Exercises for Midterm 2

1. A linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is given by $T(x, y) = (x, x + y, 2y)$. Find the matrix of $T$ with respect to the bases $\mathcal{A} = \{ (1, 3), (4, -1) \}$ in $\mathbb{R}^2$ and $\mathcal{B} = \{ (1, 0, 0), (0, 2, 0), (0, 0, -1) \}$ in $\mathbb{R}^3$.

Answer: $T_{\mathcal{A}, \mathcal{B}} = \begin{pmatrix} 1 & 4 \\ 2 & 2/3 \\ -6 & -2 \end{pmatrix}$

2. Consider the linear subspace $V = \{ (x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0 \}$ with the two bases:

$\mathcal{A} = \{ v_1 = (1, -1, 0), v_2 = (1, 1, -2) \}$ and $\mathcal{B} = \{ w_1 = (1, 0, -1), w_2 = (0, 1, -1) \}$.

a) Find coordinates of $w_1$ and $w_2$ with respect to the basis $\mathcal{A}$.
b) Find coordinates of $v_1$ and $v_2$ with respect to the basis $\mathcal{B}$.
c) Find the matrices of the base change $\mathcal{B} \rightarrow \mathcal{A}$ and $\mathcal{A} \rightarrow \mathcal{B}$.

Answer: a) $(1/2, 1/2)$ and $(-1/2, 1/2)$
b) $(1, -1)$ and $(1, 1)$
c) $S_{\mathcal{B} \rightarrow \mathcal{A}} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$, $S_{\mathcal{A} \rightarrow \mathcal{B}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

3. Let $T : \mathcal{P}_3 \rightarrow \mathcal{P}_2$ be a transformation defined by the formula $p(x) \mapsto xp''(x) + p'(x)$.

a) Show that $T$ is linear.
b) Find the matrix of $T$ with respect to the standard bases in $\mathcal{P}_3$ and $\mathcal{P}_2$.
c) Find the matrix of $T$ with respect to the basis $\{1, x + 1, (x + 1)^2, (x + 1)^3\}$ in $\mathcal{P}_3$ and $\{1, x + 1, (x + 1)^2\}$ in $\mathcal{P}_2$.
d) Find bases in the kernel and image of $T$.

Answer: a) use the definition of a linear transformation
b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix}$
4. Show that all upper triangular $2 \times 2$ matrices form a subspace of the vector space $M_2$ of all square $2 \times 2$ matrices. Find a basis of this subspace.

**Answer:** The sum of two upper triangular matrices is obviously an upper triangular matrix and the product of an upper triangular matrix by a real number is an upper triangular matrix. It means that the set of upper triangular matrices is closed with respect to linear operations and is a subspace.

A basis is \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).

5. Let \( T : U_2 \to U_2 \) be the linear transformation in the space of upper triangular $2 \times 2$ matrices defined by the formula

\[ T : M \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}^{-1} M \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \]

Find the matrix of \( T \) with respect to the basis

\[ B = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \]

of \( U_2 \). Is \( T \) an isomorphism?

**Answer:** The matrix is \( T_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} \). \( T \) is an isomorphism.

6. Let \( W = \text{span}\{ (1, 1, 0, 0), (0, 0, 1, 1) \} \subset \mathbb{R}^4 \). Find a basis in the orthogonal complement \( W^\perp \).

**Answer:** \( \{ (1, -1, 0, 0), (0, 0, 1, -1) \} \)

7. Find the orthogonal projection of the vector \( \mathbf{v} = (1, 1, 1) \in \mathbb{R}^3 \) onto the subspace of \( \mathbb{R}^3 \) which is spanned by the vectors \( \mathbf{w}_1 = (1, -1, 0) \) and \( \mathbf{w}_2 = (1, 1, -2) \).

**Answer:** \( \mathbf{0} \)

8. Let \( W \) be a subspace of \( \mathbb{R}^4 \) which is defined by

\[ W = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_3 + x_4 = 0, x_2 - x_3 = 0 \}. \]
Find an orthonormal basis in $W$ and an orthonormal basis in $W^\perp$. Find the orthogonal projection of the vector $v = (3, -2, 0, 3)$ onto $W$.

**Answer:** \( \left\{ \frac{1}{\sqrt{2}}(1, 0, 0, -1), \frac{1}{\sqrt{10}}(1, 0, 0, -1) \right\}, \left\{ \frac{1}{\sqrt{3}}(1, 0, 0, 1), \frac{1}{\sqrt{15}}(1, 3, -2, 1) \right\}, \sqrt{10}(1, -2, -2, 1) \)

9. Let $\mathcal{P}_2$ be a vector space of polynomials of degree $\leq 2$ with the inner product defined by the formula \(<p, q> = \int_{-1}^{1} p(x)q(x) \, dx\).

a) Find an orthogonal basis of $\mathcal{P}_2$ applying the Gram-Schmidt orthogonalization to the standard basis $\{1, x, x^2\}$ of $\mathcal{P}_2$.

b) Verify the Cauchy-Schwarz inequality for $p(x) = 1 - 2x$ and $q(x) = x^2$.

c) Verify the triangle inequality for $p(x) = 1 - 2x$ and $q(x) = x^2$.

d) Find a polynomial $r(x)$ of degree 1 which is orthogonal to $p(x) = 1 - 2x$. Verify the Pythagorean theorem for $p$ and $r$.

**Answer:**

a) \( \{1, x, x^2 - 1/3\} \)

b) \( \frac{2}{3} \leq \sqrt{\frac{14}{3}} \cdot \sqrt{\frac{2}{5}} \)

c) \( \frac{8}{3} \leq \sqrt{\frac{14}{3}} + \sqrt{\frac{2}{5}} \)

\( \) \( d) r(x) = x + 2/3 \) (for example). \( \frac{56}{9} = \frac{14}{3} + \frac{14}{9} \)

10. Show that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by the formula

\[ T(x, y, z) = \frac{1}{3}(x - 2y - 2z, -2x + y - 2z, -2x - 2y + z) \]

is orthogonal. Is $T$ invertible? If so, find the inverse.

**Answer:** The standard matrix of $T$ is $A = \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}$. This matrix is orthogonal since its columns (rows) are orthonormal. $T$ is invertible, the inverse is a transformation given by $A^T$. 