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Introduction to Calculus
MAT 123, Fall 2012
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Midterm 2. Solutions

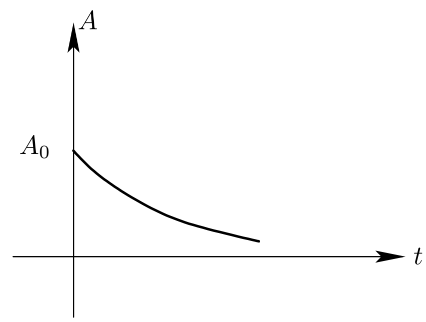
1. During a period of 70 hours, the amount of radionuclide Yttrium-93 was reduced to $\frac{1}{128}$ of the original amount. Find the half-life of Yttrium-93 and the equation describing the decay of Yttrium-93 (model of decay). Explain the meaning of all entries in the equation. Draw the graph.

Solution. The model for a radioactive decay is $A(t) = A_0\, 2^{-t/h}$, where A_0 is the initial amount the radioactive isotope, h is the half-life of the isotope, $A(t)$ is the amount of the isotope after t time units.

According to the problem, $A(70) = A_0/128$. We plug in $t = 70$ and $A(70)$ into the equation:

$A(70) = A_0\, 2^{-70/h}$ and get $A_0/128 = A_0\, 2^{-70/h}$, wherefrom $2^{-7} = 2^{-70/h}$ and $-7 = -70/h$ and, finally, $h = 10$.

So the half-life of Y-93 is $\boxed{\text{10 hours}}$ and model for the decay is $\boxed{A(t) = A_0\, 2^{-t/10}}$, where A_0 is the initial amount of Y-93, $A(t)$ is the amount of the isotope after t hours from initial moment $t = 0$.



2. If \$1,000 is invested at 0.1% interest rate compounded continuously, what will be the amount after 10 years? (To get the answer you have to use the approximation for e^t for small t.)

Solution. The model for an interest rate compounded continuously is $P(t) = P\, e^{rt}$, where P is the principal (original amount), r is the annual interest rate, and $P(t)$ is the amount after t years.

According to the problem, $P = \$1,000$, $r = 0.001$ and $t = 10$ years. We plug in these values into the equation:

$P(10) = 1000\, e^{0.001\cdot 10} = 1000\, e^{0.01} \approx 1000(1 + 0.01) = \boxed{10100}$.

The approximation $e^{0.01} \approx 1 + 0.01 = 1.01$ used in the calculation is based on the formula $e^t \approx 1 + t$, which is valid for small t.

3. A colony of bacteria *precalculucium polyspora* doubles its size each 3 hours. Now the size of the colony is 100 cells. When there will be 1,000 cells in the colony? (You may use the approximation $\log_2 10 \approx 3.32$.)

Solution. An appropriate model for population growth in the case of doubling is $A(t) = A_0\, 2^{t/d}$, where A_0 is the initial size of population at time moment $t = 0$, d is the period of doubling, and $A(t)$ is the size of population at time moment t time units.

According to the problem, $A_0 = 100$ cells and $d = 3$ hours. We have to find time moment t when $A(t)$ will be equal to 1,000 cells. We plug in the data into the equation

$1000 = 100 \cdot 2^{t/3}$ and get $10 = 2^{t/3}$. Taking the base 2 logarithm from both sides, $\log_2 10 = t/3$ or $t = 3\log_2 10 \approx 3 \cdot 3.32 = 9.96$.

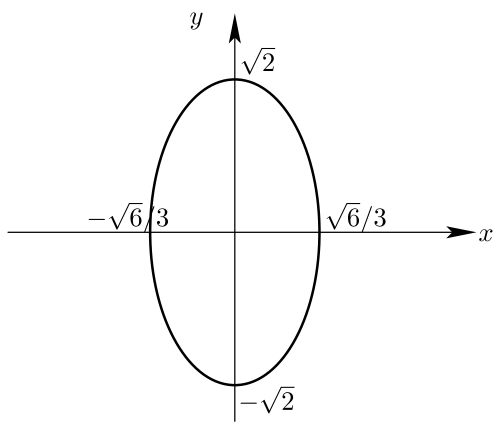
Answer: The size of the colony will reach 1,000 cells in about $\boxed{9.96 \text{ hours} = 9 \text{ hours } 57 \text{ munites and } 36 \text{ seconds}}$

4. Draw the ellipse $3x^2 + y^2 = 2$ in the coordinate plane. Find the area inside the ellipse. (Don't forget to simplify your answer, leaving no irrational expressions in the denominator.)

Solution. First, we rewrite the equation of the ellipse in the standard form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$:

$$3x^2 + y^2 = 2 \iff \frac{3x^2}{2} + \frac{y^2}{2} = 1 \iff \frac{x^2}{2/3} + \frac{y^2}{2} = 1 \iff \frac{x^2}{(\sqrt{2/3})^2} + \frac{y^2}{(\sqrt{2})^2} = 1.$$

The semi-axes of the ellipse are $a = \sqrt{2/3} = \dfrac{\sqrt{6}}{3}$ and $b = \sqrt{2}$. The area inside the ellipse is $\pi ab = \pi\dfrac{\sqrt{6}}{3} \cdot \sqrt{2} = \dfrac{\pi\sqrt{12}}{3} = \boxed{\dfrac{2\sqrt{3}\pi}{3}}$.

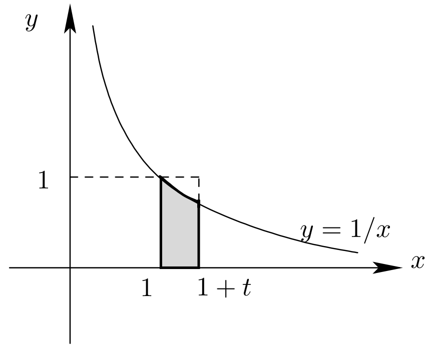


5. Find the area of the plane region under the hyperbola $y = \dfrac{1}{x}$, above the x-axis, and between the lines $x = 1$ and $x = \sqrt{e}$.

Solution. As we know, the area of the plane region located under the hyperbola $y = 1/x$, above the x-axis, and between the lines $x = 1$ and $x = c$ is equal to $\ln c$. In our case, $c = \sqrt{e}$ and therefore the area is $\ln\sqrt{e} = \ln e^{1/2} = \boxed{1/2}$.

6. Explain why for small positive values of t the following approximation is valid: $\ln(1+t) \approx t$.

Solution. As we know, $\ln(1 + t)$ is equal to the area of the plane region located under the hyperbola $y = 1/x$, above the x-axis, and between the lines $x = 1$ and $x = 1 + t$. For small t, the point $x = 1 + t$ is very close to the point $x = 1$, and the area of region under the hyperbola differs very insignificantly from the area of the rectangle with base t and height 1:



Therefore, the area under hyperbola (which is equal to $\ln(1+t)$) is approximately equal to the area of the rectangle (which is equal to $t \cdot 1$). So $\ln(1+t) \approx t$.

7. Solve the following equations:

a) $e^{2x+3} = 4$

b) $\ln(3x-2) = -4$.

Solution.

a) $e^{2x+3} = 4$

$\ln(e^{2x+3}) = \ln 4$

$2x + 3 = 2\ln 2$

$x = \dfrac{2\ln 2 - 3}{2}$

$\boxed{x = \ln 2 - \dfrac{3}{2}}$

b) $\ln(3x-2) = -4$

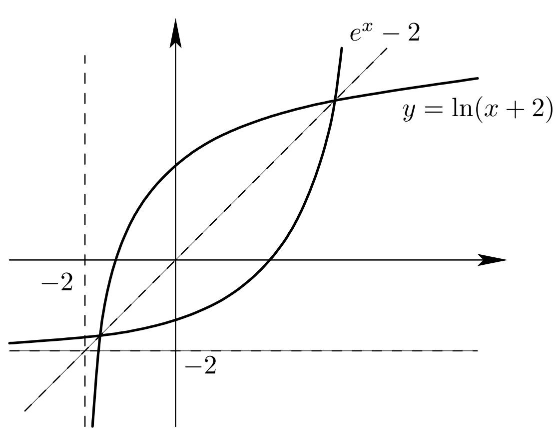
$e^{\ln(3x-2)} = e^{-4}$

$3x - 2 = e^{-4}$

$\boxed{x = \dfrac{e^{-4}+2}{3}}$

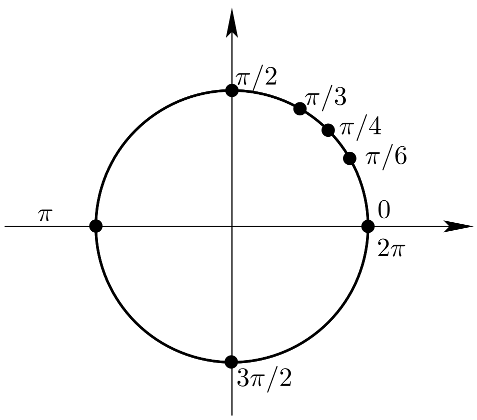
8. For the function $f(x) = e^x - 2$, find the inverse f^{-1}. Determine the domain and the range of f and f^{-1}. On the same coordinate plane, draw the graphs of f and f^{-1}. Indicate asymptotes. Are these two graphs symmetric about the line $y = x$?

Solution. Let $y = e^x - 2$, then $y + 2 = e^x$ and $\ln(y+2) = x$. Therefore, the inverse function is $f^{-1}(x) = \ln(x+2)$. The function f is defined for all values of x, so the domain of f, as well as the range of f^{-1}, is $(-\infty, \infty)$. The range of f and the domain of f^{-1} is the interval $(-2, \infty)$. Here are the graphs (observe the symmetry about the line $y = x$):



The line $y = -2$ is the horizontal asymptote for $y = e^x - 2$, the line $x = -2$ is the vertical asymptote for $y = \ln(x + 2)$.

9. On the unit circle, indicate the points corresponding to the special values of angle θ given in the table below. Fill in the table.



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.	0	undef.	0

10. This problem refers to basic trigonometric identities.

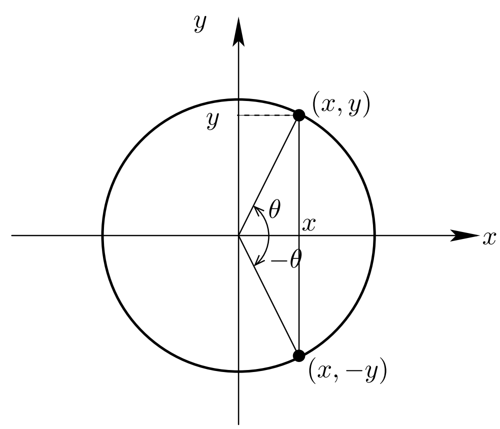
a) The fundamental trigonometric identity (Pythagorean identity): $\sin^2\theta + \cos^2\theta = 1$

b) $\cos(-\theta) = \cos\theta$ **c)** $\sin(-\theta) = -\sin\theta$ **d)** $\tan(-\theta) = -\tan\theta$

e) Choose one of the identities **a) - d)** and prove it.

a) $\cos\theta$ and $\sin\theta$ are respectively x and y coordinates of a point on the unite circle, and for each point (x, y) on the unit circle we have $x^2 + y^2 = 1$. Therefore, $\sin^2\theta + \cos^2\theta = 1$.

b), c) Consider the angles θ and $-\theta$ in standard position. The corresponding points on the unit circle are symmetric about the x-axis, they have coordinates (x, y) and $(x, -y)$ respectively. By the definition of cosine and sine, $(x, y) = (\cos\theta, \sin\theta)$ and $(x, -y) = (\cos(-\theta), \sin(-\theta))$. Therefore, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.



d) $\tan(-\theta) = \dfrac{\sin(-\theta)}{\cos(-\theta)} = \dfrac{-\sin(\theta)}{\cos(\theta)} = -\tan\theta.$

11. Let θ be an angle such that $\dfrac{\pi}{2} < \theta < \pi$ and $\sin\theta = \dfrac{2}{5}$. Find $\cos\theta$.

Solution. Since $\sin^2\theta + \cos^2\theta = 1$, we have $\left(\frac{2}{5}\right)^2 + \cos^2\theta = 1$, wherefrom $\cos^2\theta = 1 - \frac{4}{25}$ or $\cos^2\theta = \frac{21}{25}$. The angle θ has the terminal side in the third quadrant, so $\cos\theta$ is negative. Hence, $\cos\theta = -\sqrt{\frac{21}{25}} = \boxed{-\frac{\sqrt{21}}{5}}$.

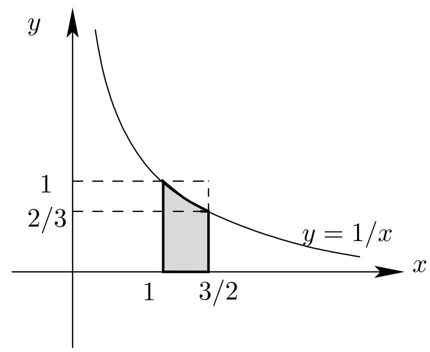
Bonus problems.

1. How to explain to a little brother what "ln 5" stands for?

Solution. Hopefully, a little brother has some understanding of area. We can draw a region on the plane situated under the hyperbola $y = 1/x$ above the x-axis and between the lines $x = 1$ and $x = 5$. The number $\ln 5$ stands for the area of this region.

2. Prove that $\frac{1}{3} < \ln\left(\frac{3}{2}\right) < \frac{1}{2}$.

Solution. $\ln(3/2)$ is the area of the plane region located under the hyperbola $y = 1/x$, above the x-axis and between the lines $x = 1$ and $x = 3/2$. This region contains the rectangle with base of $1/2$ and the hight of $2/3$, and is contained in the rectangle with base of $1/2$ and the hight of 1. So the area of this region is estimated by the areas of these two rectangles: $\frac{1}{3} < \ln\left(\frac{3}{2}\right) < \frac{1}{2}$.



3. Explain why the equation $e^{2x-1} = \ln\left(\frac{1}{2}\right)$ has no solutions.

Solution. $e^{2x-1} > 0$ for all x and $\ln 1/2 < 0$.

4. Solve the equation $\cos\theta = \ln 3$.

Solution. $\cos\theta < 1$ for any θ and $\ln 3 > 1$, so the equation has **no** solutions.

5. For $\sin(3.14)$, calculator A shows 0.0547759 and calculator B shows 0.0015927. Which calculator is set for radians and which one is set for degrees? Explain!

Solution. The angle of 3.14 radians is very close to the angle of π radians. If the calculator is set up for radians, then $\sin(3.14) \approx \sin\pi = 0$.

If the calculator is set up for degrees, then $\sin(3.14^\circ)$ will be a small number, which is though greater than $\sin(3.14)$.

Answer: calculator A is set up for degrees, calculator B is set up for radians.