ABSTRACT. We investigate the Shintani zeta functions associated to the prehomogeneous spaces, the example under consideration is the set of $2 \times 2 \times 2$ integer cubes. We show that there are three relative invariants under a certain parabolic group action, they all have arithmetic nature and completely determine the equivalence classes. We show that the associated Shintani zeta function coincides with the $A_3$ Weyl group multiple Dirichlet series. Finally, we show that the set of semi-stable integer orbits maps finitely and surjectively to a certain moduli space.

1. INTRODUCTION

The study of Dirichlet series in several complex variables has seen much development in recent years. Multiple Dirichlet series (MDS) can arise from metaplectic Eisenstein series, zeta functions of prehomogeneous vector spaces (PHVS), height zeta functions or multiple zeta values. This list is far from exhaustive. In general, completely different techniques are involved in the study of different series arising in these different contexts. Partially for this reason, it is of great interest to identify examples of multiple Dirichlet series which lie in two or more of the areas above.

This paper investigates one such example. We study in detail a Shintani zeta function associated to a certain prehomogeneous vector space and show that it coincides with a Weyl group multiple Dirichlet series (WMDS) of the form introduced in BBCFH ([BBC+06]). This latter series (in three complex variables) is also a Whittaker function of a Borel Eisenstein on the metaplectic double cover of $GL_4$.

The seminal work of Bhargava[Bha04] generalizing Gauss’s composition law on binary quadratic forms start with investigating the rich structure of integer orbits of $2 \times 2 \times 2$-cubes acted on by $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$, which refers to the $D_4$ case among the list of classification of PHVS given in Sato-Kimura [SK77]. Bhargava shows that the set of projective integer orbits with given discriminant has a group structure and it is isomorphic to the square product of narrow class group of the quadratic order with that discriminant.

We begin with some definitions. Let $G_\mathbb{C}$ be a connected complex Lie group, usually we assume that $G_\mathbb{C}$ is the complexification of a real Lie group $G_\mathbb{R}$. A PHVS $(G_\mathbb{C}, V_\mathbb{C})$ is a complex finite dimensional vector space $V_\mathbb{C}$ together with a holomorphic representation of $G_\mathbb{C}$ such that $G_\mathbb{C}$ has an open orbit in $V_\mathbb{C}$. One of the important properties is that if $V_\mathbb{C}$ is a PHVS for $G_\mathbb{C}$ then there is just one open orbit, and that orbit is dense (see [Kna02, Chapter X]). Let $P$ be a complex polynomial function on $V_\mathbb{C}$. We call it a relative invariant polynomial if $P(gv) = \chi(g)P(v)$ for some rational character $\chi$ of $G_\mathbb{C}$ and all $g \in G_\mathbb{C}$, $v \in V_\mathbb{C}$. We say the PHVS has
n relative invariants if n algebraically independent relative invariant polynomials generate the invariant ring. We define the set of semi-stable points \(V^*\) to be the subset on which no relative invariant polynomial vanishes.

Let \(V_2 = \mathbb{Z}^2 \otimes \mathbb{Z}^2 \otimes \mathbb{Z}^2\) and consider the action of \(B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})\), where \(B'_2(\mathbb{Z})\) is the subgroup of lower-triangular matrices in \(\text{SL}_2(\mathbb{Z})\) with positive diagonal elements. We denote \(A \in V_2\) by \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\) for simplicity. The complex group \(B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})\) acting on \(V_\mathbb{C}\) is again a PHVS and will have three relative invariants. They are \(m(A) = ad - bc, n(A) = ag - ce\) and the discriminant \(D(A) = \text{disc}(A)\).

The primary object of study of PHVS is the Shintani zeta function, see [SS74] for the introduction and [Shi75] for the application to the average value of \(h(d)\), the number of primitive inequivalent binary quadratic forms of discriminant \(d\). The Shintani zeta function in three variables associated to the PHVS of \(2 \times 2 \times 2\)-cubes is defined to be:

\[
Z_{\text{Shintani}}(s_1, s_2, w) = \sum_{A \in B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \setminus V^*_\mathbb{Z}} \frac{1}{|D(A)|^w |m(A)|^{s_1} |n(A)|^{s_2}}.
\]

We denote the partial sum of Shintani zeta function by

\[
Z^{\text{odd}}_{\text{Shintani}}(s_1, s_2, w) = \sum_{A \in B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \setminus V^*_\mathbb{Z}, D(A) \text{ odd}} \frac{1}{|D(A)|^w |m(A)|^{s_1} |n(A)|^{s_2}}.
\]

We will show that it is closely related to another multiple Dirichlet series which arises in the Whittaker expansion of the Borel Eisenstein series on the metaplectic double cover of \(\text{GL}_4\), which is found to be the WMDS associated to the root system \(A_3\) and it is of the form:

\[
Z_{\text{WMDS}}(s_1, s_2, w) = \sum_{D \neq 0 \atop D \text{ odd}} \frac{1}{|D|^w} \sum_{m,n > 0} \frac{\chi_D(\hat{m}) \chi_D(\hat{n})}{m^{s_1} n^{s_2}} a(D, m, n)
\]

where \(\hat{m}\) denotes the factor of \(m\) that is prime to the square-free part of \(D\). A precise formula of the coefficients \(a(D, m, n)\) will be given in section 4. We will also define the partial sum of \(Z_{\text{WMDS}}(s_1, s_2, w)\) by

\[
Z_{\text{WMDS}}^{(1)}(s_1, s_2, w) = \sum_{D \neq 0 \atop D \equiv 1 \mod 4} \frac{1}{|D|^w} \sum_{m,n > 0} \frac{\chi_D(\hat{m}) \chi_D(\hat{n})}{m^{s_1} n^{s_2}} a(D, m, n).
\]

The main results of this paper, given in Theorems 1.1, 1.2 and 1.3 below, are as follows:

1. We give an explicit description of the Shintani zeta function associated to the PHVS of \(2 \times 2 \times 2\)-cubes.
2. We show how the series is related to the quadratic WMDS associated to the root system \(A_3\).
3. We give an arithmetic meaning to the semi-stable integer orbits of the PHVS.

Denote by \(A(d, a)\) the number of solutions to the congruence \(x^2 = d \pmod{a}\).
Theorem 1.1. The Shintani zeta function associated to the PHVS of $2 \times 2 \times 2$-cubes is given by:

$$Z_{\text{Shintani}}(s_1, s_2, w) = \sum_{A \in B_2^2(\mathbb{Z}) \times B_2^2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \setminus V_2^s} \frac{1}{|D(A)|^w |m(A)|^{s_1} |n(A)|^{s_2}}$$

$$= \sum_{D = D_0 D_1^2 \neq 0} \frac{1}{|D|^w} \sum_{m,n>0} B(D, m, n) \frac{1}{m^{s_1} n^{s_2}},$$

where

$$B(D, m, n) = \sum_{d|D, d|m, d|n} d \cdot A(D/d^2, 4m/d) \cdot A(D/d^2, 4n/d).$$

Theorem 1.2. The Shintani zeta function can be related to the $A_3$-WMDS in the following way:

$$Z_{\text{Shintani}}^{\text{odd}}(s_1, s_2, w) = 4(1 - 2^{-s_1})(1 + 2^{-s_1})(1 - 2^{-s_2})(1 + 2^{-s_2}) \zeta(s_1) \zeta(s_2)$$

$$\cdot Z_{\text{WMDS}}^{(1)}(s_1, s_2, w),$$

where $\zeta(s)$ means the Riemann zeta function.

Lastly, we give the arithmetic meaning to the semi-stable integer orbits of the PHVS by showing that:

Theorem 1.3. There is a natural map, which is a surjective and finite morphism,

$$B_2^2(\mathbb{Z}) \times B_2^2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \setminus V_2^s \to \text{Is}_0 \setminus \{(R; I_1, I_2) | R/I_1 \cong N(I_1) \mathbb{Z}, R/I_2 \cong N(I_2) \mathbb{Z}\},$$

where $R$ is an oriented quadratic ring and $I_1$'s are the oriented ideals in $R$ with the norm $N(I_1)$. The cardinality $n(R; I_1, I_2)$ of the fiber is equal to

$$\sigma_1(D_1, a_1, a_2),$$

where $D = D_0 D_1^2 = \text{disc}(R)$ with $D_0$ is square-free, and $a_i = N(I_i)$. It also satisfies

$$\sum_{(R; I_1, I_2) / \sim \atop N(I_i) = a_i} n(R; I_1, I_2) = B(D, a_1, a_2).$$

The organization of this paper is as follows. In section 2, we will review a double Dirichlet series arising from three different approaches. We show the connection between Shintani’s PHVS approach and the $A_2$-WMDS approach. In section 3, we will investigate the structure of integer orbits of the PHVS of $2 \times 2 \times 2$-cubes, using the reduction theory, to derive the main formula in Thm 1.1. In section 4, we will show its connection to the $A_3$-WMDS. The main idea of the proof is to consider the generating function for the $p$-parts of $a(D, m, n)$. From the construction of an $A_3$-WMDS, the $p$-parts of the rational function which is invariant under the Weyl group action is given explicitly by taking the residue of the convolution of two rational functions of $A_2$ root system. We show it coincides with the generating function of $a(D, m, n)$. In the section 5, we will show the set of semi-stable integer orbits naturally encodes the arithmetic information by showing it maps finitely surjective to the moduli space of isomorphism classes of pairs $(R; I_1, I_2)$, where $R$ is an oriented quadratic ring and $I_1$'s are the oriented ideals in $R$. We will recall the definition of orientation of a quadratic ring and its ideal. We further show that each fiber is counted by a divisor function.
2. A2 Weyl Group Dirichlet series

In this section we will introduce three double Dirichlet series and show that they are all essentially the same. The results are not new, but this will serve as a prototype for the more involved computations involving 3-variable Dirichlet series in the latter sections.

2.1. Siegel Double Dirichlet Series. The quadratic multiple Dirichlet series first appeared in the paper of Siegel [Sie56], and its twisted Euler product with respect to one variable was given explicitly. Siegel constructed his series as the Mellin transform of an Eisenstein series of half integral weight on the congruence subgroup $\Gamma_0(4)$. In this section we will first recall the definition of Siegel’s double Dirichlet series and then, using the multiplicative property of Euler products, prove it can be expressed as a sum formed from quadratic characters.

Denote by $A(d,a)$ the number of solutions to the quadratic congruence equation $x^2 = d \pmod{a}$. Then Siegel’s double Dirichlet series is defined to be:

$$Z_{\text{Siegel}}(s,w) = \sum_{d \neq 0} \sum_{a \neq 0} \frac{A(d,a)}{|d|^{w}} |a|^{-s}.$$ 

For any positive prime integer $p$ and any integer $d$, we define a generating series for the $p$-part of the inner summation of above series to be

$$f_p(d,s) = (1 - p^{-s}) \sum_{l=0}^{\infty} A(d,p^l)p^{-ls} (p \neq 2),$$

$$f_2(d,s) = (2^s - 1) \sum_{l=1}^{\infty} A(d,2^l)2^{-ls}.$$ 

Siegel shows in [Sie56, §4] that

$$\frac{1 - \chi_d(p)p^{-s}}{1 - p^{-2s}} f_p(d,s) = \begin{cases} p^{o(1-2s)} + (1 - \chi_d(p)p^{-s}) \sum_{l=0}^{o-1} p^{l(1-2s)} & (p \neq 2), \\
\frac{1+\chi_d(2)}{1+2^{-s}} + (2^{1-s} - \chi_d(2)) \sum_{l=0}^{o} 2^{l(1-2s)} & (p = 2), \end{cases}$$

where $\chi_d$ is the quadratic character associated to the field $\mathbb{Q}(\sqrt{d})$ and $p^{2\alpha}$ is the highest power of $p$ which divides $d/d^*$, $d^*$ is the fundamental discriminant of the field $\mathbb{Q}(\sqrt{d})$. Now the inner summation of Siegel’s double Dirichlet series can be expressed as a normalized quadratic L-function.

**Proposition 2.1.** Fix an integer $d \neq 0$, then

$$\sum_{a < 0} \frac{A(d,a)}{|d|^s} = \sum_{a > 0} \frac{A(d,a)}{a^s} = \zeta(2s)^{-1} \zeta(s) L(s, \chi_d) P(d,s)$$

where the last term is

$$P(d,s) = 2^{-s} \prod_{p \neq 2} \left( p^{o(1-2s)} + (1 - \chi_d(p)p^{-s}) \sum_{l=0}^{o-1} p^{l(1-2s)} \right) \cdot \left( \frac{1 + \chi_n(2)}{1 + 2^{-s}} + \frac{1 - \chi_d(2)2^{-s}}{1 + 2^{-2s}} (2^s - 1) + (2^{1-s} - \chi_n(2)) \sum_{l=0}^{o} 2^{l(1-2s)} \right).$$
**Proof.** The first equality of the claim follows from the fact that

\[ A(d, a) = A(d, -a). \]

For the second equality, first note that the multiplicative property holds

\[ A(d, m)A(d, n) = A(d, mn) \]

for any pairs of coprime positive integers \( m \) and \( n \), by the Chinese remainder theorem.

Then the results follows from two equations

\[
\frac{1 - \chi_n(p)p^{-s}}{1 - p^{-2s}} \sum_{l=0}^{\infty} A(d, p^l)p^{-ls} = p^{\alpha(1-2s)} + (1 - \chi_n(p)p^{-s}) \sum_{l=0}^{\alpha-1} p^{l(1-2s)},
\]

\[
\frac{1 - \chi_n(2)^{-s}}{1 - 2^{-2s}} \sum_{l=1}^{\infty} A(d, 2^l)2^{-ls} = \frac{1 + \chi_n(2)}{1 + 2^{-s}} + (2^{1-s} - \chi_n(2)) \sum_{l=0}^{l} 2^{l(1-2s)},
\]

as well as the multiplicative property. \(\square\)

2.2. **Shintani Double Dirichlet Series.** Another approach to the theory of double Dirichlet series was given by Sato and Shintani using the theory of prehomogeneous vector spaces (PHVS) in [SS74]. In [Shi75], the zeta function associated to the prehomogeneous vector space of quadratic forms acted on by the Borel subgroup of \( \text{GL}_2(\mathbb{C}) \) was studied in detail. In this section, we will recall the Shintani zeta function arising from the prehomogeneous vector spaces approach.

Now we let \( B'_2(\mathbb{C}) \) be the subgroup of lower-triangular matrices in \( G_2 = \text{GL}_2(\mathbb{C}) \) and let \( \rho \) be the representation of \( \text{GL}_2(\mathbb{C}) \) acting on the three dimensional vector space \( V_2 = \{ Q(u, v) = au^2 + buv + cv^2 | (a, b, c) \in \mathbb{C}^3 \} \) of binary quadratic forms as follows

\[ \rho(g)(Q)(u, v) = Q(au + cv, bu + dv) \]

where \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). It is well known that there are two relative invariants for the action of \( B'_2(\mathbb{C}) \) on \( V_2 \), namely, the discriminant \( \text{disc}(Q) = b^2 - 4ac \) of the quadratic form and \( \alpha = Q(1, 0) \). These two invariants freely generate the ring of relative invariants.

Define \( B'_2(\mathbb{Z}) \) to be the Borel subgroup in \( \text{SL}_2(\mathbb{Z}) \) with positive diagonal elements. The Shintani zeta function associated to the PHVS of \( (B'_2(\mathbb{C}), V_2) \) is defined to be

\[
Z_{\text{Shintani}}(s, w; B'_2) = \sum_{Q \in B'_2(\mathbb{Z}) \setminus V'_2} \frac{1}{|Q(1, 0)|^s |\text{Disc}(Q)|^w}.
\]

\[
= \sum_{\alpha \neq 0} \sum_{|a|} \sum_{0 < b \leq 2a - 1} \frac{1}{|b^2 - 4ac|^w},
\]

where \( V'_2 \) is the semi-stable subset of (not necessarily primitive) quadratic forms with \( Q(1, 0) \neq 0 \) and non-zero discriminant. Alternatively, we can express the Shintani zeta function as

\[
\sum_{Q \in \text{SL}_2(\mathbb{Z}) \setminus V'_2} \frac{1}{|\text{Disc}(Q)|^w} \sum_{\gamma \in B'_2(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{Z})/\text{stab}_{\mathbb{Q}}} \frac{1}{|\gamma \circ Q(1, 0)|^s}.
\]
For the rest of this section we suppress the \( B'_2 \) from the notation. We also define
\[
\sum_{Q \in \mathcal{B}_2(\mathbb{Z}) \setminus V_{ss}^g} \frac{1}{|Q(1,0)|^s|\text{Disc}(Q)|^w}
\]
\[
= \sum_{a \neq 0} \frac{1}{|a|^s} \sum_{0 \leq b \leq 2a-1 \atop c : b^2 - 4ac \text{ odd}} \frac{1}{|b^2 - 4ac|^w}.
\]

Remark 2.2. In section 5, we will show that the orbits in \( \mathcal{B}_2(\mathbb{Z}) \setminus V_{ss}^g_{\text{primitive}} \) parametrize the isomorphism classes of the pairs \((R,I)\), where \( R \) is an oriented quadratic ring and \( I \) is an oriented ideal with cyclic quotient in \( R \). We will call \((R_1,I_1)\) and \((R_2,I_2)\) isomorphic if there is a ring isomorphism from \( R_1 \) to \( R_2 \) preserving the orientation and sending \( I_1 \) to \( I_2 \).

Lemma 2.3. \textit{The Shintani zeta function can be written as}
\[
Z_{\text{Shintani}}(s,w) = \xi_1(s,w) + \xi_2(s,w),
\]
where \( \xi_i(s,w) = \sum_{a,d > 0} \frac{A((1)^{-1}d,4a)}{a^s d^w} \).

Proof. First note that under the action of \( g = \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix} \) to the quadratic form \( Q(u,v) = au^2 + buv + cv^2 \), the middle coefficient \( b \) is mapped to \( b + 2am \). Given non-zero integers \( a \) and \( d \), the number of the solutions to \( b^2 - 4ac = d \) with \( 0 \leq d \leq 2a-1 \) and \( c \in \mathbb{Z} \) is equal to \( \frac{A(d,4a)}{2} \). So we have the equality
\[
Z_{\text{Shintani}}(s,w) = \frac{1}{2} \sum_{a,d \neq 0} \frac{A(d,4a)}{|a|^s |d|^w}.
\]
On the other hand, \( A(d,4a) = A(d,-4a) \). Therefore,
\[
Z_{\text{Shintani}}(s,w) = \sum_{a,d > 0} \frac{A(d,4a)}{a^s d^w} + \sum_{a,d > 0} \frac{A(-d,4a)}{a^s d^w}.
\]
The result follows. \( \square \)

As a corollary to Proposition 2.1, we can express the inner summation of the Shintani zeta function in terms of a quadratic Dirichlet \( L \)-function.

Corollary 2.4. \textit{Fix} \( d \neq 0 \). \textit{Then we have}
\[
\sum_{a > 0} \frac{A(d,4a)}{a^s} = \zeta(2s)^{-1} \zeta(s) L(s, \chi_d) P'(d,s),
\]
where the last term is
\[
P'(d,s) = 4^s \prod_{p \neq 2} \left( p^{\alpha(1-2s)} + (1 - \chi_d(p)) p^{-s} \sum_{l=0}^{\alpha-1} p^{l(1-2s)} \right) \cdot \left( \frac{1 + \chi_n(2)}{1 + 2^{-s}} + \frac{2^{1-s} - \chi_n(2)}{1 - 2^{-2s}} \left( 1 - 2^{-s} \right) \right).
\]
Proof. As \(\sum_{a>0} \frac{A(d,4a)}{a^s} = 4^s \sum_{a>0} \frac{A(d,4a)}{(4a)^s}\), by the proof of proposition 2.1, we need only to correct the generating function at prime \(p = 2\) in order to incorporate the factor 4, in which case the generating function should be

\[
\sum_{l=0}^{\infty} d(2^l)2^{-ls} = 4^s \sum_{l=2}^{\infty} A(d,2^l)2^{-ls}.
\]

While the function on the right hand side satisfies

\[
\frac{1 - \chi_d(2)2^{-s}}{1 - 2^{-2s}} (2^s - 1) \sum_{l=2}^{\infty} A(d,2^l)2^{-ls} = \frac{1 - \chi_d(2)2^{-s}}{1 - 2^{-2s}} f_2(d,s)
\]

\[
- \frac{1 - \chi_n(2)2^{-s}}{1 - 2^{-2s}} (2^s - 1) A(d, 2) 2^{-s}.
\]

Note that \(A(d,2) = 1\), therefore, using the multiplicative property of \(A(d,\cdot)\), we have

\[
\sum_{a>0} A(d,4a) a^s = 4^s \sum_{a>0} A(d,4a) (4a)^s = \zeta(2s)^{-1} \zeta(s) L(s, \chi_d) P'(d,s).
\]

The result follows. \(\Box\)

2.3. \(A_2\)-Weyl Group Multiple Dirichlet Series. Weyl group multiple Dirichlet series are a class of multiple Dirichlet series coming from Eisenstein series on metaplectic groups. The simplest example is the quadratic \(A_2\)-Weyl group double Dirichlet series (see [CG07]), it is defined as

\[
Z_{A_2}^{(s,w)} = \sum_{m>0, \text{odd}} \chi_D(\hat{m}) m^s |D|^{-w} a(D,m),
\]

where \(\hat{m}\) is the factor of \(m\) that is prime to the square-free part of \(D\) and \(\chi_D\) is the quadratic character associated to the field \(\mathbb{Q}(\sqrt{D})\). Moreover, the multiplicative factor \(a(D,m)\) is defined by

\[
a(D,m) = \prod_{p^k || D} a(p^k, p^l)
\]

and

\[
a(p^k, p^l) = \begin{cases} 
\min(p^{k/2}, p^{l/2}) & \text{if } \min(k,l) \text{ is even,} \\
0 & \text{otherwise.}
\end{cases}
\]

We also define

\[
Z_{A_2}^{(1)}(s,w) = \sum_{D \equiv 1 \text{ mod } 4} \frac{\chi_D(\hat{m}) m^s |D|^{-w} a(D,m)}{m^s |D|^w}.
\]

Regarding to the relation between this quadratic \(A_2\)-WMDS and the Shintani zeta function of PHVS of binary quadratic forms, we have

**Proposition 2.5.** Let \(D\) be an odd integer. We can relate the inner sum of the Shintani zeta function \(Z_{\text{Shintani}}(s,w)\) with that of the quadratic \(A_2\)-Weyl group multiple Dirichlet series \(Z_{A_2}(s,w)\) in the following way

\[
\sum_{m>0} \frac{A(D,4m)}{m^s} = \tilde{P}_2(D,s) \zeta(s) \sum_{m>0} \frac{\chi_D(\hat{m}) a(D,m)}{m^s},
\]

where \(\tilde{P}_2(D,s)\) is the value of the Shintani zeta function at \(s = 2\).
where
\[ \tilde{P}_2(D,s) = \begin{cases} 
2(1 - 2^{-s})(1 + 2^{-s}) & D \equiv 1 \pmod{4}, \\
0 & \text{otherwise.}
\end{cases} \]

Proof. Analogous to the $p$-part formula of $a(p^k, p^l)$, we will show that for an odd prime $p$

\[ A(p^k, p^l) = \begin{cases} 
2a(p^k, p^l) & \text{if } k < l, \\
\beta_{\lfloor l/2 \rfloor} & \text{otherwise.}
\end{cases} \]

First consider the case when $p \neq 2$. If $k < l$ and $k$ is an odd integer, then the congruence equation $x^2 = p^k \pmod{p^l}$ reduces to the equation of $x^2 = p \pmod{p^l}$ for some power $i$ of $p$ and there is no solution to it; while when $k$ is an even integer, the congruence equation $x^2 = p^k \pmod{p^l}$ reduces to the equation of or $x^2 = 1 \pmod{p^l}$ and there are two solutions to it. In both cases, the number of solutions are both equal to the value of $2a(p^k, p^l)$ by its definition.

If on the other hand $k \geq l$, then the set of solutions to the congruence equation $x^2 = p^k \pmod{p^l}$ is the set of multipliers of $p^{\lfloor \frac{l}{2} \rfloor}$, so the number of distinct solutions mod $p^l$ is $p^{\lfloor \frac{l}{2} \rfloor}$.

Therefore, for odd prime $p$,

\[ A(p^k, p^l) = \chi_{p^k}(p^{l/2})a(p^k, p^l) + \chi_{p^k}(p^{l/2})a(p^k, p^{l-1}) = a(p^k, p^l) + a(p^k, p^{l-1}), \]

where we set the term $a(p^k, p^{l-1})$ equal to 0 when $l = 0$.

Next, using Hensel’s lemma, an integer $d$ relatively prime to an odd prime $p$ is a quadratic residue modulo any power of $p$ if and only if it is a quadratic residue modulo $p$. In fact, if an integer $d$ is prime to the odd prime $p$, as

\[ A(d, p^l) = 2 \iff \chi_d(p) = 1 \iff A(d, p) = 2, \]

so

\[ A(d, p^l) = \chi_d(p^{l/2}) + \chi_d(p^{l-1}). \]

By the prime power modulus theory [Gau66], if the modulus is $p^l$, then $p^kd$

\[ \begin{cases} 
is a quadratic residue modulo $p^l$ if $k \geq l$, 
is a non-quadratic residue modulo $p^l$ if $k < l$ is odd,
is a quadratic residue modulo $p^l$ if $k < l$ is even and $d$ is a quadratic residue, 
is a non-quadratic residue modulo $p^l$ if $k < l$ is even and $d$ is a non-quadratic residue.
\]

Therefore, for an odd integer $d$ prime to $p \neq 2$, we have

\[ A(dp^k, p^l) = \begin{cases} 
0 & \chi_d(p^l) = -1 \text{ and } k < l \text{ even}, \\
A(p^k, p^l) & \text{otherwise.}
\end{cases} \]

In the former case, we have:

(1) \[ A(dp^k, p^l) = 0 = \chi_{dp^k}(p^{l/2})a(dp^k, p^l) + \chi_{dp^k}(p^{l/2})a(dp^k, p^{l-1}). \]

In the latter case, we also have:

(2) \[ A(dp^k, p^l) = A(p^k, p^l) = \chi_{dp^k}(p^{l/2})a(dp^k, p^l) + \chi_{dp^k}(p^{l-1})a(dp^k, p^{l-1}). \]
Now let $D$ be an arbitrary odd integer. Given a prime integer $p$, write $D = D_0p^k$, where $D_0$ is prime to $p$. Then from the equality (1) and (2) with $d$ replaced by $D_0$, it follows that

$$\sum_{l=0}^{\infty} A(D, p^l)p^{-ls} = (1 - p^{-s})^{-1} \sum_{l=0}^{\infty} \chi_D(p^l)\alpha(D, p^l)p^{-ls}.$$ 

For $p = 2$, we define $\tilde{P}_2(D, s)$ by equating

$$\sum_{l=0}^{\infty} \frac{A(D, 2^{l+2})}{2^ls} = \tilde{P}_2(D, s)(1 - 2^{-s})^{-1} \sum_{l=0}^{\infty} \frac{\chi_D(2l)}{2^ls} = \tilde{P}_2(D, s)(1 - 2^{-s})^{-1} \sum_{l=0}^{\infty} \chi_D(2l)\alpha(D, 2^l).$$ 

By the multiplicative property of $A(D, \cdot)$ and that of $\chi_D(\cdot)\alpha(D, \cdot)$,

$$\sum_{m>0} \frac{A(D, 4m)}{m^s} = \tilde{P}_2(D, s)\zeta(s) \sum_{m>0} \chi_D(m)\alpha(D, m) \frac{1}{m^s}.$$ 

It remains to compute $\tilde{P}_2$. We need the next lemma.

**Lemma 2.6.** Let $D$ be an odd integer. With $\tilde{P}_2(D, s)$ defined in the above proposition, we have

$$\tilde{P}_2(D, s) = \begin{cases} 2(1 - 2^{-s})(1 + 2^{-s}) & D \equiv 1 \pmod{4}, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** Write

$$\sum_{l=0}^{\infty} \frac{A(D, 2^{l+2})}{2^ls} = A(D, 4) + \sum_{l=1}^{\infty} \frac{A(D, 2^{l+2})}{2^ls},$$

and note that

$$A(D, 4) = \begin{cases} 2 & D \equiv 1 \text{ or } 5 \pmod{8}, \\ 0 & \text{otherwise}. \end{cases}$$

To simplifying the second term, note that if $D$ is an odd integer and $m = 8, 16$, or some higher power of 2, then $D$ is a quadratic residue modulo $m$ if and only if $D \equiv 1 \pmod{8}$, therefore for $l \geq 1$,

$$A(D, 2^{l+2}) = \begin{cases} 4 & D \equiv 1 \pmod{8}, \\ 0 & \text{otherwise}. \end{cases}$$

Also note that

$$\chi_D(2) = \begin{cases} 1 & D \equiv 1 \pmod{8}, \\ -1 & D \equiv 5 \pmod{8}, \\ 0 & \text{otherwise}. \end{cases}$$

Then direct computation gives the results. \hfill \Box

This finishes the proof of the proposition. \hfill \Box

**Theorem 2.7.** The Shintani zeta function $Z_{Shintani}^{odd}(s, w)$ and the quadratic $A_2$-Weyl group multiple Dirichlet series $Z_{A_2}^{(1)}(s, w)$ satisfy the relation

$$Z_{Shintani}^{odd}(s, w) = 2(1 - 2^{-s})(1 + 2^{-s})\zeta(s)Z_{A_2}^{(1)}(s, w).$$
3. Bhargava Integer Cubes and Zeta Functions

3.1. Prehomogeneous vector space of parabolic subgroups. Let $V_{\mathbb{Z}}$ be the $\mathbb{Z}$-module of $2 \times 2 \times 2$ integer matrices, which we also call Bhargava integer cubes. There are three ways to form pairs of matrices by taking the opposite sides out of 6 sides. Denote them by

$$A^P = M_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; A^Q = N_1 = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

$$A^L = M_2 = \begin{pmatrix} a & c \\ e & g \end{pmatrix}; A^R = N_2 = \begin{pmatrix} b & d \\ f & h \end{pmatrix},$$

$$A^U = M_3 = \begin{pmatrix} a & e \\ b & f \end{pmatrix}; A^O = N_3 = \begin{pmatrix} c & g \\ d & h \end{pmatrix}.$$

For each pair $(M_i, N_i)$ we can associate to it a binary quadratic forms by taking

$$Q_i(u, v) = \det(M_i u - N_i v).$$

Explicitly for $A$ as above,

$$Q_1(u, v) = u^2(ad - bc) + uv(-ah + bg + cf - de) + v^2(ce - fg),$$

$$Q_2(u, v) = u^2(ag - ce) + uv(-ah - bg + cf + de) + v^2(bh - df),$$

$$Q_3(u, v) = u^2(af - be) + uv(-ah + bg - cf + de) + v^2(ch - dg).$$

We call $A$ projective if the associated binary quadratic forms are all primitive. The action of $G_{\mathbb{Z}} = \text{GL}_2(\mathbb{Z}) \times \text{GL}_2(\mathbb{Z}) \times \text{GL}_2(\mathbb{Z})$ on $V_{\mathbb{Z}}$ is defined by letting the $g_i$ in $(g_1, g_2, g_3) \in G_{\mathbb{Z}}$ act on the matrix pair $(M_i, N_i)$. It is easy to check that the actions of the three components commute with each other, thereby giving an action of the product group. For example, if $g_1 = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$, then it acts on the pair $(M_1, N_1)$ by

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \cdot \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}.$$  

The action extends to the complex group $G_{\mathbb{C}} = \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})$ on the complex vector space $V_{\mathbb{C}}$. We denote by $(G_{\mathbb{C}}, V_{\mathbb{C}})$ a complex vector space acted on by a (connected) complex group. In our setting, the $(G_{\mathbb{C}}, V_{\mathbb{C}})$ is a PHVS which refers to the $D_4$ case discussed in [WY92]. Now we consider the Borel subgroup $B'_2(\mathbb{C}) \subset G_{\mathbb{C}}$ consisting of the lower-triangular matrices. The action of $B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C})$ induced from $G_{\mathbb{C}}$ on the vector space $V_{\mathbb{C}}$ has three relative invariants, explicitly for $A \in V_{\mathbb{C}}$, given as follows

$$D(A) = \text{disc}(A) = (-ah + bg + cf - de)^2 - 4(ad - be)(eh - fg),$$

$$m(A) = \det(A^P) = ad - bc,$$

$$n(A) = \det(A^L) = ag - ce.$$  

Furthermore, we can show that

**Proposition 3.1.** The pair $(B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times \text{GL}_2(\mathbb{C}), V_{\mathbb{C}})$ is a prehomogeneous vector space.

**Proof.** Let $\mathcal{H}$ be the hypersurface in $V_{\mathbb{C}}$ defined as the zero locus of the single equation

$$\text{disc}(A) \det(A^P) \det(A^L) = 0.$$
Any $A \in V_C H$ is $B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times GL_2(\mathbb{C})$ equivalent to some element with the form
\[(A^F, A^B) = \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \begin{pmatrix} 0 & f \\ g & h \end{pmatrix} \right),\]

where $a, d, g, f \neq 0$. Furthermore, we can show that they are all in the single orbit of
\[(A^F, A^B) = \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).\]

This follows from finding solutions to the following system of equations, and then taking the proper scaling:
\[
\begin{align*}
\lambda_{i1}f + \lambda_{2i}a + \lambda_{32}d &= 0, \\
\lambda_{13}a + \lambda_{22}f + \lambda_{32}g &= 0, \\
\lambda_{12}g + \lambda_{22}d + \lambda_{33}a &= 0, \\
\lambda_{13}d + \lambda_{23}g + \lambda_{33}d &= -h - \lambda_{14}h - \lambda_{24}h - \lambda_{34}h,
\end{align*}
\]

where $\left( \begin{pmatrix} \lambda_{i1} & \lambda_{i2} \\ \lambda_{i3} & \lambda_{i4} \end{pmatrix} \right)$ is in the $i$-th place of $B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times GL_2(\mathbb{C})$. So $(B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times GL_2(\mathbb{C}), V_C)$ is a prehomogeneous vector space. \(\square\)

Let $B'_2(\mathbb{Z})$ be the Borel subgroup of $B'_2(\mathbb{C})$ in $SL_2(\mathbb{Z})$ with positive diagonal elements. The Shintani zeta function associated to the prehomogeneous vector space $(B'_2(\mathbb{C}) \times B'_2(\mathbb{C}) \times GL_2(\mathbb{C}), V_C)$ is defined to be

\[(3)\]
\[
Z_{\text{Shintani}}(s_1, s_2, w) = \sum_{A \in B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \setminus V'^*_k} \frac{1}{[\text{disc}(A)]^{w} |\det(A^F)|^{s_1}|\det(A^L)|^{s_2}},
\]

where $V'^*_k$ is the subset of semi-stable points of $V_2$ consisting of those orbits on which none of the three relative invariants vanishes. Denote by $\text{Stab}(A)$ the stabilizer group in $B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ of the cube $A$. We need to show that the order $|\text{Stab}(A)|$ is finite.

**Proposition 3.2.** For any $A \in V'^*_k$, the stabilizer group $\text{Stab}(A)$ in $B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ is:
\[
\mathbb{Z}_2 \cong \{(I_2, I_2, I_2)\},
\]

where $I_2$ is the $2 \times 2$ identity matrix. Therefore
\[
|\text{Stab}(A)| = 1
\]

for any $A \in V'^*_k$.

**Proof.** For a given $2 \times 2 \times 2$ integer cube $A \in V'^*_k$, we write
\[
Q_1(A) = mu^2 + xuv + sv^2,
\]
\[
Q_2(A) = nu^2 + yuv + tv^2
\]
to be first two binary quadratic forms associated to it. Under the action of $B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ we can change $A$ to another cube satisfying
\[
c(A) = 0 \text{ and } 0 \leq x \leq 2|m| - 1 \text{ and } 0 \leq y \leq 2|n| - 1.
\]
For such $A$, note that the entries $a(A), d(A), g(A) \neq 0$. Therefore the stabilizer group in $B_2^\prime(Z) \times B_2^\prime(Z) \times SL_2(Z)$ must have the form
\[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
\pm 1 & 0 \\
0 & \pm 1
\end{pmatrix}.
\]
Now it becomes obvious that the diagonal elements in the last matrix have to be both positive. □

We can rewrite the Shintani zeta function as
\[
Z_{\text{Shintani}}(s_1, s_2, w) = \sum_{D \neq 0} \frac{1}{|D|^w} \sum_{m, n > 0} \frac{B(D, m, n)}{m^{s_1} n^{s_2}},
\]
where
\[
B(D, m, n) = \sharp \{ A \in V_2^{ss}/\sim: \text{disc}(A) = D, |\det(A^F)| = m, |\det(A^L)| = n \}.
\]

3.2. Reduction theory. In this section, we will consider the geometry of integer orbits of the PHVS under the parabolic group action. This will help to reduce the sum over semi-stable orbits of $2 \times 2 \times 2$ integer cubes in the Shintani zeta function $Z_{\text{Shintani}}(s_1, s_2, w)$ to a sum over the tuples of integers $(D, m, n)$ satisfying certain relations. First we need a lemma which establishes the existence of a $2 \times 2 \times 2$ integer cube with the required arithmetic invariants.

**Lemma 3.3.** Let $D$, $m$ and $n$ be non-zero integers. For each solution $(x, y)$ to the congruence equations
\[
x^2 \equiv D \pmod{4m} \text{ for } 0 \leq x \leq 2|m| - 1,
\]
\[
y^2 \equiv D \pmod{4n} \text{ for } 0 \leq y \leq 2|n| - 1,
\]
there exists a $2 \times 2 \times 2$ integer cube $A$ such that
\[
\text{disc}(A) = D,
\]
\[
Q_1(A)(u, v) = mu^2 + xuv + sv^2,
\]
\[
Q_2(A)(u, v) = nu^2 + yuv + tv^2.
\]
Moreover, the required $2 \times 2 \times 2$ integer cube $A$ can be chosen such that in the bottom side $A^D$ of $A$
\[
c = 0 \text{ and } \text{g.c.d.}(d, g, h) = 1.
\]

**Proof.** If there are solutions to the congruence equations, we have
\[
D = x^2 - 4ms = y^2 - 4nt
\]
for some integers $s$ and $t$. It implies that $D$ is congruent to 0 or 1 (mod 4), and the integers $x, y$ have the same parity. Take
\[
a = |\text{g.c.d.}(m, n, \frac{x + y}{2})|,
\]
from
\[
\frac{x - y}{2} \cdot \frac{x + y}{2} = ms - nt,
\]
it follows that
\[
\text{g.c.d.}(\frac{m}{a}, \frac{n}{a}) | \frac{x - y}{2}.
\]
Set
\[ d = \frac{m}{a}, \quad g = \frac{n}{a}, \quad \text{and} \quad h = -\frac{x + y}{2a}, \]
then
\[ \text{g.c.d.}(d,g,h) = 1, \]
and (5) can be written as
\[ \frac{x - y}{2} \cdot (-h) = ds - gt. \]

We claim that there is an integer \( f \), such that
\[ s + fg \equiv 0 \pmod{h}, \]
\[ t + fd \equiv 0 \pmod{h}. \]
This can be proved as follows: First if \( h = 0 \), then \( ds = gt \). As in this case \( \text{g.c.d.}(d,g) = 1 \), we conclude that there exists such an integer \( f \) such that \( s = -fg \) and \( t = -fd \). If \( h \neq 0 \), for any prime divisor \( p \) of \( h \), we have
\[ p \mid ds - gt \quad \text{and} \quad \text{g.c.d.}(d,g,p) = 1, \]
it follows that there is a unique solution (mod \( p \)) to the congruences
\[ s + fg \equiv 0 \pmod{p}, \]
\[ t + fd \equiv 0 \pmod{p}. \]
Using the Chinese remainder theorem, we conclude that there are integers \( b \) and \( e \) such that
\[ s = eh - fg, \]
\[ t = bh - fd. \]
It follows that
\[ ds - gt = (de - bg) \cdot h, \]
combined with (7), we have
\[ \frac{x - y}{2} = bg - de. \]
If in the case of \( h = 0 \), from (6), we know that there always exists integers \( b \) and \( e \) such that the above equation holds.

Now we define a \( 2 \times 2 \times 2 \) integer cube \( A \) by
\[ (A^F, A^B) = \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right). \]
Then the associated two binary quadratic forms are
\[ Q_1(A)(u, v) = adu^2 + (-ah + bg - de) uv + (eh - fg)v^2 = mu^2 + xuv + sv^2, \]
\[ Q_2(A)(u, v) = agu^2 + (-ah - bg + de) uv + (bh - df)v^2 = nu^2 + yuv + tv^2. \]
So \( A \) is the cube required. In particular, we have showed that \( \text{g.c.d.}(d,g,h) = 1. \) □

We next want to show that under the assumption that \( D \) is square-free, the data \( (D, m, n, x, y) \) uniquely determines a \( B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \)-orbit.
Proposition 3.4. Let $D$ be a non-zero square-free integer and $m$, $n$ be non-zero integers. Each solution to the congruence equations
\begin{align*}
x^2 &\equiv D \pmod{4m} \text{ for } 0 \leq x \leq 2|m| - 1, \\
y^2 &\equiv D \pmod{4n} \text{ for } 0 \leq y \leq 2|n| - 1,
\end{align*}
determines a unique orbit of integer cubes in $B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \backslash V^*_Z$ such that
\begin{align*}
\text{disc}(A) &= D, \\
Q_1(A)(u,v) &= mu^2 + xuv + sv^2, \\
Q_2(A)(u,v) &= nu^2 + yuv + ty^2.
\end{align*}
Moreover, two different solutions to the congruence equations correspond to two different orbits of integer cubes in $B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \backslash V^*_Z$.

Proof. From the above lemma, we know that there always exists such a $2 \times 2 \times 2$ integer cube $A$. We want to show that such an $2 \times 2 \times 2$ integer cube $A$ is uniquely determined by the data $(D, m, n, x, y)$ satisfying the congruence equations.

Denote $A$ by the pair of matrices
\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix}. \]

We first show that the $(a, d, g, h)$ is uniquely determined up to the sign of $a$. As $\{1\} \times \{1\} \times SL_2(\mathbb{Z}) \subset B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$, we can assume that $c(A) = 0$. Then we have equations
\begin{align*}
bg - ah - ed &= x, \\
ed - ah - bg &= y, \\
ad &= m, \\
ag &= n.
\end{align*}

after adding the first two equations,
\begin{align*}
ah &= -(x + y)/2, \\
ad &= m, \\
ag &= n.
\end{align*}

Therefore we have
\[ a \times \text{g.c.d.}(h, d, g) = \text{g.c.d.}((x + y)/2, m, n). \]

As
\[ D = (bg - ah - ed)^2 - 4ad(eh - fg), \]
$D$ is square-free implies that $\text{g.c.d.}(h, d, g) = 1$, therefore $a = |\text{g.c.d.}((x + y)/2, m, n)|$ as we can make $a$ to be positive.

We next show that for two $2 \times 2 \times 2$ integer cubes with $c = 0$ and fixed discriminant $D$ such that they have the same tuples $(a, d, g, h)$ and $(Q_1(u, v), Q_2(u, v))$, then they are equal up to the action of $\{1\} \times \{1\} \times B_2(\mathbb{Z})$, the third one is an upper-triangular matrix in $SL_2(\mathbb{Z})$. Here in this statement we again require $D$ to be square-free.
We already showed that for a given $A$,
\[
Q_1(A)(u, v) = mu^2 + xuv + sv^2 = adu^2 + (bg - ah - ed)uv + (eh - fg)v^2,
\]
\[
Q_2(A)(u, v) = nu^2 + yuv + tv^2 = agu^2 + (ed - ah - bg)uv + (bh - df)v^2.
\]
As $(Q_1(A_1)(u, v), Q_2(A_1)(u, v))$ are the same for two integer cubes by assumption, we can make the following equations by setting the coefficients are equal
\[
x(A_1) = x(A_2) : b_1g - ah - e_1d = b_2g - ah - e_2d,
\]
\[
y(A_1) = y(A_2) : e_1d - ah - b_1g = e_2d - ah - b_2g,
\]
\[
s(A_1) = s(A_2) : e_1h - f_1g = e_2h - f_2g,
\]
\[
t(A_1) = t(A_2) : h_1d = b_2h - df_2.
\]
Taking the subtraction of right side from the left side on each equation, we have:
\[
\Delta(b)g - \Delta(e)d = 0,
\]
\[
\Delta(e)h - \Delta(f)g = 0,
\]
\[
\Delta(b)h - \Delta(f)d = 0,
\]
here $\Delta(b) = b_1 - b_2$. Then we have solutions to the above equation system:
\[
\Delta(e); \Delta(b) = \Delta(e)d/g; \Delta(f) = \Delta(e)h/g.
\]
Under the assumption of $D$ being square-free, we have shown that $g.c.d.(d, g, h) = 1$. Therefore $g|\Delta(e)$, i.e., the solution has form
\[
\Delta(e) = gk; \Delta(b) = dk; \Delta(f) = hk.
\]
This implies that we can make $A_1$ equivalent to $A_2$ under the action of $\{1\} \times \{1\} \times B_2(\mathbb{Z})$.

Now we prove the second part of the proposition by contradiction. Given a square-free integer $D$ and two non-zero integers $m$ and $n$, if $(x_1, y_1)$ and $(x_2, y_2)$ are two different solutions to the congruence equations in the proposition. Let $A_1$ and $A_2$ be the two corresponding $2 \times 2 \times 2$ integer cubes satisfying $\text{disc}(A_i) = D$, and
\[
Q_1(A_i)(u, v) = mu^2 + x_iuv + s_iu^2,
\]
\[
Q_2(A_i)(u, v) = nu^2 + y_iuv + t_iu^2.
\]
Suppose that $A_1$ and $A_2$ are $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times SL_2(\mathbb{Z})$ equivalent. We assume $c(A_1) = c(A_2) = 0$ as before. Since $0 \leq x_i \leq 2m - 1$ and $0 \leq y_i \leq 2n - 1$, the group action actually belongs to $\{1\} \times \{1\} \times B_2(\mathbb{Z})$, which implies that $(Q_1(A_i)(u, v), Q_2(A_i)(u, v))$ are the same. In particular $x_1 = x_2$ and $y_1 = y_2$, contradiction. So $A_1$ and $A_2$ are not $B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times SL_2(\mathbb{Z})$ equivalent.

Now we turn to the general case without assuming square-free discriminant. First we consider the non-zero integer $D = D_0p^2$, where $D_0$ is square-free.

**Proposition 3.5.** Let $m$ and $n$ be non-zero integers, and $D = D_0p^2$ where $D_0$ is square-free and $p$ is a prime integer. The orbits counting number
\[
B(D, m, n) = A(D, 4m) A(D, 4n) + b(D, \frac{m}{p}, \frac{n}{p}).
\]
where the second term
\[ b(D_0, \frac{m}{p}, \frac{n}{p}) = \begin{cases} pA(D_0, \frac{4m}{p})A(D_0, \frac{4n}{p}) & \text{if } p \text{ divides } \gcd(D,m,n), \\ 0 & \text{otherwise}. \end{cases} \]

Proof. By the existence Lemma 3.3, we can find an \(2 \times 2 \times 2\) integer cube \(A\) satisfying
\[ c = 0 \text{ and } \gcd(d,g,h) = 1, \]
such that
\[ \text{disc}(A) = D, \quad \det(A^p) = m \quad \text{and} \quad \det(A^L) = n. \]

From the proof of the last proposition, the condition of \(\gcd(d,g,h) = 1\) implies that the integer cube \(A\) determines a unique orbit in \(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \SL_2(\mathbb{Z}) \setminus V_{2}^s(D)\).

We denote by \(V_{2}^s(D)\) the semi-stable subset of \(V_{2}\) with discriminant \(D\).

If \(p | \gcd(D,m,n)\), we write \(m_0 = \frac{m}{p}\), \(n_0 = \frac{n}{p}\). Applying the existence Lemma 3.3 to the non-zero integers \(D_0, m_0, n_0\), we can find an \(2 \times 2 \times 2\) integer cube \(A_0\) satisfying
\[ c_0 = 0 \text{ and } \gcd(d_0, g_0, h_0) = 1, \]
such that
\[ \text{disc}(A_0) = D_0, \quad \det(A_0^p) = m_0 \quad \text{and} \quad \det(A_0^L) = n_0. \]

The condition \(\gcd(d_0, g_0, h_0) = 1\) implies that \(A_0\) determines a unique orbit in \(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \SL_2(\mathbb{Z}) \setminus V_{2}^s(D_0)\).

For each action of \(g\) in \(\{(1) \times (1) \times \left( \begin{array}{cc} 1 & j \\ 0 & p \end{array} \right) : 0 \leq j \leq p - 1 \}\) on \(A_0\), the \(2 \times 2 \times 2\) integer cube \(g \cdot A_0\) determines uniquely the orbit in \(B_2'(\mathbb{Z}) \times B_2'(\mathbb{Z}) \times \SL_2(\mathbb{Z}) \setminus V_{2}^s(D)\).

\(\square\)

Corollary 3.6. If \(D\) is a fundamental discriminant, then we have
\[ B(D, m, n) = A(D, 4m)A(D, 4n). \]

Proof. If \(D\) is square-free then this follows from the above proposition we proved. Otherwise \(D = 4D_0\) where \(D_0\) is square-free, using the above proposition applied to the case \(p = 2\), we have
\[ B(D, m, n) = A(D, 4m)A(D, 4n) + b(D_0, \frac{m}{2}, \frac{n}{2}). \]

The second term is 0 as the discriminant of a quadratic form should be \(D_0 = f^2d_K\) where \(d_K\) is a fundamental discriminant, but \(D\) is already a fundamental discriminant. \(\square\)

Proposition 3.7. Let \(m\) and \(n\) be non-zero integers, and \(D = D_0p^{2k}\) where \(D_0\) is square-free and \(p\) is a prime integer. The orbits counting number
\[ B(D, m, n) = A(D, 4m)A(D, 4n) + b(D, \frac{m}{p^i}, \frac{n}{p^i}) + \ldots + b\left(\frac{D}{p^{2i}}, \frac{m}{p^{2i}}, \frac{n}{p^{2i}}\right) + \ldots + b(D_0, \frac{m}{p^i}, \frac{n}{p^i}), \]
for some \(1 \leq i \leq k,
\[ b\left(\frac{D}{p^{2i}}, \frac{m}{p^{2i}}, \frac{n}{p^{2i}}\right) = \begin{cases} p^iA\left(\frac{D}{p^{2i}}, \frac{4m}{p^{2i}}\right)A\left(\frac{D}{p^{2i}}, \frac{4n}{p^{2i}}\right) & \text{if } p^i \text{ divides } \gcd(D, m, n), \\ 0 & \text{otherwise}. \end{cases} \]
Proof. By the existence Lemma 3.3, we can find an $2 \times 2$ integer cube $A$ satisfying
\[ c = 0 \text{ and } \text{g.c.d.}(d, g, h) = 1, \]
such that
\[ \text{disc}(A) = D, \quad \text{det}(A^F) = m \text{ and } \text{det}(A^L) = n. \]
The condition of $\text{g.c.d.}(d, g, h) = 1$ implies that the integer cube $A$ determines a unique orbit in $B_2^d(\mathbb{Z}) \times B_2^d(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \backslash V_2^d(D)$.

If $p^1 | \text{g.c.d.}(D, m, n)$, we write $m_i = \frac{m}{p^1}$, $n_i = \frac{n}{p^1}$. Applying the existence Lemma 3.3 to the non-zero integers $\frac{D}{p^{2i}}, \frac{m}{p^1}$, and $\frac{n}{p^1}$, we can find an $2 \times 2 \times 2$ integer cube $A_i$ satisfying
\[ c_i = 0 \text{ and } \text{g.c.d.}(d_i, g_i, h_i) = 1, \]
such that
\[ \text{disc}(A_i) = \frac{D}{p^{2i}}, \quad \text{det}(A_i^F) = \frac{m}{p^i} \text{ and } \text{det}(A_i^L) = \frac{n}{p^i}. \]
The condition $\text{g.c.d.}(d_i, g_i, h_i) = 1$ implies that $A_i$ determines a unique orbit in $B_2^d(\mathbb{Z}) \times B_2^d(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \backslash V_2^d(D)$.

For each action of $g$ in $\{(1) \times (1) \times \left( \begin{array}{cc} 1 & j \\ 0 & p^i \end{array} \right) : 0 \leq j \leq p^i - 1\}$ on $A_i$, the $2 \times 2 \times 2$ integer cube $g \cdot A_i$ determines uniquely the orbit in $B_2^d(\mathbb{Z}) \times B_2^d(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \backslash V_2^d(D)$.

In summary we have the full general formula for the orbits counting number $B(D, m, n)$.

Proposition 3.8. Let $m$ and $n$ be non-zero integers, and $D = D_0^2 \cdot D_1^2$ where $D_0$ is square-free. The orbits counting number
\begin{equation}
B(D, m, n) = \sum_{d|D_1} b \left( \frac{D}{d^2}, \frac{m}{d}, \frac{n}{d} \right),
\end{equation}
where
\[ b \left( \frac{D}{d^2}, \frac{m}{d}, \frac{n}{d} \right) = \begin{cases} d \cdot A \left( \frac{D}{d^2}, \frac{4m}{d}, \frac{4n}{d} \right) & \text{if } d \text{ divides } \text{g.c.d.}(D_1, m, n), \\ 0 & \text{otherwise.} \end{cases} \]

Proof. By the existence Lemma 3.3, we can find an $2 \times 2 \times 2$ integer cube $A$ satisfying
\[ c = 0 \text{ and } \text{g.c.d.}(d, g, h) = 1, \]
such that
\[ \text{disc}(A) = D, \quad \text{det}(A^F) = m \text{ and } \text{det}(A^L) = n. \]
The condition of $\text{g.c.d.}(d, g, h) = 1$ implies that the integer cube $A$ determines a unique orbit in $B_2^d(\mathbb{Z}) \times B_2^d(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \backslash V_2^d(D)$.

If $d | \text{g.c.d.}(D_1, m, n)$, we write $m' = \frac{m}{d}$, $n' = \frac{n}{d}$. Applying the existence Lemma 3.3 to the non-zero integers $\frac{D}{d^2}, \frac{m}{d}$, and $\frac{n}{d}$, we can find an $2 \times 2 \times 2$ integer cube $A'$ satisfying
\[ c' = 0 \text{ and } \text{g.c.d.}(d', g', h') = 1, \]
such that
\[ \text{disc}(A') = \frac{D}{d^2}, \quad \det((A')^F) = \frac{m}{d} \quad \text{and} \quad \det((A')^L) = \frac{n}{d}. \]

The condition \( g.c.d.(d', g', h') = 1 \) implies that \( A' \) determines a unique orbit in \( B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \setminus V^*_2(\mathbb{Z}) \).

For each action of \( g \) in \( \{(1) \times (1) \times \begin{pmatrix} 1 & j \\ 0 & d \end{pmatrix} : 0 \leq j \leq d-1 \} \) on \( A' \), the \( 2 \times 2 \times 2 \) integer cube \( g \cdot A' \) determines uniquely the orbit in \( B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \setminus V^*_2(\mathbb{Z}) \). \( \square \)

Finally we obtain the explicit formula for the Shintani zeta function associated to the PHVS of \( 2 \times 2 \times 2 \) cubes.

**Theorem 3.9.** The Shintani zeta function \( Z_{\text{Shintani}}(s_1, s_2, w) \) can be expressed as

\[
\sum_{A \in B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \setminus V^*_2} \frac{1}{|\text{disc}(A)|^w |\det(A^F)|^{s_1} |\det(A^L)|^{s_2}}
\]

\[
= \sum_{D=D_1D_2} \frac{1}{|D|^w} \sum_{m,n>0} \frac{d \cdot A(D, 4m/d)}{d \cdot A(D, 4n/d)} m^{s_1} n^{s_2}.
\]

In particular, if \( D \) is an odd integer, then (8) becomes

\[
B(D, m, n) = \sum_{d \mid D_1} d \cdot A(D, 4m/d) \cdot A(D, 4n/d),
\]

as when \( d \) is an odd integer it is a divisor of \( m \) if and only if it is a divisor of \( 4m \).

So we have the following

**Corollary 3.10.** The partial Shintani zeta function \( Z_{\text{Shintani}}^{\text{odd}}(s_1, s_2, w) \) defined by

\[
Z_{\text{Shintani}}^{\text{odd}}(s_1, s_2, w) = \sum_{A \in B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \setminus V^*_2} \frac{1}{|\text{disc}(A)|^w |\det(A^F)|^{s_1} |\det(A^L)|^{s_2}}
\]

can be written as

\[
\sum_{D=D_1D_2} \frac{1}{|D|^w} \sum_{m,n>0} \frac{d \cdot A(D, 4m/d)}{d \cdot A(D, 4n/d)} m^{s_1} n^{s_2}.
\]

4. **A\text{3} Weyl Group Multiple Dirichlet Series**

In this section, we will relate the Shintani zeta function \( Z_{\text{Shintani}}(s_1, s_2, w) \) to the quadratic \( A_3 \)-Weyl group multiple Dirichlet series. The idea is first to construct a multiple Dirichlet series \( Z_{\text{WMDS}}(s_1, s_2, w) \) and then show its relation to the Shintani zeta function of PHVS of \( 2 \times 2 \times 2 \) cubes, using the results we did for the relation between \( Z_{A_2}(s, w) \) and \( Z_{\text{Shintani}}(s, w; B'_2) \). Finally we show the multiple Dirichlet series \( Z_{\text{WMDS}}(s_1, s_2, w) \) is the desired \( A_3 \)-Weyl group multiple Dirichlet series by computing the generating function of its \( p \)-parts.

In the \( A_2 \)-WMDS

\[
Z_{A_2}(s, w) = \sum_{m \geq 0, D \text{ odd}} \frac{\chi_D(m) a(D, m)}{m^s |D|^w},
\]
we let \( \hat{A}(D, m) = \chi_D(\hat{m})a(D, m) \). Write \( D = D_0D_1^2 \) where \( D_0 \) stands for the square-free part of \( D \). Define

\[
Z_{WMDS}(s_1, s_2, w) = \sum_{D = D_0D_1^2 \neq 0 \atop D \text{ odd}} \frac{1}{|D|^w} \sum_{m, n > 0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1}n^{s_2}} a(D, m, n),
\]

where

\[
a(D, m, n) = \sum_{d \mid D_1} d \cdot a\left(\frac{D}{d^2}, \frac{m}{d}, \frac{n}{d}\right).
\]

**Lemma 4.1.** The multiple Dirichlet series \( Z_{WMDS}(s_1, s_2, w) \) can be expressed as

\[
Z_{WMDS}(s_1, s_2, w) = \sum_{D = D_0D_1^2 \neq 0 \atop D \text{ odd}} \frac{1}{|D|^w} \sum_{m, n > 0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1}n^{s_2}} a(D, m, n)
\]

\[
= \sum_{D = D_0D_1^2 \neq 0 \atop D \text{ odd}} \frac{1}{|D|^w} \sum_{d \mid D_1} d \cdot \hat{A}(\frac{D}{d^2}, \frac{m}{d}, \frac{n}{d}) \cdot \hat{A}(\frac{D}{d^2}, \frac{m}{d}, \frac{n}{d}).
\]

**Proof.** This follows from the fact that the quadratic character \( \chi_d(\cdot) \) is the same as \( \chi_{D/d^2}(\cdot) \) by definition, and the factor of \( m \) prime to \( D \) is the same as the factor of \( \frac{m}{d} \) prime to \( \frac{D}{d^2} \). Therefore \( \chi_D(\hat{m})\chi_D(\hat{n}) \) is a common factor for fixed integers \( D, m \) and \( n \).

We also define the partial sum of \( Z_{WMDS}(s_1, s_2, w) \) by

\[
Z^{(1)}_{WMDS}(s_1, s_2, w) = \sum_{D = D_0D_1^2 \neq 0 \atop D \equiv 1 \mod 4} \frac{1}{|D|^w} \sum_{m, n > 0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1}n^{s_2}} a(D, m, n).
\]

As we did in the Proposition 2.5, we will show the relation between the inner sum of \( Z^{(1)}_{WMDS}(s_1, s_2, w) \) and \( Z_{WMDS}(s_1, s_2, w) \).

**Proposition 4.2.** Let \( D \) be an odd integer. The inner sum of the Shintani zeta function \( Z^{odd}_{\text{Shintani}}(s_1, s_2, w) \) can be expressed by

\[
\sum_{m, n > 0} \frac{B(D, m, n)}{m^{s_1}n^{s_2}} = \hat{P}_2(D, s_1, s_2)\zeta(s_1)\zeta(s_2) \sum_{m, n > 0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1}n^{s_2}} a(D, m, n),
\]

where

\[
\hat{P}_2(D, s_1, s_2) = \hat{P}_2(D, s_1)\hat{P}_2(D, s_2)
\]

and \( \hat{P}_2(D, s) \) is defined in Proposition 2.5.

**Proof.** Recall that in the proof of Proposition 2.5, we have shown that

\[
\sum_{m > 0} \frac{A(D, 4m)}{m^s} = \hat{P}_2(D, s)\zeta(s) \sum_{m > 0} \frac{\chi_D(\hat{m})a(D, m)}{m^s}.
\]

Therefore,

\[
\sum_{m, n > 0} \frac{A(D, 4m)A(D, 4n)}{m^{s_1}n^{s_2}} = \hat{P}_2(D, s_1)\hat{P}_2(D, s_2)\zeta(s_1)\zeta(s_2)
\]

\[
\cdot \sum_{m, n > 0} \frac{\chi_D(\hat{m})\chi_D(\hat{n})}{m^{s_1}n^{s_2}} a(D, m)a(D, n).
\]
In particular for any $d^2 | D$, replace $D$ by $\frac{D}{d^2}$, $m$ by $\frac{m}{d}$ and $n$ by $\frac{n}{d}$, we have
\[
\sum_{m,n > 0} A\left(\frac{D}{d^2}, \frac{4m}{d}\right) A\left(\frac{D}{d^2}, \frac{4n}{d}\right) = \hat{P}_2\left(\frac{D}{d^2}, s_1\right) \hat{P}_2\left(\frac{D}{d^2}, s_2\right) \zeta(s_1) \zeta(s_2)
\cdot \sum_{m,n > 0} \frac{\chi_D(m) \chi_D(n)}{m^{s_1} n^{s_2}} a\left(\frac{D}{d^2}, \frac{m}{d}\right) a\left(\frac{D}{d^2}, \frac{n}{d}\right).
\]
As $D$ is odd and $D \equiv D/d^2 \pmod{8}$, by the definition of $\hat{P}_2(D, s)$, it follows that
\[
\hat{P}_2(D, s) = \hat{P}_2\left(\frac{D}{d^2}, s\right).
\]
Finally, taking the sum over all $d$ with $d^2 | D$, we have
\[
\sum_{m,n > 0} \sum_{d^2 | D} d \cdot A\left(\frac{D}{d^2}, \frac{4m}{d}\right) A\left(\frac{D}{d^2}, \frac{4n}{d}\right) = \hat{P}_2(D, s_1) \hat{P}_2(D, s_2) \zeta(s_1) \zeta(s_2)
\cdot \sum_{m,n > 0} \frac{\chi_D(m) \chi_D(n)}{m^{s_1} n^{s_2}} a(D, m, n).
\]

Recall the definition of $\hat{P}_2(D, s)$,
\[
\hat{P}_2(D, s) = \begin{cases} 
2(1 - 2^{-s})(1 + 2^{-s}) & D \equiv 1 \pmod{4}, \\
0 & \text{otherwise.}
\end{cases}
\]

Now we can give an explicit relation between $Z_{\text{Shintani}}(s_1, s_2, w)$ and $Z_{\text{WMDS}}(s_1, s_2, w)$. We further relate the Shintani zeta function $Z_{\text{Shintani}}(s_1, s_2, w)$ to a Weyl group multiple Dirichlet series by showing that $Z_{\text{WMDS}}(s_1, s_2, w)$ is a quadratic $A_3$-WMDS.

**Theorem 4.3.** The Shintani zeta function of PHVS of $2 \times 2 \times 2$ cubes can be related to the multiple Dirichlet series $Z_{\text{WMDS}}(s_1, s_2, w)$ by

\[
Z_{\text{Shintani}}^{\text{odd}}(s_1, s_2, w) = 4(1 - 2^{-s_1})(1 + 2^{-s_1})(1 - 2^{-s_2})(1 + 2^{-s_2}) \zeta(s_1) \zeta(s_2) 
\cdot Z_{\text{WMDS}}^{(1)}(s_1, s_2, w).
\]

**Theorem 4.4.** $Z_{\text{WMDS}}(s_1, s_2, w)$ is a quadratic $A_3$-Weyl group multiple Dirichlet series.

**Proof.** Consider the $p$-parts of our $a(D, m, n)$ defined by
\[
a_{kl}(p) = a(p^k, p^l, p^l).
\]
Explicit from its definition,
\[
a_{kl}(p) = a(p^k, p^l,a(p^k, p^l) + pa(p^{k-2}, p^{l-1})a(p^{k-2}, p^{l-1}) + \cdots.
\]
In [CG07] the author developed a systematical way to construct the Weyl group multiple Dirichlet series. The idea is to construct a rational function invariant under the Weyl group action. In the case of root system of $A_3$ type, the $p$-parts of the rational function is given by
\[
f_{A_3}(x, y, z) = \frac{(1 - xy - yz + xy + pxy^2z - px^2y^2z - pxy^2z^2 + px^2y^3z^2)}{(1 - x)(1 - y)(1 - z)(1 - py^2z^2)(1 - py^2z^2)(1 - px^2y^2z^2)}.
\]
Write the expansion
\[
f_{A_3}(x, y, z) = \sum b_{kl}(p)x^ky^lz^l,
\]
then we can compare our \( \{a_{klt}\} \) with \( \{b_{klt}\} \). They coincide with each other as follows: we know that for \(|x|, |y|, |z| < 1/p\) there is

\[
 f_{A_3}(x, y, z) = \frac{1}{1 - pxy^2z} \int f_{A_2}(x, t) f_{A_2}(yt^{-1}, z) \frac{dt}{t},
\]

where

\[
 f_{A_2}(x_1, x_2) = \sum_{k, l \geq 0} a(p^k, p^l)x_1^k x_2^l
\]

and the integral is taken over the circle \(|t| = 1/p\) ([CG07, Example 3.7]). Substituting the above expansion of \( f_{A_2} \) into \( f_{A_3} \), note that \( a(p^k, p^l) = a(p^l, p^k) \), we have

\[
 f_{A_3}(x, y, z) = \frac{1}{1 - pxy^2z} \sum_{k, l, t \geq 0} a(p^k, p^l)a(p^k, p^l)x^t y^k z^t
\]

\[
 = \sum_{s=0}^{\infty} p^s x^s y^{2s} z^s \sum_{k, l, t \geq 0} a(p^k, p^l)a(p^k, p^l)x^t y^k z^t
\]

\[
 = \sum_{k, l, t \geq 0} (a(p^k, p^l)a(p^k, p^l) + pa(p^k-2, p^l-1)a(p^k-2, p^l-1) + \cdots)x^t y^k z^t
\]

\[
 = \sum_{k, l, t \geq 0} a(p^k, p^l)a(p^k, p^l)x^t y^k z^t.
\]

Therefore, \( a_{klt} = b_{klt} \).

\[\Box\]

5. MODULI PARAMETRIZING IDEALS OF A QUADRATIC RING

We first recall the classical results in the theory of binary quadratic forms [Cox89]. Given a primitive integral binary quadratic form

\[ Q(x, y) = ax^2 + bxy + cy^2 \]

with discriminant \( D = b^2 - 4ac \) such that \( K = \mathbb{Q}(\sqrt{D}) \) a quadratic field, there is a canonical way to associate it a proper ideal \( I \) in the quadratic order \( R = R(D) \).

Let \( \tau \) be one of the two roots of the quadratic function \( Q(x, 1) = 0 \), then

\[ R = \langle 1, a\tau \rangle \text{ and } I = \langle a, a\tau \rangle. \]

If \( f \) is the conductor of the quadratic order \( R \), then we can express \( a\tau \) as:

\[ a\tau = \frac{b \mp f d_K}{2} \pm f w_K, \]

where \( w_K = \frac{d_K + \sqrt{d_K}}{2} \), and \( d_K \) is the fundamental discriminant of the quadratic field \( K \). It follows that \( R \) has a \( \mathbb{Z} \)-basis \([1, f w_K]\), which only depends on the discriminant \( D = f^2 d_K \) of \( R \). Notice that \( I \) is the ideal of \( R \) satisfying \( R/I \cong \mathbb{N}(I)\mathbb{Z} \), where the norm \( \mathbb{N}(I) = |a| \).

In order to extend the above construction to an arbitrary integral binary quadratic form

\[ Q(x, y) = ax^2 + bxy + cy^2 \]

with discriminant \( D = b^2 - 4ac \neq 0 \), we need to first recall the definition of oriented quadratic ring introduced in the paper [Bha04]. A quadratic ring is the commutative ring with unity whose underlying additive group is \( \mathbb{Z}^2 \). There is a unique automorphism for a quadratic ring \( R \). With the automorphism, we can
define the trace of an element \( x \in R \) by taking \( \text{Tr}(x) = x + x' \), where \( x' \) denotes the image of \( x \) under the automorphism. Alternatively, the trace function \( \text{Tr} : R \to \mathcal{Z} \) is defined as the trace of the endomorphism \( R \stackrel{x}{\to} R \). We also define the norm of an element \( x \in R \) by taking \( N(x) = x \cdot x' \). The discriminant \( \text{disc}(R) \) of \( R \) is defined to be the determinant \( \det(\text{Tr}(\alpha_i \alpha_j)) \) where \( \{\alpha_i\} \) is any \( \mathcal{Z}\)-basis of \( R \). As the \( \mathcal{Z}\)-basis of any quadratic \( R \) has the form \([1, \tau]\), where \( \tau \) satisfies the equation \( \tau^2 + \tau r + s = 0 \), the discriminant of \( R \) is given explicitly by \( \text{disc}(R) = r^2 - 4s \). Conversely, given any integer \( D \equiv 0 \) or \( 1 \) (mod \( 4 \)), there exists a unique quadratic ring \( R(D) \) with discriminant \( D \). Canonically, \( R(D) \) has a \( \mathcal{Z}\)-basis \([1, \tau_D]\), where \( \tau_D \) is determined by

\[
\tau_D^2 = \frac{D}{4} \quad \text{or} \quad \tau_D^2 = \frac{D - 1}{4} + \tau_D,
\]

in accordance to whether \( D \equiv 0 \) (mod \( 4 \)) or \( D \equiv 1 \) (mod \( 4 \)). We call \( R \) non-degenerate if \( \text{disc}(R) \neq 0 \). From now on, we only consider the case of non-degenerate quadratic rings.

For an integer \( D \neq 0 \), the quadratic ring \( R(D) \) has a unique non-trivial automorphism. The quadratic ring \( R(D) \) is oriented if we specify the choice of \( \tau_D \). For an oriented quadratic ring \( R(D) \), the specific choice of \( \tau \) in any \( \mathcal{Z}\)-basis \([1, \tau]\) is made such that the change-of-basis matrix from the basis \([1, \tau]\) to the canonical basis \([1, \tau_D]\) has positive determinant. We call such a basis \([1, \tau]\) positively oriented. For the rest of the section, we always assume that the quadratic ring \( R(D) \) is oriented with each \( \mathcal{Z}\)-basis \([1, \tau]\) positively oriented.

Finally, for a quadratic ring \( R \) with non-zero discriminant \( D \), we define for it the narrow class group \( \text{Cl}^+(R) \), the group of oriented ideal classes. Recall that an oriented ideal is the pair \((I, \epsilon)\), where \( I \) is a (fractional) ideal of \( R \) in \( K(R) = R \otimes \mathbb{Q} \), and \( \epsilon = \pm 1 \) gives the orientation of the ideal \( I \). For an element \( k \in K(R) \), the product \( k \cdot (I, \epsilon) \) is defined to be the oriented ideal \( (kI, \text{sgn}(N(k)) \epsilon) \). Two oriented ideal \((I_1, \epsilon_1)\) and \((I_2, \epsilon_2)\) belong to the same oriented ideal class if they satisfy \((I_1, \epsilon_1) = k \cdot (I_2, \epsilon_2)\). We will suppress \( \epsilon \) for the rest of the section and assume \( I \) always oriented. For an oriented ideal \( I \subset R \), the unoriented norm of \( I \) is defined to be \( N(I) = |R/I| \); while the oriented norm of \( I \) is denoted by \( \epsilon \cdot N(I) \).

Now for the binary quadratic form \((10)\), the oriented quadratic ring \( R \) is defined to be

\[
R = (1, \tau),
\]

where the choice of \( \tau \) is specified and it satisfies

\[
\tau^2 + b \tau + ac = 0.
\]

Further, the oriented ideal \( I \) is defined to be

\[
I = (a, \tau)
\]

with the orientation given by the ordered basis \([a, \tau]\). It is easy to see that \( I \) is the ideal contained in \( R \) with the norm \( N(I) = |a| \). If the binary quadratic form is primitive, i.e., \( \text{g.c.d.}(a,b,c) = 1 \), then the ideal \( I \) is proper, which means it has an inverse in the quadratic algebra \( K(R) \).

Let \( \text{SL}_2(\mathbb{Z}) \subset \text{SL}_2(\mathcal{Z}) \) be the subgroup of lower-triangular integer matrices with positive diagonal elements. Then we will show the following result which says the set of pairs \((R, I)\) with oriented ideal \( I \) with cyclic quotient in \( R \) can be parametrized
by the integer orbits of the PHVS of binary quadratic forms acted on by the Borel subgroup $B_2'(\mathbb{C})$.

**Proposition 5.1.** The natural map

$$B_2'(\mathbb{Z}) \setminus \{Q(u,v) = au^2 + buv + cv^2 : b^2 - 4ac \neq 0, a \neq 0\}$$

$$\rightarrow \text{Iso}\setminus\{(R, I) : R/I \cong N(I)\mathbb{Z}\}$$

defined above is a bijection. The isomorphism $f$ from the pair $(R_1, I_1)$ to another $(R_2, I_2)$ is defined to be the orientation-preserving isomorphism from $R_1$ to $R_2$ and sending $I_1$ to $I_2$.

**Proof.** From the construction above, the map is well defined. We first prove the surjectivity. Given an oriented quadratic ring $R$ with $\text{disc}(R) = D$ and an oriented ideal $I \subset R$ defined by:

$$R = \langle 1, \tau_D \rangle \text{ and } I = \langle \alpha, \beta \rangle,$$

where $\tau_D$ is defined in (11) and the orientation of $I$ is determined by the ordered basis $[\alpha, \beta]$, we can always assume that the norm $N(\alpha) = \alpha \cdot \alpha' \neq 0$. This is trivial in the number field case. To prove it in the general case, we define a binary quadratic form by

$$\text{(12)} \quad (\alpha \cdot \alpha')u^2 - (\alpha' \cdot \beta + \alpha \cdot \beta')uv + (\beta \cdot \beta')v^2,$$

then it is easy to see that the discriminant of (12) is exactly the discriminant of $I$ given by

$$\text{disc}(I) = \left( \det \begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \right)^2.$$ 

As

$$\text{disc}(I) = (N(I))^2 \cdot \text{disc}(R),$$

and $R$ is non-degenerate, it implies that $\text{disc}(I) \neq 0$. So at least one of the coefficients of $u^2$ and $v^2$ is non-zero. We can assume $\alpha \cdot \alpha' \neq 0$ by changing the order of $\alpha$ and $\beta$. As $\alpha, \beta \in I$, $N(\alpha) \mid N(I)$ and $N(\beta) \mid N(I)$, it follows that $N(I)$ is the common factor of all coefficients of (12). After canceling this common factor, we write it as

$$\text{(13)} \quad mu^2 + nuv + lv^2$$

with $D = n^2 - 4ml$. So the quadratic ring $R(D)$ can be also written as

$$R(D) = [1, \tau_1],$$

where the choice of $\tau_1$ is specified and it satisfies

$$\text{(14)} \quad \tau_1^2 + n\tau_1 + ml = 0.$$ 

We write

$$(\alpha, \beta) = (1, \tau_1) \begin{pmatrix} p & r \\ q & s \end{pmatrix}.$$ 

Substituting $u = \beta$ and $v = \alpha$ into (13), it becomes zero, by comparing it with the defining equation (14) of $\tau_1$, then

$$p = ms + nq \quad \text{or} \quad p = -ms \quad \text{and} \quad r = -ql.$$ 

As $I$ is an ideal with cyclic quotient, we must have $\text{g.c.d.}(q, s) = 1$. Then by the elementary divisor theorem, we can transform the matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ by a left multiplication in $B_2(\mathbb{Z})$ and a right multiplication in $\text{SL}_2(\mathbb{Z})$ to the matrix has the
form \( \begin{pmatrix} a & * \\ 0 & 1 \end{pmatrix} \) with \( a = \pm N(I) \). Therefore under a certain basis, \( R \) and \( I \) can be written as

\[
R(D) = \langle 1, \tau_2 \rangle \quad \text{and} \quad I = \langle a, \tau_2 \rangle,
\]

where the choice of \( \tau_2 \) is made such that the basis \([1, \tau_2]\) is positively oriented. With \( \alpha \) replaced by \( a \) and \( \beta \) replaced by \( \tau_2 \), by (12) there is a binary quadratic form. From the discussion above, all coefficients of it are divisible by \( |a| = N(I) \). After canceling this common factor, we have

\[
a u^2 + b u v + c v^2,
\]

with \( D = b^2 - 4ac \). This is the required binary quadratic form.

To prove the injectivity of the map, suppose that two binary quadratic forms

\[
Q_i(u, v) = a_i u^2 + b_i u v + c_i v^2,
\]

where \( a_1 = a_2 = a \neq 0 \) and \( D = b_i^2 - 4a_i c_i \neq 0 \) for \( i = 1, 2 \), have the same image. We want to prove that \( b_2 = b_1 + 2na \) and \( c_2 = n^2a + b_1n + c_1 \) for some integer \( n \). From the definition of the map, the oriented quadratic ring and the oriented ideal can be written as

\[
R_1 = \langle 1, \tau_1 \rangle, \quad I_1 = \langle a, \tau_1 \rangle,
\]

\[
R_2 = \langle 1, \tau_2 \rangle, \quad I_2 = \langle a, \tau_2 \rangle
\]

respectively, where the choice of \( \tau_i \)'s are made such that both bases \([1, \tau_1]\) and \([1, \tau_2]\) are positively oriented, and they satisfy

\[
\tau_i^2 + b_i \tau_i + ac_i = 0.
\]

As they have the same image, there exists an isomorphism \( f \) from \( R_1 \) to \( R_2 \) preserving the orientation, so it has the form:

\[
f(\tau_1) = \tau_2 + s,
\]

As it also satisfies \( f(I_1) = I_2 = \mathbb{Z}a + \mathbb{Z}\tau_2 \), so we have

\[
s = na,
\]

\[
b_2 = b_1 + 2na,
\]

\[
c_2 = n^2a + b_1n + c_1,
\]

for some integer \( n \). \( \square \)

Now we return to the case of \( 2 \times 2 \times 2 \) integer cubes. Given a \( 2 \times 2 \times 2 \) integer cube \( A \), suppose that the \( D = \text{disc}(A) \neq 0 \), we consider the two binary quadratic forms \( Q_1(A)(u, v) \) and \( Q_2(A)(u, v) \) associated to \( A \). Suppose that the coefficients \( a_i \) of \( u^2 \) are not zero, then applying the map in the last proposition to each \( Q_i(A)(u, v) \), we get the pairs \((R, I_1, I_2)\) where \( I_i \) is the oriented ideal with \( R/I_i \cong |a_i|\mathbb{Z} \). Explicitly, the map is given by

\[
\{ A : Q_i(A)(u, v) = a_i u^2 + b_i u v + c_i v^2, i = 1, 2 \}
\]

\[
\rightarrow \{(R; I_1, I_2) : I_i = \langle a_i, \tau_i \rangle, \tau_i^2 + b_i \tau + a_i c_i = 0 \}.
\]
Theorem 5.2. With the notation above, then the natural map of (16) defines a surjective and finite morphism
\[ B'_2(\mathbb{Z}) \times B'_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z}) \smallsetminus V^*_Z \rightarrow \text{Iso}\{(R; I_1, I_2) \mid R/I_1 \cong N(I_1)\mathbb{Z}, R/I_2 \cong N(I_2)\mathbb{Z}\} \]
The cardinality \( n(R; I_1, I_2) \) of the fiber is equal to
\[ \sigma_1(D_1, a_1, a_2), \]
where \( D = D_0D_1^2 = \text{disc}(R) \), and \( D_0 \) is square-free. And it satisfies
\[ \sum_{(R; I_1, I_2)/N(I_1)=[a_i]} n(R; I_1, I_2) = B(D, a_1, a_2). \]

Proof. The map is described above and it is easy to see well defined. We first prove the surjectivity of the map. Given a pair \((R; I_1, I_2)\) with \( I_i \) an oriented ideal of \( R \) and \( R/I_i \cong N(I_i)\mathbb{Z} \), by Proposition 5.1, we know that there are two binary quadratic forms
\[ Q_i(u, v) = a_iu^2 + b_iuv + c_iv^2 \]
for \( i = 1, 2 \) with \( D = \text{disc}(R) = b_1^2 - 4a_1c_1 \), such that \( R \) and \( I_i \) are determined by
\[ R = \langle 1, \tau_D \rangle = \langle 1, \tau_1 \rangle = \langle 1, \tau_2 \rangle, \]
\[ I_1 = \langle a_1, \tau_1 \rangle \quad \text{and} \quad I_2 = \langle a_2, \tau_2 \rangle, \]
where \( \tau_D \) is defined in (11), and \( \tau_i \) satisfies
\[ \tau_i^2 + b_i\tau_i + a_ic_i = 0 \]
for \( i = 1, 2 \). By the Lemma 3.3, we know that there exists a \( 2 \times 2 \times 2 \) integer cube \( A \) such that
\[ Q_i(A)(u, v) = Q_i(u, v). \]
Under the correspondence of (16), we conclude that the integer cube \( A \) maps to the given pair \((R; I_1, I_2)\).

To prove the second part of the theorem, let \( A \) and \( A' \) be the two \( 2 \times 2 \times 2 \) integer cubes which are in the fiber of \((R; I_1, I_2)\) with \( N(I_i) = |a_i| \), then they have the following arithmetic property: \( \text{disc}(R) = \text{disc}(A) = \text{disc}(A') = D \), and the two quadratic forms associated to them are the same \( Q_i(A)(u, v) = Q_i(A')(u, v) = a_iu^2 + b_iuv + c_iv^2 \), where \( 0 \leq b_i \leq 2|a_i| - 1 \) is assumed. Write \( D = D_0D_1^2 \). From our general formula of \( B(D, m, n) \) in the Shintani zeta function \( Z_{\text{Shintani}}(s_1, s_2, w) \), we know that the fiber counting function is equal to
\[ n(R; I_1, I_2) = \sum_{d|D, d|a_1, d|a_2} f(d), \text{where} \quad f(d) = \begin{cases} d & \text{if } \left( \frac{b_i}{d} \right)^2 \equiv D \pmod{4a_i}, \\ 0 & \text{otherwise.} \end{cases} \]

Note that as \( b_i^2 \equiv D \pmod{4a_i} \) already holds, so it automatically implies \( \left( \frac{b_i}{d} \right)^2 \equiv \frac{D}{d^2} \pmod{4a_i} \) if \( d|g.c.d.(D_1, a_1, a_2) \). It follows that
\[ n(R; I_1, I_2) = \sigma_1(D_1, a_1, a_2). \]
Furthermore, the sum of cardinalities over the fixed norms of \( N(I_1) = |a_1| \) and \( N(I_2) = |a_2| \) is exactly
\[ B(D, a_1, a_2) = \sharp \{ A \in V^*_Z / \sim \mid \text{Disc}(A) = D, \det(A^F) = |a_1|, \det(A^L) = |a_2| \}. \]


References


