Solutions for Midterm 2

Problem 1. A linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is defined by \( T(x, y) = (-x + 2y, 3x - y) \).

Find a matrix of \( T \) with respect to the basis \( B = \{(2, 1), (1, 1)\} \) in \( \mathbb{R}^2 \). Is \( T \) an isomorphism? Explain!

Solution. A matrix of \( T \) with respect to the basis \( B = \{(2, 1), (1, 1)\} \) is

\[
T_B = \begin{pmatrix}
T(2, 1) & T(1, 1)
\end{pmatrix},
\]

where the coordinates of the column vectors are given in the basis \( B \). We calculate

\[
T(2, 1) = (0, 5) = -5 \cdot (2, 1) + 10 \cdot (1, 1)
\]

\[
T(1, 1) = (1, 2) = -1 \cdot (2, 1) + 3 \cdot (1, 1)
\]

and get the matrix:

\[
T_B = \begin{pmatrix}
-5 & 10 \\
-1 & 3
\end{pmatrix}.
\]

Another way to solve the problem is to consider the composition

\[
\mathbb{R}^2 \xrightarrow{id} \mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2 \xrightarrow{id} \mathbb{R}^2
\]

of \( T \) and the identity transformations of \( \mathbb{R}^2 \). The matrix of the composition is the product of three matrices:

\[
T_B = S_{St-B} \cdot T_{St} \cdot S_{B-St} = S^{-1} T_B S,
\]

where \( S = S_{B-St} \) is the transition matrix from basis \( B \) to the standard basis and \( T_{St} = \begin{pmatrix}
-1 & 2 \\
3 & -1
\end{pmatrix} \) is the standard matrix of \( T \). So

\[
T_B = \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}^{-1} \cdot \begin{pmatrix}
-1 & 2 \\
3 & -1
\end{pmatrix} \cdot \begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix} = \begin{pmatrix}
-5 & -1 \\
10 & 3
\end{pmatrix}.
\]

\( T \) is an isomorphism since \( \det T = -5 \neq 0 \).

Answer: \( \begin{pmatrix}
-5 & -1 \\
10 & 3
\end{pmatrix} \), \( T \) is an isomorphism.
Problem 2. Show that the set of traceless $2 \times 2$ matrices $U = \{ A \in M_2 \mid \text{tr}A = 0 \}$ is a subspace of the space of $2 \times 2$ matrices $M_2$. Find a basis of $U$.

Solution. Let $A$ and $B$ be two arbitrary matrices from $U$ and $k$ be a real number. Then $\text{tr} (A + B) = \text{tr} A + \text{tr} B = 0 + 0 = 0$ and $\text{tr} (kA) = k \text{tr} A = 0 \cdot 0 = 0$. This shows that $U$ is closed under matrix addition and multiplication by a scalar. Hence $U$ is a subspace of $M_2$.

Any traceless matrix $A$ can be written (in a unique way) as

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

where $a, b$ and $c$ are arbitrary real numbers. So

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and a basis of $U$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$  

Answer: $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$
Problem 3. Let $\mathcal{P}_2$ be the vector space of polynomials of degree $\leq 2$ equipped with the inner product
\[ < p, q > = \int_{-1}^{1} p(x)q(x) \, dx. \]
Let $W$ be a subspace of $\mathcal{P}_2$ generated by the polynomial $x - 1$. Find an orthogonal basis of the orthogonal complement of $W$.

Solution. Let $W = \text{span}\{x - 1\}$. Then $W^\perp = \{p(x) = a+bx+cx^2 \in \mathcal{P}_2 \mid < p(x), x-1 > = 0\}$.

Since
\[ < p(x), x - 1 > = \int_{-1}^{1} (a + bx + cx^2)(x - 1) \, dx = 2 \int_{0}^{1} -a + (b-c)x^2 \, dx = 2 \left( -a + \frac{b-c}{3} \right), \]
the condition $< p(x), x - 1 > = 0$ implies $a = \frac{b-c}{3}$ and the orthogonal complement $W^\perp$ consists of polynomials of form
\[ p(x) = \frac{b-c}{3} + bx + cx^2 = b \left( \frac{1}{3} + x \right) + c \left( -\frac{1}{3} + x^2 \right). \]
So $W^\perp = \text{span} \left\{ \frac{1}{3} + x, -\frac{1}{3} + x^2 \right\} = \text{span} \{1 + 3x, -1 + 3x^2\}$. Since
\[ < 1 + 3x, -1 + 3x^2 > = \int_{-1}^{1} (1 + 3x)(-1 + 3x^2) \, dx = 2 \int_{0}^{1} -1 + 3x^2 = 0, \]
the polynomials $1 + 3x$ and $-1 + 3x^2$ are orthogonal, and an orthogonal basis of $W^\perp$ is $\{1 + 3x, -1 + 3x^2\}$.

Answer: $\{1 + 3x, -1 + 3x^2\}$
Problem 4.

The former secret agent is now a contractor. His company works in the Euclidean space $\mathbb{R}^3$. He has got a tool called the Gram-Schmidt orthogonalization and a basis $\{(-1,0,1), (1,1,3), (1,3,-1)\}$ of $\mathbb{R}^3$. Help him to construct an orthonormal basis of $\mathbb{R}^3$ out of the given basis!

We use Gram-Schmidt orthogonalization to make an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Here are formulae:

\[
\begin{align*}
\mathbf{u}_1 &= \mathbf{v}_1 \\
\mathbf{u}_2 &= \mathbf{v}_2 - \frac{\mathbf{u}_1^\top \mathbf{v}_2}{\mathbf{u}_1^\top \mathbf{u}_1} \mathbf{u}_1 \\
\mathbf{u}_3 &= \mathbf{v}_3 - \frac{\mathbf{u}_1^\top \mathbf{v}_3}{\mathbf{u}_1^\top \mathbf{u}_1} \mathbf{u}_1 - \frac{\mathbf{u}_2^\top \mathbf{v}_3}{\mathbf{u}_2^\top \mathbf{u}_2} \mathbf{u}_2.
\end{align*}
\]

In our case, $\mathbf{v}_1 = (-1,0,1)$, $\mathbf{v}_2 = (1,1,3)$, $\mathbf{v}_3 = (1,3,-1)$. We substitute the given data:

\[
\begin{align*}
\mathbf{u}_1 &= (-1,0,1) \\
\mathbf{u}_2 &= (1,1,3) - \frac{(-1,0,1) \cdot (1,1,3)}{(-1,0,1) \cdot (-1,0,1)} (-1,0,1) = (1,1,3) - \frac{2}{2} (-1,0,1) = (2,1,2) \\
\mathbf{u}_3 &= (1,3,-1) - \frac{(-1,0,1) \cdot (1,3,-1)}{(-1,0,1) \cdot (-1,0,1)} (-1,0,1) - \frac{(2,1,2) \cdot (1,3,-1)}{(2,1,2) \cdot (2,1,2)} (2,1,2) \\
&= (1,3,-1) + (-1,0,1) - \frac{1}{3} (2,1,2) = (-\frac{2}{3}, -\frac{8}{3}, -\frac{2}{3}) = -\frac{2}{3} (1,-4,1).
\end{align*}
\]

The constructed basis $\{\mathbf{u}_1 = (-1,0,1), \mathbf{u}_2 = (2,1,2), \mathbf{u}_3 = -\frac{2}{3} (1,-4,1)\}$ is orthogonal. Let us normalize it.

\[
\begin{align*}
\mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{\mathbf{u}_1}{\sqrt{\mathbf{u}_1^\top \mathbf{u}_1}} = \frac{(-1,0,1)}{\sqrt{(-1,0,1) \cdot (-1,0,1)}} = \frac{1}{\sqrt{2}} (-1,0,1) \\
\mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{\mathbf{u}_2}{\sqrt{\mathbf{u}_2^\top \mathbf{u}_2}} = \frac{(2,1,2)}{\sqrt{(2,1,2) \cdot (2,1,2)}} = \frac{1}{3} (2,1,2) \\
\mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{\mathbf{u}_3}{\sqrt{\mathbf{u}_3^\top \mathbf{u}_3}} = \frac{-\frac{2}{3} (1,-4,1)}{\sqrt{-\frac{2}{3} (1,-4,1) \cdot (-\frac{2}{3}) (1,-4,1)}} = -\frac{1}{3\sqrt{2}} (1,-4,1).
\end{align*}
\]

Finally, the basis $\{\mathbf{e}_1 = \frac{1}{\sqrt{2}} (-1,0,1), \mathbf{e}_2 = \frac{1}{3} (2,1,2), \mathbf{e}_3 = -\frac{1}{3\sqrt{2}} (1,-4,1)\}$ is orthonormal.

Answer: $\left\{ \frac{1}{\sqrt{2}} (-1,0,1), \frac{1}{3} (2,1,2), -\frac{1}{3\sqrt{2}} (1,-4,1) \right\}$