Problem 1 (15pt). Find derivatives of the following functions. Simplify your answer whenever it is possible.

a) $y = \sin \frac{1}{x^2}$, 
   $y' = \left(\cos \frac{1}{x^2}\right) \cdot \frac{-2}{x^3} = \frac{-2\cos \frac{1}{x^2}}{x^3}$

b) $y = x \cdot 3^x$, 
   $y' = 3^x + x \cdot 3^x \ln 3 = 3^x (1 + x \ln 3)$

c) $y = \frac{x^4 - 2x^2 + 1}{x^2 - 1}$
   $= \frac{(x^2 - 1)^2}{x^2 - 1} = x^2 - 1$, 
   $y' = 2x$

d) $y = \cot \sqrt{x^2 + 1}$
   $y' = -\frac{1}{\sin^2 \sqrt{x^2 + 1}} \cdot \frac{x}{\sqrt{x^2 + 1}} = -\frac{x}{\sqrt{x^2 + 1} \sin^2 \sqrt{x^2 + 1}}$
e) \( y = \arctan \frac{\pi}{x} \), 
\[
y' = \frac{1}{1 + \frac{\pi^2}{x^2}} \cdot \left(-\frac{\pi}{x^2}\right) = -\frac{\pi}{x^2 + \pi^2}
\]

f) \( y = \left(\frac{1}{x}\right)^{\ln x} = e^{-\ln x} \)
\[
y' = -\frac{\ln x}{x^2}
\]
\[
y_{y'y} = -\left(\frac{\ln x}{x^2}\right)^2
\]
\[
y_{y'y} = -2 \frac{\ln x}{x}
\]
\[
y' = -2 \left(\frac{1}{x}\right) \cdot \frac{\ln x}{x}
\]
\[
y' = -2 \frac{\ln x}{x^{1+\ln x}}
\]
Problem 2 (15pt). Show that the function \( y = e^{4x} + 2e^{-x} \) is a solution of the differential equation \( y'' - 13y' - 12y = 0 \).

\[
\begin{align*}
\textbf{1 pt} & \quad y' = 4e^{4x} - 2e^{-x} \\
\textbf{2 pt} & \quad y'' = 16e^{4x} + 2e^{-x} \\
\textbf{2 pt} & \quad y''' = 64e^{4x} - 2e^{-x} \\
\textbf{3 pt} & \quad \frac{64e^{4x} - 2e^{-x}}{y''} - 13\left(\frac{4e^{4x} - 2e^{-x}}{y''}\right) - 12\left(\frac{e^{4x} + 2e^{-x}}{y}\right) = 0 \\
\textbf{3 pt} & \quad \frac{64e^{4x} - 2e^{-x}}{y''} - \frac{52e^{4x} + 26e^{-x}}{y''} - \frac{12e^{4x} - 24e^{-x}}{y} = 0 \\
\textbf{6 pt} & \quad 0 = 0 \quad \checkmark
\end{align*}
\]
Problem 3 (15pt). Use a linearization to find an approximate value of \( \frac{1}{e^{0.001}} \). Is the true value greater than or less than your approximation? Explain!

\[
\frac{1}{e^{0.01}} = e^{-0.01}
\]

A linearization formula

\[
f(a+h) \approx f(a) + f'(a) \cdot h
\]

written for \( f(x) = e^x \), \( a = 0 \), \( h = -0.01 \)

becomes

\[
e^{0+(-0.01)} \approx e^0 + \left. \frac{de^x}{dx} \right|_{x=0} \cdot (-0.01)
\]

So

\[
e^{-0.01} \approx 1 + 1 \cdot (-0.01) = 0.99
\]

The true value of \( \frac{1}{e^{0.01}} \) is greater than the approximation:

\[
\frac{1}{e^{0.01}} = e^{-0.01} > 0.99
\]

OR

\[
f(x) = e^{-x}
\]

\[
e^{-0.01} = e^{0+(-0.01)} \approx e^0 + \left. \frac{de^{-x}}{dx} \right|_{x=0} \cdot (0.01) =
\]

\[
= 1 - 0.01 = 0.99
\]
Problem 4 (15pt). Find the equation of the tangent line to the curve \( x^2y^3 + x\ln y = 1 \) at the point \( x = 1, \ y = 1 \).

Point \((1,1)\) belongs to the curve since \( 1^3 + 1 + \ln 1 = 1 \)

The eq. of the tangent line at \((1,1)\) is

\[
y - 1 = \frac{dy}{dx} \bigg|_{(1,1)} (x-1).
\]

The derivative \( \frac{dy}{dx} \) can be obtained by implicit differentiation:

\[
dx \left( x^2y^3 + x\ln y = 1 \right) = 0,
\]

where \( y' = \frac{dy}{dx} \)

\[
y' \left( 3x^2y^2 + \frac{x}{y} \right) = -2xy^3 - x\ln y
\]

\[
y' = -\frac{2xy^3 + \ln y}{3x^2y^2 + \frac{x}{y}}
\]

\[
\frac{dy}{dx} \bigg|_{x=1, y=1} = y' \bigg|_{x=1, y=1} = -\frac{2 \cdot 1 \cdot 1 + \ln 1}{3 \cdot 1 + \frac{1}{1}} = -\frac{2}{4} = -\frac{1}{2}
\]

The eq. of the tang. line is

\[
y - 1 = -\frac{1}{2} (x-1)
\]

\[
y = -\frac{1}{2}x + \frac{3}{2}
\]
Problem 5 (20pt). Show that the curve \( x = t^3 - t, \ y = t^2 \) has two different tangent lines at the point \((x, y) = (0, 1)\) and find their slopes.

First, find the values of \( t \) corresponding to the point \( x=0, \ y=1 \):

\[
\begin{align*}
0 &= t^3 - t \\
1 &= t^2 
\end{align*}
\]

\[
\Rightarrow \quad \begin{cases} t(t-1)(t+1) = 0 \\ (t-1)(t+1) = 0 \end{cases} \Rightarrow t = \pm 1
\]

Two values of \( t \) show that the curve passes through \((0, 1)\) twice.

The slopes are

\[
\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2 - 1}
\]

\[
\left. \frac{dy}{dx} \right|_{t=1} = \frac{2(1)}{3(1) - 1} = \frac{2}{3-1} = 1
\]

\[
\left. \frac{dy}{dx} \right|_{t=-1} = \frac{2(-1)}{3(-1) - 1} = \frac{-2}{-3-1} = -1
\]

Answer: The slopes are \(-1\) and \(+1\).
Problem 6 (20pt). How fast is the area of a magic rectangle changing if one side of it is 20 ft long and is decreasing at a rate of 1 ft/sec and the other side is 15 ft long and is increasing at a rate 2 ft/sec?

Let \( x(t), y(t) \) be the sides of the rectangle. Then the area is

\[
A(t) = x(t) \cdot y(t)
\]

Given:

\[
\frac{dx}{dt} \bigg|_{x=20} = -1 \text{ ft/sec}
\]

\[
\frac{dy}{dt} \bigg|_{y=15} = 2 \text{ ft/sec}
\]

Find:

\[
\frac{dA}{dt} \bigg|_{x=20 \ y=15} = ?
\]

\[
A(t) = x(t) \cdot y(t) \Rightarrow \frac{dA}{dt} = \frac{dx}{dt} \cdot y(t) + x(t) \frac{dy}{dt}
\]

\[
\frac{dA}{dt} \bigg|_{x=20 \ y=15} = -1 \cdot 15 + 20 \cdot 2 = 25 \text{ ft/sec}
\]

Answer: the area is increasing at a rate 25 ft/sec.