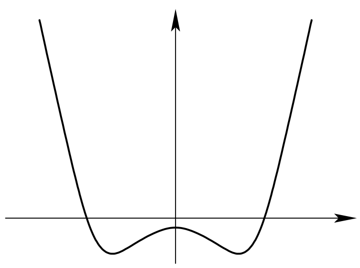
Stony Brook University
Mathematics Department
J. Viro

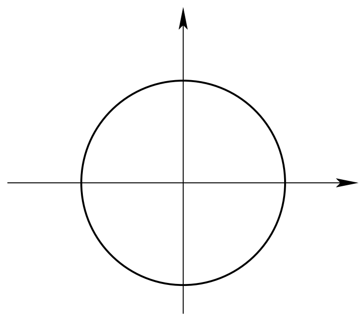
Introduction to Calculus
MAT 123, Fall 2012
October 17th, 2012

Midterm 1. Solutions

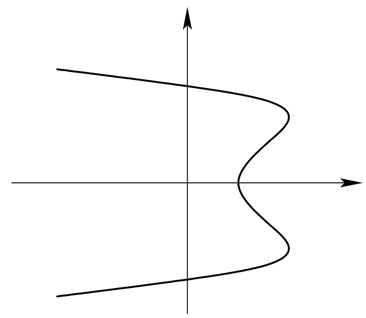
1 (6pt). Under each picture state whether it is the graph of a function or not a function. If it is a function state whether it is an even or odd function or neither.



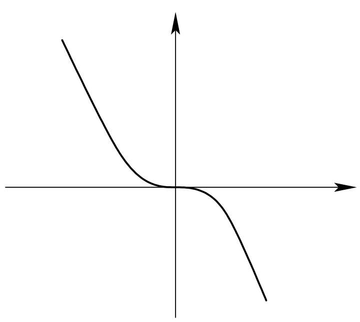
even function



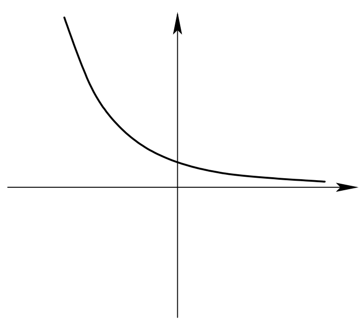
not a function



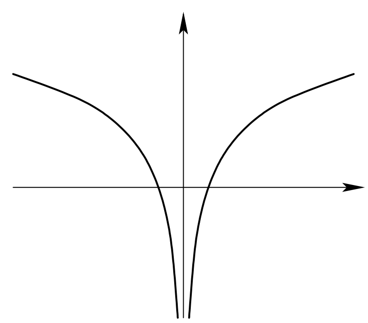
not a function



odd function



neither even nor odd function



even function

2 (6pt). Find an equation of a line which passes through the point $(1, -2)$ and is parallel to the line $x - 2y = 3$. Find the coordinates of the x- and y- intercepts.

Solution. First, rewrite the equation of the given line in the slope-intercept form:
$x - 2y = 3 \iff y = x/2 - 3/2$. Parallel lines have the same slopes, therefore the line we are looking for has the slope of $1/2$. Using the slope-point form of a line, we get
$y - (-2) = \dfrac{1}{2}(x - 1)$. Or, equivalently, $y = x/2 - 5/2$.

The x-intercept (where $y = 0$) can be found from the equation $0 = x/2 - 5/2$. From here, $x = 5$.

The y-intercept (where $x = 0$) is $y = 0/2 - 5/2 = -5/2$.

Answer: $\boxed{y = x/2 - 5/2,\ x\text{-intercept is } (5, 0),\ y\text{-intercept is } (0, -5/2).}$

3 (8pt). For the parabola $y = -x^2 + 2x + 2$, find the coordinates of the vertex, an equation of the axis of symmetry, the x- and y- intercepts. (Simplify all the expressions containing radicals!) Draw the graph. Label your picture properly, that is indicate the vertex, the axis of symmetry, the x- and y- intercepts.

Solution. The x-coordinate of the vertex of the parabola $y = ax^2 + bx + c$ is given by the formula $x = -\dfrac{b}{2a}$. In our case, $a = -1$, $b = 2$, $c = 2$ and the vertex is at $x = -\dfrac{2}{2 \cdot (-1)} = 1$. The y-coordinate of the vertex is $y(1) = -1^2 + 2 \cdot 1 + 2 = 3$. Therefore, the vertex is located at $(1, 3)$.

Alternatively, for the lovers of completing the square, one can rewrite the quadratic expression as $-x^2 + 2x + 2 = -(x-1)^2 + 1 + 2 = -(x-1)^2 + 3$. From here, $y = -(x-1)^2 + 3$ and the vertex is obviously at $(1, 3)$.

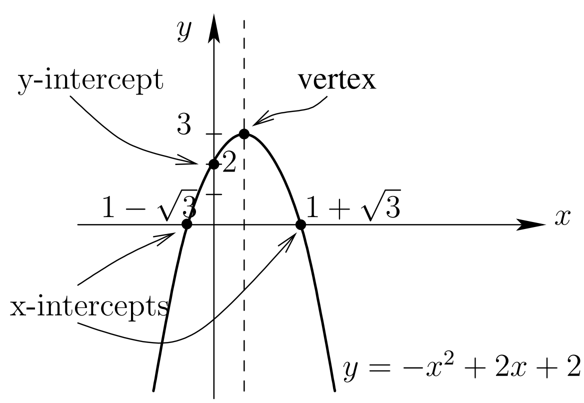
The equation of the axis of symmetry is $x = 1$. This is a vertical line passing through the vertex.

The x-intercepts (where $y = 0$) can be found from the equation $-x^2 + 2x + 2 = 0$. As we remember, the solution of the quadratic equation $ax^2 + bx + c = 0$ is given by the formula $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In our case,
$x_{1,2} = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot (-1) \cdot 2}}{2 \cdot (-1)} = \dfrac{-2 \pm \sqrt{12}}{-2} = \dfrac{-2 \pm 2\sqrt{3}}{-2} = 1 \mp \sqrt{3}$. So x-intercepts are $x_1 = 1 + \sqrt{3}$ and $x_2 = 1 - \sqrt{3}$. Note: FOIL method doesn't work here, since the solutions aren't rational numbers.

Alternatively, we can reuse our result of completing the square. Since $y = -(x-1)^2 + 3$, the x-intercepts satisfy the equation $0 = -(x-1)^2 + 3$, from which we get $(x-1)^2 = 3$ and, finally, $x_{1,2} = 1 \pm \sqrt{3}$.

The y-intercept (where $x = 0$) is, obviously, $y = 2$.

We use all the information for the graphing. Our parabola has horns facing down since $a = -1 < 0$. On the coordinate system, we place the vertex and draw the axis of symmetry, locate the y-intercept and draw a parabola.



Answer: vertex is at $(1, 3)$, axis of symmetry is $x = 1$,
x-intercepts are $(1 \pm \sqrt{3}, 0)$, y-intercept is $(0, 2)$,
the graph is above.

4 (8pt). Find the domain and the range of each the following functions. Write down your answer in a complete form: the domain is . . . , the range is . . . .

Solution. a) $y = |x+1|$. The domain is $\mathbb{R}$, the range is $[0, \infty)$.

b) $y = 3^{x+1}$ The domain is $\mathbb{R}$, the range is $(0, \infty)$.

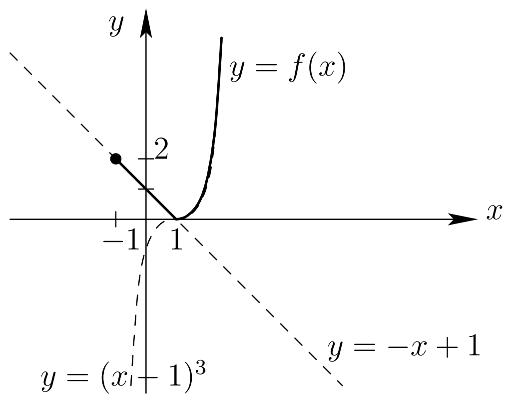
c) $y = \sqrt[3]{x+1}$. The domain is $\mathbb{R}$, the range is $\mathbb{R}$.

d) $y = (x+1)^{\frac{1}{4}}$. The domain is $[-1, \infty)$, the range is $[0, \infty)$.

5 (10pt). Draw the graph of the function $f(x) = \begin{cases} -x+1, & \text{if } \; -1 \le x < 1 \\ (x-1)^3, & \text{if } \; x \ge 1. \end{cases}$

Determine the intervals where f is increasing. Determine the intervals where f is decreasing.

Solution. The graph consist of two pieces: a part of the line $y = -x+1$ with $-1 \le x < 1$ and a part of the cubic parabola $y = (x-1)^3$ where $x \ge 1$:



As seen on the picture, f increases on $[1, \infty)$ and decreases on $[-1, 1]$.

6 (6pt). Show that the function $f(x) = \dfrac{1}{x^2} - x^4$ is even, and the function $g(x) = x^3 + x$ is odd. (You have to use the definition of even/odd function.)

Solution.

$f(-x) = \dfrac{1}{(-x)^2} - (-x)^4 = \dfrac{1}{x^2} - x^4 = f(x)$ for any $x \ne 0$. Therefore, $f(x)$ is an even function.

$g(-x) = (-x)^3 + (-x) = -x^3 - x = -(x^3 + x) = -g(x)$ for any x. Therefore, $g(x)$ is an odd function.

7 (6pt). Let $f(x) = 5^x$, $g(x) = 2x+3$. Find $f \circ g$, $g \circ f$ and $g \circ g$.

Solution. $(f \circ g)(x) = f(g(x)) = f(2x+3) = \boxed{5^{2x+3}}$, $(g \circ f)(x) = g(f(x)) = g(5^x) = \boxed{2 \cdot 5^x + 3}$, $(g \circ g)(x) = g(g(x)) = g(2x+3) = 2(2x+3) + 3 = \boxed{4x+9}$.

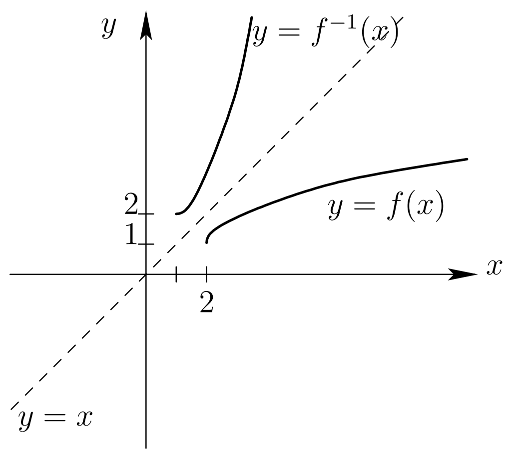
8 (10pt). Let $f(x) = \sqrt{x-2} + 1$. Find a formula for the inverse function $f^{-1}(x)$. Determine the domain and the range of f and f^{-1}. On the same coordinate system, draw the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

Solution. Let $y = \sqrt{x-2}+1$. To find the inverse function, we have to solve this equation for x:
$y = \sqrt{x-2}+1 \implies y-1 = \sqrt{x-2} \implies (y-1)^2 = x-2 \implies x = (y-1)^2+2$. Therefore, the formula for the inverse function is $f^{-1}(y) = (y-1)^2+2$, or, with x as a variable, $f^{-1}(x) = (x-1)^2+2 = x^2-2x+3$.

The domain of f^{-1} is the range of f. To find the range of $f(x) = \sqrt{x-2}+1$, we observe that $\sqrt{x-2} \geq 0$ and, therefore $\sqrt{x-2}+1 \geq 1$. So the range of f is the interval $[1, \infty)$.

The range of f^{-1} is the domain of f. Since $f(x) = \sqrt{x-2}+1$ is defined for $x \geq 2$ only, the domain of f is the interval $[2, \infty)$.

To draw the picture, we use graphs transformations. The graph of $y = \sqrt{x-2}+1$ is the graph of the radical function $y = \sqrt{x}$ shifted to the right by 2 units and than shifted up by 1 unit. The graph of $y = (x-1)^2+2$, $x \geq 1$ is the graph of a half of the parabola $y = x^2$ shifted to the right by 1 unit and than shifted up by 2 units. Note that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are symmetric about the line $y = x$.



Answer. $\boxed{\begin{array}{l} f^{-1}(x) = (x-1)^2+2 = x^2-2x+3, \\ \text{the domain of } f^{-1} \text{ is } [1,\infty), \quad \text{the range of } f^{-1} \text{ is } [2,\infty), \\ \text{the picture is above.} \end{array}}$

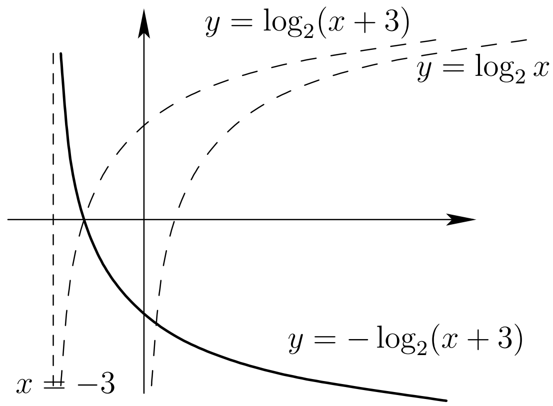
9 (10pt). Simplify the following expressions:

a) $\log_3 \sqrt{27} = \log_3 (27)^{1/2} = \log_3 (3^3)^{1/2} = \log_3 (3)^{3/2} = \boxed{\frac{3}{2}}$

b) $2^{\log_{1/2} \sqrt[3]{64}} = 2^{\log_{1/2} 4} = 2^{\log_{1/2} (\frac{1}{2})^{-2}} = 2^{-2} = \boxed{\frac{1}{4}}$.

10 (10pt). For the function $y = \log_2 \left(\frac{1}{x+3} \right)$, find the domain and the range. Draw the graph. Indicate the asymptote.

Solution. First, we use properties of logarithms to observe that $\log_2 \left(\frac{1}{x+3} \right) = -\log_2 (x+3)$. Now the graphing is simple: the graph of $y = -\log_2 (x+3)$ is obtained from the standard graph of $y = \log_2 x$ by horizontal shift to the left followed be a flip about the x-axis. The line $x = -3$ is the vertical asymptote. The domain is $(-3, \infty)$, the range is $(-\infty, \infty)$.



Answer. $\boxed{\begin{array}{l}\text{the domain is } (-3,\infty), \quad \text{the range of is } (-\infty,\infty), \\ \text{the asymptote is } x=-3, \text{ the picture is above.}\end{array}}$

11 (10pt). For the function $f(x) = 2^{x-1} + 3$, find the inverse.

Solution. $y = 2^{x-1} + 3 \implies y - 3 = 2^{x-1} \implies \log_2(y-3) = x - 1 \implies x = 1 + \log_2(y-3)$. The inverse function is $\boxed{f^{-1}(y) = 1 + \log_2(y-3)}$.

12. Solve the following equations:

a) $9^x - 3^x - 2 = 0$ **b)** $3\log x - \log(2x) = 2$.

Solution. a) $9^x - 3^x - 2 = 0 \iff 3^{2x} - 3^x - 2 = 0$. Let $t = 3^x$. Note that $t > 0$ since $3^x > 0$. Substituting t for 3^x in the equation, we get $t^2 - t - 2 = 0$. By factoring, $(t+1)(t-2) = 0$. Therefore $t = -1$ or $t = 2$. The negative solution $t = -1$ is rejected, and we get the only solution $t = 2$, which gives us $3^x = 2$ or $\boxed{x = \log_3 2}$.

b) The logarithms involved into the equation are common logarithms, that is logarithms with base 10. Both of them make sense only for positive values of x, so $x > 0$.

$$3\log x - \log(2x) = 2 \iff \log(x^3) - \log(2x) = 2 \iff \log\left(\frac{x^3}{2x}\right) = 2 \iff \log\left(\frac{x^2}{2}\right) = 2$$

$\iff \dfrac{x^2}{2} = 10^2 \iff x^2 = 200 \iff x = \pm\sqrt{200} = \pm 10\sqrt{2}$. Since $x > 0$, we reject the negative solution and get the only solution $\boxed{x = 10\sqrt{2}}$.

Bonus problems (2pt each). Pick up a problem (or problems) which appeals to you. Give a complete solution. No partial credit will be given for these problems.

1. Is any linear function invertible? Explain!

Solution. Not any linear function $y = mx + b$ is invertible. All constant functions $y = b$ are not invertible. Notice, that all other linear functions, that is linear functions with non-zero slopes, are invertible.

2. Prove that a composition of two even functions is an even function.

Solution. Let f and g be even functions, that is $f(-x) = f(x)$ and $g(-x) = g(x)$ for all x. If the composition $g \circ f$ is defined for all x, then $(g \circ f)(-x) = g(f(-x)) = g(f(x)) = (g \circ f)(x)$, which shows that $g \circ f$ is an even function.

3. Does there exist an invertible even function? Explain!

Solution. Yes, it does. Any function f with the domain consisting of the only number 0 and $f(0) = a$ for any number a is even (since $f(-x) = f(x)$ for all x in the domain, that is for the only $x = 0$) and invertible (the inverse is defined by $f^{-1}(a) = 0$).

4. The sum of two positive numbers is 13/3. What is the largest possible value for their product?

Solution. Let us denote one number by x. Then the other number will be $(13/3 - x)$. Since both numbers are positive, $0 < x < 13/3$. The product these two numbers is $x(13/3 - x)$. The function $f(x) = x(13/3 - x)$ attains its maximum at the vertex of the parabola $y = -x^2 + \frac{13}{3}x$, which is at the point $(x, y) = (13/6, 169/36)$. Therefore, the largest possible value for the product is $\boxed{169/36}$.

5. Let $f(x) = \log_2(x^2)$ and $g(x) = 2\log_2 x$. Is it true that $f = g$? Explain!

Solution. $f \neq g$ since these functions have different domains: the domain of f consists of all numbers but 0, and the domain of g consists of positive numbers.