 1, d, m, n \text{ odd}} \frac{\chi_d(\hat{m}) \chi_d(\hat{n})}{m^{s_1} n^{s_2} d^w} a(d, m, n).
\]
In these two examples, the coefficient \( a(d, m) \) or \( a(d, m, n) \) satisfy the multiplicative property, so the description of them reduces to the coefficients \( a(p^k, p^l) \) or
a(p^k, p^l, p^m). For example, for fixed prime p, the coefficients of A3-Weyl group multiple Dirichlet series can by organized into the generating function

$$\sum_{k,l,t \geq 0} \frac{a(p^k, p^l, p^m)}{p^{kw} p^{ls} p^{ts}}$$

or simply the function

$$f_{A_3}(x, y, z) = \sum_{k,l,t \geq 0} a_{klt}(p)x^ky^lz^t$$

with x, y, z replaced by p^{−s_1}, p^{−w}, p^{−s_2} respectively. From [CG07], we have

$$f_{A_3}(x, y, z) = \frac{1}{(1-x)(1-y)(1-z)(1-pxy^2z^2)(1-py^2xz^2)(1-p^2x^2y^2z^2)}.$$

In our previous paper, we have obtained the relation between the Shintani zeta function of 2 × 2 × 2 integer cubes with the A3-Weyl group multiple Dirichlet series. The main result is the following

**Proposition 3.12.** Let D be an odd integer. Then

$$\sum_{m,n \geq 0} B^{\text{cube}}_{m,n}(D, m, n) = \hat{P}(D, s_1) \hat{P}(D, s_2) \zeta(s_1) \zeta(s_2) \sum_{m,n \geq 0} \chi_D(\hat{m}) \chi_D(\hat{n}) a(D, m, n)$$

where

$$a(D, m, n) = \sum_{d \mid D} d \cdot a(D, \frac{m}{d^2}, \frac{n}{d}),$$

and

$$\hat{P}(D, s) = (1 - 2^{-s}) \sum_{l=0}^{1} \frac{A(D, 2^{(l+2)})}{2^l} \left(1 - \frac{\chi_D(2)}{2^s}\right)^{-1}.$$  

As in the Shintani zeta function Z_{Shintani}(s_1, s_2, w), for fixed D, m, n, the coefficient $B(D, m, n)$ is the linear combination among $\frac{1}{4} A(D, \frac{m}{d^2}, \frac{n}{d}) A(D, \frac{m}{d^2}, \frac{n}{d})$, where d is the divisor of D. Therefore, with $\frac{1}{4} A(D, \frac{m}{d^2}, \frac{n}{d}) A(D, \frac{m}{d^2}, \frac{n}{d})$ replaced by $a(D, \frac{m}{d^2}, \frac{n}{d})$, we have

**Proposition 3.13.** Let D be an odd integer. We have

$$\sum_{m,n \geq 0} B_{m,n}(D, m, n) = \hat{P}(D, s_1) \hat{P}(D, s_2) \zeta(s_1) \zeta(s_2) \sum_{m,n \geq 0} \chi_D(\hat{m}) \chi_D(\hat{n}) b(D, m, n)$$

where

$$b(D, m, n) = \sum_{d \mid D} f(d) \cdot a(D, \frac{m}{d^2}, \frac{n}{d}),$$

the counting function $f(d)$ depends only on D and d, and

$$\hat{P}(D, s) = (1 - 2^{-s}) \sum_{l=0}^{1} \frac{A(D, 2^{(l+2)})}{2^l} \left(1 - \frac{\chi_D(2)}{2^s}\right)^{-1}.$$  

Finally, we obtain the explicit formula for the Shintani zeta function associated to the paris of quaternary alternating 2-forms.
Theorem 3.14. Take the series with sum over odd discriminants, then the Shintani zeta function $Z_{\text{Shintani}}(s_1, s_2, w)$ can be written as

$$Z_{\text{Shintani}}(s_1, s_2, w) = 2^{-1} (1 - 2^{-s_1})(2 - 2^{-s_1})(1 - 2^{-s_2})(2 - 2^{-s_2}) \zeta(s_1) \zeta(s_2)$$

$$
\cdot Z^{(1)}_{\text{WMD}}(s_1, s_2, w) \\
+ 2^{-1} (1 - 2^{-s_1})(1 + 2^{-s_1})(1 - 2^{-s_2})(1 + 2^{-s_2}) \zeta(s_1) \zeta(s_2)
$$

$$
\cdot Z^{(2)}_{\text{WMD}}(s_1, s_2, w),
$$

where

$$Z^{(1)}_{\text{WMD}}(s_1, s_2, w) = \sum_{D \equiv 1 \pmod{8}} \sum_{m,n \geq 1} \frac{\chi_D(m) \chi_D(n)}{m^{s_1} n^{s_2} D^w} b(D, m, n),$$

$$Z^{(2)}_{\text{WMD}}(s_1, s_2, w) = \sum_{D \equiv 5 \pmod{8}} \sum_{m,n \geq 1} \frac{\chi_D(m) \chi_D(n)}{m^{s_1} n^{s_2} D^w} b(D, m, n).$$

The generating function of coefficients $b(p^k, p^l, p^j)$ satisfies

$$H(x, y, z) = f_{A3}(x, y, z) + \frac{(1 - p^{-2})}{4} \left( f_{A3}(x, y, z) \frac{pxy^2 z}{1 - pxy^2 z} - f_{A3}(-x, y, z) \frac{pxy^2 z}{1 + pxy^2 z} \right)$$

$$+ \frac{p(1 - p^{-2})}{4} \left( f_{A3}(x, y, z) \frac{pxy^2 z}{1 - pxy^2 z} - f_{A3}(-x, y, z) \frac{pxy^2 z}{1 + pxy^2 z} \right).$$

Proof. This follows directly from Proposition 3.9 and the calculus of formal series expansion. From

$$f_{A3}(x, y, z) = \frac{1}{1 - pxy^2 z} \sum_{k,l,t \geq 0} a(p^k, p^l) a(p^k, p^l) x^l y^k z^t$$

$$= \sum_{s=0}^{\infty} p^s x^{2s} y^{2s} z^s \sum_{k,l,t \geq 0} a(p^k, p^l) a(p^k, p^l) x^l y^k z^t$$

$$= \sum_{k,l,t \geq 0} \left( a(p^k, p^l) a(p^k, p^l) + pa(p^{k-2}, p^{l-1}) a(p^{k-2}, p^{l-1}) + \ldots \right) x^l y^k z^t$$

$$= \sum_{k,l,t \geq 0} a(p^k, p^l) x^l y^k z^t,$$

by comparing it with

$$H(x, y, z) - f_{A3}(x, y, z) = \sum_{k,l,t \geq 0} ((p^2 - 1) a(p^{k-4}, p^{l-2}) a(p^{k-4}, p^{l-2})$$

$$+ (p^3 - p) a(p^{k-6}, p^{l-3}) a(p^{k-6}, p^{l-3})$$

$$+ 2(p^4 - p^2) a(p^{k-8}, p^{l-4}) a(p^{k-8}, p^{l-4})$$

$$+ 2(p^5 - p^3) a(p^{k-10}, p^{l-5}) a(p^{k-10}, p^{l-5}) + \ldots$$

$$+ j(p^{2j} - p^{2j-2}) a(p^{k-4j}, p^{l-2j}) a(p^{k-4j}, p^{l-2j})$$

$$+ j(p^{2j+1} - p^{2j-1}) a(p^{k-4j-2}, p^{l-2j-1}) a(p^{k-4j-2}, p^{l-2j-1})$$

$$+ \ldots) x^l y^k z^t,$$
we obtain that

\[
RHS = \frac{(1 - p^{-2})}{4} \left( f_{A^3}(x, y, z) \frac{pxy^2z}{1 - pxy^2z} - f_{A^3}(-x, y, z) \frac{pxy^2z}{1 + pxy^2z} \right) \\
+ \frac{p(1 - p^{-2})}{4} \left( f_{A^3}(x, y, z) \frac{pxy^2z}{1 - pxy^2z} - f_{A^3}(-x, y, z) \frac{pxy^2z}{1 + pxy^2z} \right).
\]

□

References


Department of Mathematics, Stony Brook University