A LOCAL-GLOBAL PRINCIPLE FOR WEAK APPROXIMATION ON VARIETIES OVER FUNCTION FIELDS

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Abstract. We present a new perspective on the weak approximation conjecture of Hassett and Tschinkel: formal sections of a rationally connected fibration over a curve can be approximated to arbitrary order by regular sections. The new approach involves the study of how ideal sheaves pullback to Cartier divisors.

1. Statement of results sec-intro

Let k be an algebraically closed field. Let B be a smooth, projective, connected curve over k.

notat-rings

Notation 1.1. For every closed point b of B, denote the Henselian local ring of B at b by $\mathcal{O}^h_{B,b}$ and denote its fraction field $K(\mathcal{O}^h_{B,b})$ by $K^h_{B,b}$. Denote the completion of the local ring by $\widehat{\mathcal{O}}_{B,b}$, and denote its fraction field by $\widehat{K}_{B,b}$.

More generally, for an effective Cartier divisor E in B with $\mathrm{Supp}(E) = \{b_1, \dots, b_N\}$, denote by $\mathcal{O}_{B,E}$ the semilocal ring of B at $\{b_1, \dots, b_N\}$, denote by $\mathcal{O}^h_{B,E}$ the product $\mathcal{O}^h_{B,b_1} \times \cdots \times \mathcal{O}^h_{B,b_N}$, and denote its total ring of fractions of $\mathcal{O}^h_{B,E}$ by $K^h_{B,E} = K^h_{B,b_1} \times \cdots \times K^h_{B,b_N}$. Denote by $\widehat{\mathcal{O}}_{B,E}$ the product $\widehat{\mathcal{O}}_{B,b_1} \times \cdots \times \widehat{\mathcal{O}}_{B,b_N}$, and denote by $\widehat{K}_{B,E}$ its total ring of fractions $\widehat{K}_{B,E} \times \cdots \times \widehat{K}_{B,E}$.

For each of the B-algebras, $\mathcal{O} = \mathcal{O}_{B,E}$, resp., $\mathcal{O} = \mathcal{O}^h_{B,E}$, $\mathcal{O} = \widehat{\mathcal{O}}_{B,E}$, the morphism

$$\mathrm{Spec}\ \mathcal{O} \times_B E \to E$$

is an isomorphism. Thus we identify E with its inverse image in each of these affine B-schemes.

Let X be a normal, projective, connected k-scheme and let $\pi : X \to B$ be a surjective k-morphism whose geometric generic fiber is normal.

defn-comblike

Definition 1.2. Let $s : B \to X$ be a section of π. A *comb-like curve with handle s in X* is a connected, closed curve C in X such that $C \to X$ is a regular immersion and such that the intersection of C with the generic fiber of B equals $s(\eta_B)$. In other words, $s(B)$ is the only irreducible component of C dominating B, and C is reduced at the generic point of $s(B)$.

rmk-comblike

Remark 1.3. Among all comb-like curves are the *combs* with handle s as defined by Kollár-Miyaoka-Mori, and the *combs with broken teeth* as defined by Hassett-Tschinkel. There are many comb-like curves which are neither combs nor combs with broken teeth, most of which are uninteresting. We are interested only in those

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comb-like curves which deform to smooth, connected curves. This imposes many local constraints which we have not attempted to characterize.

Definition 1.4. For an effective Cartier divisor E in B, an *E-section of* π, or just *E-section*, is a B-morphism $s_E : E \to X$. A comb-like curve C in X *deforms to section curves agreeing with s_E Zariski locally*, resp. *étale locally*, *formally locally*, if for $B' = \text{Spec } \mathcal{O}_{B,E}$, resp. for $B' = \text{Spec } \mathcal{O}^h_{B,E}$, $B' = \text{Spec } \widehat{\mathcal{O}}_{B,E}$, there exists a DVR $(R, \mathfrak{n})$ with coefficient field k and an R-flat, closed subscheme $\mathcal{C} \subset \text{Spec } R \times_{\text{Spec } k} B' \times_B X$ such that

(i) the fiber of $\mathcal{C}$ over the residue field Spec $R/\mathfrak{n}$ equals the base change of C to $B' \times_B X$, and
(ii) the fiber of $\mathcal{C}$ over the fraction field Spec $K(R)$ equals the image of a section $s_\eta : \text{Spec } K(R) \times_{\text{Spec } k} B' \to \text{Spec } K(R) \times_{\text{Spec } k} B' \times_B X$ whose restriction to Spec $K(R) \times_{\text{Spec } k} E$ equals the base change of s_E.

In other words, on a Zariski neighborhood, resp. étale neighborhood, formal neighborhood, of the fiber $\pi^{-1}(E)$ in X, C' deforms to section curves all of which contain $s_E(E)$.

rmk-jet

Remark 1.5. Let $E = E_1 \sqcup \cdots \sqcup E_N$ be the connected components of E. Then C deforms to section curves agreeing with s_E étale locally, resp. formally locally, if and only if for every $i = 1, \dots, N$, C deforms to section curves agreeing with $s_E|_{E_i}$ étale locally, resp. formally locally. Thus the étale and formal local conditions can each be studied fiber-by-fiber.

thm-main

Theorem 1.6. *Let $s : B \to X$ be a section of π mapping the geometric generic point of B into the very free locus of the geometric generic fiber of π. (In particular, the very free locus is nonempty and π is a separably rationally connected fibration.) Let E be an effective Cartier divisor in B. Let s_E be an E-section of π. Let C be a comb-like curve with handle s. Assume C deforms to section curves agreeing with s_E étale locally or formally locally. Then C deforms to section curves agreeing with s_E Zariski locally. To be precise, after attaching to C sufficiently many very free teeth in fibers of π at general points of $s(B)$ and with general normal directions, C deforms to comb-like curves all of which are section curves agreeing with s_E over a Zariski open neighborhood of E in B.*

This reduces weak approximation to the problem of finding comb-like curves C with handle s deforming to section curves agreeing with s_E either étale locally or formally locally. There is a criterion due to Hassett guaranteeing existence of such curves. Originally formulated as a criterion in characteristic 0, the extension to positive characteristic poses some minor issue in characteristics 2, 3 and 5. In order to avoid these issues, we use a slight trick.

defn-conic

Definition 1.7. Let B be a smooth, connected, projective k-curve and let E be an effective Cartier divisor in B. An *E-conic* is a smooth, projective k-surface S together with a birational morphism $\nu : P \to \mathbb{P}^1 \times_k B$ such that the restriction

$$\nu : \nu{-}1(\mathbb{P}^1 \times_k (B - E)) \to \mathbb{P}^1 \times_k (B - E)$$

is an isomorphism. The *zero section* of P is the unique B-morphism

$$0_P : B \to P$$

whose composition with ν is the zero section of $\mathbb{P}^1 \times_k B$.

Remark 1.8. **(i)** Given a section s and an E-section s_E, suppose there exists an E-conic $\nu : P \to X \times_B \mathbb{P}^1$ and a comb-like curve C in $X \times_B \mathbb{P}^1$ with handle $(s, 0_P)$ which étale locally or formally locally deforms to section curves agreeing with a given E-section (s_E, t_E). Then Theorem 1.6 applies to produce a section in $X \times_B \mathbb{P}^1$ agreeing with (s_E, t_E). Thus the projection into X is a section of π agreeing with s_E. Therefore, to prove the existence of sections of π agreeing with s_E, it suffices to prove the local condition for $X \times_B \mathbb{P}^1$

(ii) Also,for a divisor E with support $\{b_1, \dots, b_N\}$, the modification $\nu : P \to \mathbb{P}^1 \times_k B$ and the comb-like curve C can both be defined in the neighborhood of every fiber $\pi^{-1}(b_i)$ and then these local pieces can be assembled into a global modification, resp. a global comb-like curve. Thus the existence of ν and C with the required properties can be verified near each fiber $\pi^{-1}(b_i)$ separately.

prop-Hassett

Proposition 1.9 (Hassett). *Let B be a smooth, connected, projective k-curve, let X be a normal, projective, connected k-scheme, and let $\pi : X \to B$ be a surjective k-morphism. Let $s : B \to X$ be a section of π whose image $s(B)$ is contained in the smooth locus of π. Let E be an effective Cartier divisor in B supported at a single point b. Let $\mathcal{O}$ denote $\mathcal{O}^h_{B,b}$, resp. $\widehat{\mathcal{O}}_{B,b}$, and let K denote $K^h_{B,b}$, resp. $\widehat{K}_{B,b}$. Let s_E be an E-section of π.*

Roughly stated, if the étale, resp. formal, germ of s over b gives a K-point of $X(K)$ which is R-equivalent to the K-point of a germ agreeing with s_E, then for some E-conic P, there exists a comb-like curve C in $X \times_B P$ with handle $(s, 0_P)$ which étale locally, resp. formally locally, deforms to section curves agreeing with an E-section (s_E, t_E).

Following is one precise formulation (certainly not the most general). Assume there exists a morphism of K-schemes

$$f : \mathbb{P}^1_K \to \mathit{Spec}\ K \times_B X$$

whose closure S in $\mathit{Spec}\ \mathcal{O} \times_B X$ intersects the closed fiber $\pi^{-1}(b)$ only at smooth points of X and sending the K-points 0, resp. ∞, of $\mathbb{P}^1_K(K)$ to the K-point of the germ of s, resp. to a K-point whose unique extension

$$s' : \mathit{Spec}\ \mathcal{O} \to \mathit{Spec}\ \mathcal{O} \times_B X$$

agrees with s_E. Then there exists an E-conic P and a comb-like curve C in $X \times_B P$ with handle $(s, 0_P)$ which étale locally, resp. formally locally, deforms to section curves agreeing with an E-section (s_E, t_E).

Moreover, if f is an immersion and if the closed fiber of S is smooth, then there exists a comb-like curve C in X with handle s which étale locally, resp. formally locally, deforms to section curves agreeing with s_E (i.e., there is no need for the E-conic).

When combined with Proposition 1.9, Theorem 1.6 unifies and extends several known results regarding weak approximation. Most of these follow immediately; we content ourselves with the following.

Corollary 1.10. [HT06] *With hypotheses as in Theorem 1.6, assume also that* cor-HT *$s(B)$ is contained in the smooth locus of X. Let E be an effective Cartier divisor in B and let s_E be an E-section of π. If the relative dimension of π is ≥ 3 and if every fiber $\pi^{-1}(b)$ in $\pi^{-1}(E)$ is smooth and separably rationally connected, then* *there exists a comb-like curve C with handle s which Zariski locally deforms to section curves agreeing with s_E. If the relative dimension is ≤ 2 and if every fiber $\pi^{-1}(b)$ in $\pi^{-1}(E)$ is smooth and rationally connected, then for the E-conic $P = B \times_k \mathbb{P}^1$ there exists a comb-like curve C in $X \times_B P$ with handle $(s, 0_P)$ which Zariski locally deforms to section curves agreeing with an E-section (s_E, t_E).*

More generally, the two conclusions above hold if for every point b in Supp(E) there exists an immersed, resp. not necessarily immersed, very free rational curve in the smooth locus of $\pi^{-1}(b)$ containing both $s(b)$ and $s_E(b)$ (possibly both the same point).

cor-rational

Corollary 1.11. *With hypotheses as in Theorem 1.6, assume also that $s(B)$ is in the smooth locus of X. Let E be an effective Cartier divisor in B such that X is smooth along $\pi^{-1}(E)$. Assume that for each b in Supp(E), either Spec $\widehat{K}_{B,b} \times_B X$ is $\widehat{K}_{B,b}$-rational or $\widehat{K}_{B,b}$-stably rational. Then for every E-section s_E there exists an E-conic P over B and a comb-like curve C in $X \times_B P$ with handle $(s, 0_P)$ which Zariski locally deforms to section curves agreeing with an E-section (s_E, t_E).*

In particular, for a smooth cubic surface over a local field specializing to a cubic surface with only rational double points and no more than 3 of these, the cubic surface is rational.

cor-cubic

Corollary 1.12. [HT07] *Let S be a smooth cubic surface over Spec $k(\!(t)\!)$ extending to a flat, proper scheme over Spec $k[\![t]\!]$ whose closed fiber is a cubic surface with only rational double points, and at most 3 of these. Then S is $k(\!(t)\!)$-rational.*

Assume X is smooth and the geometric generic fiber of π is a smooth cubic surface. Let s be a section of π. Let E be a Cartier divisor in B such that for every point b in Supp(E) the basechange Spec $\widehat{K}_{B,b} \times_B X$ extends to a flat, proper scheme over Spec $\widehat{\mathcal{O}}_{B,b}$ whose closed fiber is a cubic surface with only rational double points, and at most 3 of these. Then for every E-section s_E, there exists an E-conic P and a comb-like curve C in $X \times_B P$ with handle $(s, 0_P)$ which Zariski locally deforms to section curves agreeing with an E-section (s_E, t_E).

rmk-cubic

Remark 1.13. In fact Hassett and Tschinkel prove this result for families of cubic surfaces whose singular fibers have rational double points, but possibly more than 3 such double points. There is only one cubic surface with 4 rational double points, the zeroset of $X_0X_1X_2 + X_0X_1X_3 + X_0X_2X_3 + X_1X_2X_3$ which has 4 ordinary double points at the vertices of the coordinate tetrahedron. The method in [HT07] uses *strong rational connectedness*, i.e., proving the smooth locus of each singular fiber is rationally connected, combined with Corollary 1.10. For the unique cubic surface with 4 ordinary double points, one can explicitly construct rational curves in the smooth locus connecting any pair of points.

sec-G

2. Extension of two results of Grothendieck

One of Grothendieck's results which is frequently useful in studying coherent sheaves on proper schemes is [Gro63, Corollaire 7.7.8]. The original proof includes a hypothesis which Grothendieck refers to as "*surabondante*" (Grothendieck's emphasis). This hypothesis has subsequently been removed, cf. [Lie06, Proposition 2.1.3]. We would like to briefly explain another way to remove these hypotheses based on the representability of the Quot scheme. Using a general such representability result of Olsson, this leads to the following formulation.

Theorem 2.1. [Gro63, Corollaire 7.7.8], [Lie06, Proposition 2.1.3] *Let Y be an algebraic space. Let $f : \mathcal{X} \to Y$ be a separated, locally finitely presented, algebraic stack over the category (Aff/Y) of affine Y-schemes. Let $\mathcal{F}$ and $\mathcal{G}$ be locally finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules. Assume that $\mathcal{G}$ is Y-flat and has proper support over Y. Then the covariant functor of quasi-coherent $\mathcal{O}_Y$-modules,* thm-Hom

$$\mathcal{T}(\mathcal{M}) := f_*(Hom_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{M})),$$

is representable by a locally finitely presented, quasi-coherent $\mathcal{O}_Y$-module $\mathcal{N}$, i.e., there is a natural equivalence of functors

$$\mathcal{T}(\mathcal{M}) \xrightarrow{\cong} Hom_{\mathcal{O}_Y}(\mathcal{N}, \mathcal{M}).$$

Grothendieck deduced his version of this result as a corollary of another theorem. In fact the theorem also follows from the corollary.

Corollary 2.2. [Gro63, Théorème 7.7.6] *Let $f : \mathcal{X} \to Y$ be as in Theorem 2.1. Let $\mathcal{G}$ be a locally finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module which is Y-flat and has proper support over Y. Then there exists a locally finitely presented, quasi-coherent $\mathcal{O}_Y$-module $\mathcal{Q}$ and a natural equivalence of covariant functors of quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{M}$,* cor-Hom

$$f_*(\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{M}) \xrightarrow{\cong} Hom_{\mathcal{O}_Y}(\mathcal{Q}, \mathcal{M}).$$

Grothendieck in turn deduces [Gro63, Théorème 7.7.6] as a special case of a more general representability result regarding the hyperderived pushforwards of a bounded below complex of locally finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules which are Y-flat and have proper support over Y. Undoubtedly this general representability result extends to the setting of stacks. However, to deduce [Gro63, Corollaire 7.7.8] from [Gro63, Théorème 7.7.6], Grothendieck requires a hypothesis that $\mathcal{F}$ has a presentation by locally free $\mathcal{O}_{\mathcal{X}}$-modules. This is quite a restrictive hypothesis.

Instead one can try to prove Theorem 2.1 directly using Artin's representability theorems. This is precisely what Lieblich does in [Lie06, Proposition 2.1.3]. Here we point out that Theorem 2.1 also follows from from representability of the Quot functor as proved by Olsson, [Ols05, Theorem 1.5].

lem-Hom1

Lemma 2.3. *Let $f : \mathcal{X} \to Y$ be as in Theorem 2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be locally finitely presented, quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules. And let*

$$\phi : \mathcal{F} \to \mathcal{G}$$

be a homomorphism of $\mathcal{O}_{\mathcal{X}}$-modules. Assume $\mathcal{G}$ has proper support over Y, resp. $\mathcal{F}$ and $\mathcal{G}$ have proper support over Y and $\mathcal{G}$ is Y-flat. Then there exists an open subspace U, resp. V, of S with the following property. For every morphism $g : Y' \to Y$ of algebraic spaces, the pullback morphism of sheaves on $Y' \times_Y \mathcal{X}$,

$$(g, Id_{\mathcal{X}})^*\phi : (g, Id_{\mathcal{X}})^*\mathcal{F} \to (g, Id_{\mathcal{X}})^*\mathcal{G}$$

is surjective, resp. an isomorphism, if and only if g factors through U, resp. V.

Proof. Since $\mathrm{Coker}(\phi)$ is a locally finitely presented, quasi-coherent sheaf supported on $\mathrm{Supp}(\mathcal{G})$, $\mathrm{Supp}(\mathrm{Coker}(\phi))$ is closed in $\mathrm{Supp}(\mathcal{G})$. Since $\mathrm{Supp}(\mathcal{G})$ is proper over Y, $f(\mathrm{Supp}(\mathrm{Coker}(\phi)))$ is a closed subset of Y. Define U to be the open complement. For a morphism $g : Y' \to Y$, $(g, \mathrm{Id}_{\mathcal{X}})^*\phi$ is surjective if and only if $\mathrm{Coker}((g, \mathrm{Id}_{\mathcal{X}})^*\phi)$ is zero, i.e., if and only if the support of $\mathrm{Coker}((g, \mathrm{Id}_{\mathcal{X}})^*\phi)$ is empty. Formation of the cokernel is compatible with pullback, i.e.,

$$\mathrm{Coker}((g, \mathrm{Id}_{\mathcal{X}})^*\phi) \cong (g, \mathrm{Id}_{\mathcal{X}})^*\mathrm{Coker}(\phi).$$

Thus the support of $\mathrm{Coker}((g, \mathrm{Id}_{\mathcal{X}})^*\phi)$ is empty if and only if $g(Y')$ is disjoint from $f(\mathrm{Supp}(\mathrm{Coker}(\phi)))$, i.e., if and only if $g(Y')$ is contained in U.

Next assume that $\mathcal{F}$, $\mathcal{G}$ each have proper support over Y and $\mathcal{G}$ is Y-flat. Since $\mathcal{G}$ is Y-flat, the kernel of ϕ is locally finitely presented by [Gro67, Lemme 11.3.9.1] (the property of being locally finitely presented can be checked locally in the fppf topology on $\mathcal{X}$, thus reduces to the case of a morphism of schemes). The support of the kernel is contained in the support of the kernel of $\mathcal{F}$. Thus $\mathrm{Ker}(\phi)$ is a locally finitely presented, quasi-coherent sheaf on the support of $\mathcal{F}$, which is proper over Y. So the support of $\mathrm{Ker}(\phi)$ is also proper over Y. Thus its image under f is a closed subset of Y. Define V to be the open complement of this closed subset in U.

For every morphism $g : Y' \to U$, since $\mathcal{G}$ is Y-flat, the following sequence is exact

$$0 \to (g, \mathrm{Id}_{\mathcal{X}})^*\mathrm{Ker}(\phi) \to (g, \mathrm{Id}_{\mathcal{X}})^*\mathcal{F} \to (g, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \to 0.$$

Thus $(g, \mathrm{Id}_{\mathcal{X}})^*\phi$ is an isomorphism if and only if $(g, \mathrm{Id}_{\mathcal{X}})^*\mathrm{Ker}(\phi)$ is zero, i.e., if and only if $g(Y')$ is contained in V. $\square$

lem-Hom2

Lemma 2.4. *Let $f : \mathcal{X} \to Y$, $\mathcal{F}$ and $\mathcal{G}$ be as in Theorem 2.1. There exists a locally finitely presented, separated morphism of algebraic spaces $h : Z \to Y$ and a morphism of quasi-coherent sheaves on $Z \times_Y \mathcal{X}$*

$$\phi : (h, Id_{\mathcal{X}})^*\mathcal{F} \to (h, Id_{\mathcal{X}})^*\mathcal{G}$$

which represents the contravariant functor associating to every morphism $g : Y' \to Y$ the set of morphisms of quasi-coherent sheaves on $Y' \times_Y \mathcal{X}$

$$\psi : (g, Id_{\mathcal{X}})^*\mathcal{F} \to (g, Id_{\mathcal{X}})^*\mathcal{G}.$$

Proof. By [Ols05, Theorem 1.5], there exists a locally finitely presented, separated morphism $i : W \to Y$ of algebraic spaces and a quotient

$$\theta : (i, \mathrm{Id}_{\mathcal{X}})^*(\mathcal{F} \oplus \mathcal{G}) \to \mathcal{H}$$

representing the Quot functor of flat families of locally finitely presented, quasi-coherent quotients of the pullback of $\mathcal{F} \oplus \mathcal{G}$ having proper support over the base. Denote by

$$\theta_{\mathcal{G}} : (i, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \to \mathcal{H}$$

the composition of the summand

$$e_{\mathcal{G}} : (i, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \hookrightarrow (i, \mathrm{Id}_{\mathcal{X}})^*(\mathcal{F} \oplus \mathcal{G})$$

with θ.

By Lemma 2.3, there is an open subspace Z of W such that for every morphism $j : Y' \to W$, the pullback

$$(j, \mathrm{Id}_{\mathcal{X}})^*\theta_{\mathcal{G}} : (i \circ j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \to (j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{H}$$

is an isomorphism if and only if $j(Y')$ factors through Z. Denote by $h : Z \to Y$ the restriction of i to Z. If $j(Y')$ factors through Z, then $(j, \mathrm{Id}_{\mathcal{X}})^*\theta$ equals the composition

$$(i \circ j, \mathrm{Id}_{\mathcal{X}})^*(\mathcal{F} \oplus \mathcal{G}) \xrightarrow{(\psi,\mathrm{Id})} (i \circ j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G} \xrightarrow{j^*\theta_{\mathcal{G}}} (j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{H}$$

for a unique morphism of quasi-coherent sheaves

$$\psi : (i \circ j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{F} \to (i \circ j, \mathrm{Id}_{\mathcal{X}})^*\mathcal{G}.$$

In particular, applied to $\mathrm{Id}_Z : Z \to Z$, this produces the homomorphism ϕ. And it is straightforward to see that the natural transformation associating to every morphism $j : Y' \to Z$ the homomorphism ψ is an equivalence of functors, i.e., $(h : Z \to Y, \phi)$ is universal. □

Proof of Theorem 2.1. Let $h : Z \to Y$ and ϕ be as in Lemma 2.4. The zero homomorphism $0 : \mathcal{F} \to \mathcal{G}$ defines a Y-morphism $z : Y \to Z$. Since Z is separated over Y, the Y-morphism z is a closed immersion. Since Z is locally finitely presented over Y, the ideal sheaf $\mathcal{I}$ of this closed immersion is a locally finitely presented, quasi-coherent $\mathcal{O}_Z$-module. Thus $z^*\mathcal{I}$ is a locally finitely presented, quasi-coherent $\mathcal{O}_Y$-module. Denote this $\mathcal{O}_Y$-module by $\mathcal{N}$.

Consider the closed subspace Z_1 of Z with ideal sheaf $\mathcal{I}^2$. The restriction of h to Z_1 is a finite morphism, thus equivalent to the locally finitely presented $\mathcal{O}_Y$-algebra $h_*\mathcal{O}_{Z_1}$. Of course this fits into a short exact sequence

$$0 \to z^*\mathcal{I} \to h_*\mathcal{O}_{Z_1} \to \mathcal{O}_Y \to 0$$

where the injection is an ideal sheaf and the surjection is a homomorphism of $\mathcal{O}_Y$-algebras. And the morphism z defines a splitting of this surjection of $\mathcal{O}_Y$-algebras. Thus, as an $\mathcal{O}_Y$-algebra, there is a canonical isomorphism

$$h_*\mathcal{O}_{Z_1} \cong \mathcal{O}_Y \oplus z^*\mathcal{I}.$$

The restriction of ϕ to Z_1 together with adjunction of h^* and h_* defines a homomorphism of $\mathcal{O}_Y$-modules

$$\mathcal{F} \to \mathcal{G} \otimes_{\mathcal{O}_Y} h_*\mathcal{O}_{Z_1}.$$

Using the canonical isomorphism, this homomorphism is of the form

$$\mathcal{F} \xrightarrow{(0,\chi)} \mathcal{G} \oplus (\mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{N}).$$

The homomorphism χ defines a natural transformation of covariant functors of quasi-coherent $\mathcal{O}_Y$-modules $\mathcal{M}$

$$\mathit{Hom}_{\mathcal{O}_Y}(\mathcal{N}, \mathcal{M}) \to f_*(\mathit{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G} \otimes_{\mathcal{O}_Y} \mathcal{M}).$$

By the same argument used to prove the equivalence of parts (a) and (d) of [Gro63, Théorème 7.7.5], this is an equivalence of functors and the induced Y-morphism

$$\underline{\mathrm{Spec}}_Y(\mathrm{Sym}^\bullet(\mathcal{N})) \to Z$$

is an isomorphism. □

Proof of Corollary 2.2. This follows from Theorem 2.1 by taking $\mathcal{F}$ to be $\mathcal{O}_{\mathcal{X}}$. □

3. Pseudo-ideal sheaves

sec-pis

Let $f : X \to Y$ be a flat, locally finitely presented, proper morphism of algebraic spaces. For every morphism of algebraic spaces, $g : Y' \to Y$, denote by $f_{Y'} : X_{Y'} \to Y'$ the basechange of f. Using results of Martin Olsson, the following definitions and results are still valid whenever X is a flat, locally finitely presented, proper algebraic stack over Y. But since our application is to algebraic spaces, we leave the case of stacks to the interested reader.

defn-pis

Definition 3.1. For every morphism $g : Y' \to Y$ of algebraic spaces, a flat family of *pseudo-ideal sheaves of* X/Y *over* Y' is a pair $(\mathcal{F}, u)$ consisting of

(i) a Y'-flat, locally finitely presented, quasi-coherent $\mathcal{O}_{X_{Y'}}$-module $\mathcal{F}$, and
(ii) an $\mathcal{O}_{X_{Y'}}$-homomorphism $u : \mathcal{F} \to \mathcal{O}_{X_{Y'}}$

such that the following induced morphism is zero,

$$u' : \bigwedge^2 \mathcal{F} \to \mathcal{F}, \quad f_1 \wedge f_2 \mapsto u(f_1)f_2 - u(f_2)f_1.$$

For every pair $g_1 : Y_1' \to Y$, $g_2 : Y_2' \to Y$ of morphisms of algebraic spaces, for every pair $(\mathcal{F}_1, u_1)$, resp. $(\mathcal{F}_2, u_2)$, of flat families of pseudo-ideal sheaves of X/Y over Y_1', resp. over Y_2', and for every Y-morphism $h : Y_1' \to Y_2'$, a *pullback map* from $(\mathcal{F}_1, u_1)$ to $(\mathcal{F}_2, u_2)$ over h is an isomorphism of $\mathcal{O}_{X_{Y_1'}}$-modules

$$\eta : \mathcal{F}_1 \to h^*\mathcal{F}_2$$

such that $h^*u_2 \circ \eta$ equals u_1.

The *category of pseudo-ideal sheaves of* X/Y, $\mathrm{Pseudo}_{X/S}$, is the category whose objects are data $(g : Y' \to Y, (\mathcal{F}, u))$ of an affine Y-scheme Y' together with a flat family of pseudo-ideal sheaves of X/Y over Y', and whose Hom sets

$$\mathrm{Hom}((g_1 : Y_1' \to Y, (\mathcal{F}_1, u_1)), (g_2 : Y_2' \to Y, (\mathcal{F}_2, u_2)))$$

are the sets of pairs (h, η) of a Y-morphism $h : Y_1' \to Y_2'$ together with a pullback map η from $(\mathcal{F}_1, u_1)$ to $(\mathcal{F}_2, u_2)$ over h. Identity morphisms and composition of morphisms are defined in the obvious manner. There is an obvious functor from $\mathrm{Pseudo}_{X/Y}$ to the category (Aff/Y) of affine Y-schemes sending every object $(g : Y' \to Y, (\mathcal{F}, u))$ to $(g : Y' \to Y)$ and sending every morphism (h, η) to h.

prop-algebraic

Proposition 3.2. *The category* $\mathrm{Pseudo}_{X/Y}$ *is a limit-preserving algebraic stack over the category* (Aff/Y) *of affine* Y*-schemes. Moreover, the diagonal is quasi-compact and separated.*

Proof. Denote by $\mathrm{Coh}_{X/Y}$ the category of coherent sheaves on X/Y, cf. [LMB00, (2.4.4)]. There is a functor $G : \mathrm{Pseudo}_{X/Y} \to \mathrm{Coh}_{X/Y}$ sending every object $(g : Y' \to Y, (\mathcal{F}, u))$ to $(g : Y' \to Y, \mathcal{F})$ and sending every morphism (h, η) to (h, η). This is a 1-morphism of categories over (Aff/Y). By [LMB00, Théorème 4.6.2.1] and [Sta06, Proposition 4.1], $\mathrm{Coh}_{X/Y}$ is a limit-preserving algebraic stack over (Aff/Y) with quasi-compact, separated diagonal. Thus to prove the proposition, it suffices to prove that G is representable by locally finitely presented, separated algebraic spaces.

Let Y' be a Y-algebraic space and let $\mathcal{F}$ be a locally finitely presented, quasi-coherent $\mathcal{O}_{X_{Y'}}$-module. Since $f_{Y'} : X_{Y'} \to Y'$ is flat, locally finitely presented and proper, by Lemma 2.4, there exists a locally finitely presented, separated morphism $h : Z \to Y'$ of algebraic spaces and a universal homomorphism

$$u : (h, \mathrm{Id}_{X_{Y'}})^* \mathcal{F} \to \mathcal{O}_{X_Z}.$$

In fact, as follows from the proof of Theorem 2.1, there is a locally finitely presented, quasi-coherent $\mathcal{O}_{Y'}$-module $\mathcal{N}$ and a homomorphism of $\mathcal{O}_{X_{Y'}}$-modules

$$\chi : \mathcal{F} \to \mathcal{O}_{X_{Y'}} \otimes_{\mathcal{O}_{Y'}} \mathcal{N}$$

such that $Z = \underline{\mathrm{Spec}}_{Y'}(\mathrm{Sym}^\bullet(\mathcal{N}))$ and such that u is the homomorphism induced by χ.

Since $\mathcal{F}$ is a locally finitely presented, quasi-coherent $\mathcal{O}_{X_{Y'}}$-module which is Y'-flat and has proper support over Y', the same is true of $(h, \mathrm{Id}_{X_{Y'}})^* \mathcal{F}$ relative to Z. Thus, by Theorem 2.1, there exists a locally finitely presented, quasi-coherent $\mathcal{O}_Z$-module $\mathcal{N}$ and a universal homomorphism

$$\chi : \bigwedge^2 (h, \mathrm{Id}_{X_{Y'}})^* \mathcal{F} \to (h, \mathrm{Id}_{X_{Y'}})^* \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{N}.$$

Thus there exists a unique homomorphism of $\mathcal{O}_Z$-modules,

$$\zeta : \mathcal{N} \to \mathcal{O}_Z$$

such that the induced homomorphism of $\mathcal{O}_{X_Z}$-modules

$$\bigwedge^2 (h, \mathrm{Id}_{X_{Y'}})^* \mathcal{F} \xrightarrow{\chi} (h, \mathrm{Id}_{X_{Y'}})^* \mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{N} \xrightarrow{\mathrm{Id} \otimes \zeta} (h, \mathrm{Id}_{X_{Y'}})^* \mathcal{F}$$

equals u'. Let P denote the closed subspace of Z whose ideal sheaf equals $\zeta(\mathcal{N})$. Chasing diagrams, the restriction morphism $h|_P : P \to Y'$ together with the pull-back of u to P is a pair representing $Y' \times_{\mathrm{Coh}_{X/Y}} \mathrm{Pseudo}_{X/Y} \to Y'$. Since $Z \to Y'$ is a locally finitely presented, separated morphism of algebraic spaces (also schematic), and since P is a closed subspace of Y' whose ideal sheaf is locally finitely generated (being the image of the locally finitely presented sheaf $\mathcal{N}$), also $P \to Y$ is a locally finitely presented, separated morphism of algebraic spaces (also schematic). $\square$

Denote by $(g : \mathrm{Hilb}_{X/Y} \to Y, C)$ a universal pair of a morphism of algebraic spaces g and a closed subspace C of $\mathrm{Hilb}_{X/Y} \times_Y X$ which is flat, locally finitely presented, and proper over $\mathrm{Hilb}_{X/Y}$, i.e., an object representing the Hilbert functor, cf. [Art69, Corollary 6.2]. Denote by

$$0 \to \mathcal{I} \xrightarrow{u} \mathcal{O}_{\mathrm{Hilb}_{X/Y} \times_Y X} \to \mathcal{O}_C \to 0.$$

the natural exact sequence, where $\mathcal{I}$ is the ideal sheaf of C in $\mathrm{Hilb}_{X/Y} \times_Y X$.

prop-Hilbopen

Proposition 3.3. *The pair $(\mathcal{I}, u)$ is a family of pseudo-ideal sheaves of X/Y over $\mathit{Hilb}_{X/Y}$. The induced 1-morphism*

$$\iota : \mathit{Hilb}_{X/Y} \to \mathit{Pseudo}_{X/Y}$$

is representable by open immersions.

Proof. Since the kernel of a surjection of flat modules is flat, $\mathcal{I}$ is flat over $\mathrm{Hilb}_{X/Y}$. By [Gro67, Lemme 11.3.9.1], $\mathcal{I}$ is a locally finitely presented, quasi-coherent sheaf. Since the homomorphism u is injective, to prove that u' is zero it suffices to prove the composition $u \circ u'$ is zero. This follows immediately from the definition of u'. Thus the pair $(\mathcal{I}, u)$ is a family of pseudo-ideal sheaves of X/Y over $\mathrm{Hilb}_{X/Y}$.

To prove that ι is representable by open immersions, it suffices to prove that it is representable by quasi-compact, étale monomorphisms of schemes. To this end, let Y' be an affine Y-scheme and let $(\mathcal{F}, v)$ be a flat family of pseudo-ideal sheaves of X/Y over Y'. Denote the cokernel of v by

$$w : \mathcal{O}_{X_{Y'}} \to \mathcal{G}.$$

By [OS03, Theorem 3.2], there is a morphism of algebraic spaces $\sigma : \Sigma \to Y'$ such that $(\sigma, \mathrm{Id}_{X_{Y'}})^*\mathcal{G}$ is flat over Σ and such that Σ is universal among Y'-spaces with this property. Moreover, Σ is a surjective, finitely-presented, quasi-affine monomorphism (in particular schematic). As an aside, please note that the remark preceding [OS03, Theorem 3.2] is incomplete – [OS03, Proposition 3.1] should be properly attributed to Laumon and Moret-Bailly, [LMB00, Théorème A.2].

By [Gro67, Lemma 11.3.9.1], $\mathrm{Ker}((\sigma, \mathrm{Id}_{X_{Y'}})^*w)$ is a locally finitely presented, quasi-coherent sheaf. Moreover, because it is the kernel of a surjection of sheaves which are flat over Σ, it is also flat over Σ. By Lemma 2.3, there is an open subscheme W of Σ such that a morphism $S \to \Sigma$ factors through W if and only if the pullback of

$$(\sigma, \mathrm{Id}_{X_{Y'}})^*\mathcal{F} \to \mathrm{Ker}((\sigma, \mathrm{Id}_{X_{Y'}})^*w)$$

is an isomorphism. Chasing universal properties, it is clear that $W \to Y'$ represents

$$Y' \times_{\mathrm{Pseudo}_{X/Y}} \mathrm{Hilb}_{X/Y}.$$

Thus ι is representable by finitely-presented, quasi-affine monomorphisms of schemes.

It only remains to prove that ι is étale. Because ι is a finitely-presented it remains to prove that ι is formally étale. Thus, let $Y' = \mathrm{Spec}\ A'$ where A' is a local Artin $\mathcal{O}_Y$-algebra with maximal ideal $\mathfrak{m}$ and residue field κ. And let

$$0 \to J \to A' \to A \to 0$$

be an infinitesimal extension, i.e., $\mathfrak{m}J$ is zero. Let $(\mathcal{F}, u)$ be a pseudo-ideal sheaf of X/Y over Y', and assume the basechange to Spec A is an ideal sheaf with A-flat cokernel. Since ι is a monomorphism, formal étaleness for ι precisely says that $Y' \to \mathrm{Pseudo}_{X/Y}$ factors through ι, i.e., u is injective and $\mathrm{Coker}(u)$ is A'-flat.

To prove this use the local flatness criterion, e.g., as formulated in [Gro67, Proposition 11.3.7]. This criterion is an equivalence between the conditions of

(i) injectivity of u is injective and A'-flatness of $\mathrm{Coker}(u)$
(ii) and injectivity of

$$u \otimes_{A'} \kappa : \mathcal{F} \otimes_{A'} \kappa \to \mathcal{O}_{X_{Y'}} \otimes_{A'} \kappa$$

By hypothesis, (i) holds after basechange to A. Thus (ii) holds after basechange to A. But since $A'/\mathfrak{m}$ equals $A/\mathfrak{m}$, (ii) for the original family over A' is precisely the same as (ii) for the basechange family over A. Thus also (i) holds over A'. □

The significance of pseudo-ideal sheaves has to do with restriction to Cartier divisors. Let D be an effective Cartier divisor in X, considered as a closed subscheme of X, and assume D is flat over Y. Denote by $\mathcal{I}_D$ the pullback

$$\mathcal{I}_D := \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_D$$

on $\mathrm{Hilb}_{X/Y} \times_X D$. And denote by

$$u_D : \mathcal{I}_D \to \mathcal{O}_{\mathrm{Hilb}_{X/Y} \times_X D}$$

the restriction of u.

prop-flat

Proposition 3.4. *The locally finitely presented, quasi-coherent sheaf $\mathcal{I}_D$ is flat over $Hilb_{X/Y}$. Thus the pair $(\mathcal{I}_D, u_D)$ is a flat family of pseudo-ideal sheaves of D/Y over $Hilb_{X/Y}$.*

Proof. Associated to the Cartier divisor D there is an injective homomorphism of invertible sheaves
$$t' : \mathcal{O}_X(-D) \xrightarrow{\mathcal{O}}_X .$$
This induces a morphism of locally finitely presented, quasi-coherent sheaves
$$t : \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to \mathcal{I}.$$
The cokernel of t is $\mathcal{I}_D$. By the local flatness criterion, [Gro67, Proposition 11.3.7], to prove that t is injective and $\mathcal{I}_D$ is flat over $\mathrm{Hilb}_{X/Y}$, it suffices to prove that the "fiber" of t over every point of $\mathrm{Hilb}_{X/Y}$ is injective. Thus, let κ be a field, let $y : \mathrm{Spec}\ \kappa Y$ be a morphism, and let $\mathcal{I}_y \xrightarrow{\mathcal{O}}_{X_y}$ be an ideal sheaf. Since D is Y-flat, the homomorphism of locally free sheaves
$$t'_y : \mathcal{O}_X(-D) \otimes_{\mathcal{O}_Y} \kappa \to \mathcal{O}_X \otimes_{\mathcal{O}_Y} \kappa$$
is injective, thus a flat resolution of $\mathcal{O}_D \otimes_{\mathcal{O}_Y} \kappa$. In particular, $Tor_2^{\mathcal{O}_{X_y}}(\mathcal{O}_{X_y}/\mathcal{I}_y, \mathcal{O}_{D_y})$ equals zero because there is a flat resolution of $\mathcal{O}_{D_y}$ with amplitude $[-1, 0]$. By the long exact sequence of Tor associated to the short exact sequence
$$0 \to \mathcal{I}_y \to \mathcal{O}_{X_y} \to \mathcal{O}_{X_y}/\mathcal{I}_y \to 0,$$
there is an isomorphism
$$Tor_1^{\mathcal{O}_{X_y}}(\mathcal{I}_y, \mathcal{O}_{D_y}) \cong Tor_2^{\mathcal{O}_{X_y}}(\mathcal{O}_{X_y}/\mathcal{I}_y, \mathcal{O}_{D_y}) = 0.$$
But this Tor sheaf is precisely the kernel of
$$t_y : \mathcal{I}_y \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \to \mathcal{I}_y.$$
Thus t_y is injective, and so $\mathcal{I}_D$ is flat over $\mathrm{Hilb}_{X/Y}$. □

notat-iotaD

Notation 3.5. Denote by
$$\iota_D : \mathrm{Hilb}_{X/Y} \to \mathrm{Pseudo}_{D/Y}$$
the 1-morphism associated to the flat family $(\mathcal{I}_D, u_D)$ of pseudo-ideal sheaves of D/Y over $\mathrm{Hilb}_{X/Y}$. This is the *divisor restriction map.*

Since $\mathrm{Hilb}_{X/Y}$ and $\mathrm{Pseudo}_{D/Y}$ are both locally finitely presented over Y, ι_D is locally finitely presented. Since $Hilb_{X/Y}$ is an algebraic space, ι_D is representable (by morphisms of algebraic spaces). Since the diagonal morphism of $\mathrm{Pseudo}_{D/Y}$ over Y is separated, and since $\mathrm{Hilb}_{X/Y}$ is separated over Y, ι_D is separated.

ssec-inf

3.1. **Infinitesimal study of the divisor restriction map.** Let A' be a local Artin $\mathcal{O}_Y$-algebra with maximal ideal $\mathfrak{m}$ and residue field κ. And let
$$0 \to J \to A' \to A \to 0$$
be an infinitesimal extension, i.e., $\mathfrak{m}J$ is zero. Denote by $X_{A'}$, resp. X_A, X_κ, the fiber product of $X \to Y$ with $\mathrm{Spec}\ A' \to Y$, resp. $\mathrm{Spec}\ A \to Y$, $\mathrm{Spec}\ \kappa \to Y$.

Let $(\mathcal{F}_{A'}, u_{A'})$ be a pseudo-ideal sheaf of D/Y over $\mathrm{Spec}\ A'$. Denote by $(\mathcal{F}_A, u_A)$, resp. $(\mathcal{F}_\kappa, u_\kappa)$, the restriction of $(\mathcal{F}_{A'}, u_{A'})$ to A, resp. to κ. Let $\mathcal{I}_A$ be the ideal sheaf of a flat family C_A of closed subschemes of X/Y over Spec A. Denote by $\mathcal{I}_\kappa$, resp. C_κ, the restriction of $\mathcal{I}_A$ to κ, resp. of C_A to κ. And assume that ι_D sends $\mathcal{I}_A$ to $(\mathcal{F}_A, u_A)$.

prop-inf

Proposition 3.6. *Let n be a nonnegative integer. Assume that C_κ is a regular immersion of codimension n in X_κ, cf.* [Gro67, Définition 16.9.2] *(since X_κ is an algebraic space, in that definition one must replace the Zariski covering by affine schemes by an étale covering by affine schemes).*

(i) *The morphism ι_D is* locally unobstructed *at C_κ in the following sense. For every étale morphism Spec $R_{A'} \to X_{A'}$ such that the pullback I_κ of $\mathcal{I}_\kappa$ in R_κ is generated by a regular sequence, there exists an ideal $I_{A'}$ in $R_{A'}$ whose restriction I_A to R_A equals the pullback of $\mathcal{I}_A$ and whose "local pseudo-ideal sheaf" $(I_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D, v)$ equals the pullback of $(\mathcal{F}_{A'}, u_{A'})$. Moreover, the set of such ideals $I_{A'}$ is naturally a torsor for the R_κ-submodule*

$$J \otimes_\kappa \mathcal{O}_X(-D) \cdot \mathit{Hom}_{R_\kappa}(I_\kappa, R_\kappa/I_\kappa)$$

of

$$J \otimes_\kappa \mathit{Hom}_{R_\kappa}(I_\kappa, R_\kappa/I_\kappa)$$

(here $\mathcal{O}_X(-D)\cdot$ denote multiplication by the inverse image ideal of $\mathcal{O}_X(-D)$).

(ii) *There exists an element ω in*

$$J \otimes_\kappa H^1(C_\kappa, \mathcal{O}_X(-D) \cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$$

which equals 0 if and only if there exists a flat family $C_{A'}$ of closed subschemes of X/Y over Spec A' whose restriction to Spec A equals C_A and whose image under ι_D equals $(\mathcal{F}_{A'}, u_{A'})$. When it equals 0, the set of such families $C_{A'}$ is naturally a torsor for the κ-vector space

$$J \otimes_\kappa H^0(C_\kappa, \mathcal{O}_X(-D) \cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa})).$$

In particular, if $h^1(C_\kappa, \mathcal{O}_X(-D)\cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}_\kappa^2, \mathcal{O}_{C_\kappa}))$ equals 0, then ι_D is smooth at $[C_\kappa]$.

Proof. **(i)** Let $v_{(A,1)}, \dots, v_{(A,n)}$ be a regular sequence in R_A generating I_A. Denote by v_A the R_A-module homomorphism

$$v_A : R_A^{\oplus n} \to R_A, \quad v_A(\mathbf{e}_i) = v_{(A,i)}.$$

Denote the associated R_A-module homomorphism by

$$v'_A : R_A^{\oplus \binom{n}{2}} \to R_A^{\oplus n}, \quad v'_A(\mathbf{e}_i \wedge \mathbf{e}_j) = v_{(A,i)}\mathbf{e}_j - v_{(A,j)}\mathbf{e}_i.$$

Since $(v_{(A,1)}, \dots, v_{(A,n)})$ is a regular sequence, the following sequence is exact

$$R_A^{\oplus \binom{n}{2}} \xrightarrow{v'_A} R_A^{\oplus n} \to I_A \to 0.$$

Thus there is an exact sequence

$$(R_A \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}} \xrightarrow{v'_A \otimes \mathrm{Id}} (R_A \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \to (I_A \otimes_{\mathcal{O}_X} \mathcal{O}_D) \to 0.$$

Denote by $F_{A'}$ the $R_{A'}$-module whose associated quasi-coherent sheaf is the pullback of $\mathcal{F}_{A'}$. Since $\mathcal{F}_A$ equals $\mathcal{I}_A \otimes_{\mathcal{O}_X} \mathcal{O}_D$, there is an isomorphism

$$F_{A'}/JF_{A'} \cong I_A \otimes_{\mathcal{O}_X} \mathcal{O}_D$$

compatible with the maps $u_{A'}$ and u_A. Since $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n}$ is a projective $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-module, there exists an $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-module homomorphism

$$a : (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \to F_{A'}$$

whose restriction to A is the surjection above. So, by Nakayama's lemma, this map is also surjective. Since both the source and target of the surjection are A'-flat, also the kernel is A'-flat. Thus there is also a lifting of the set of generators of the kernel, i.e., there is an exact sequence of $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-modules

$$(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}} \xrightarrow{b} (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \xrightarrow{a} F_{A'} \to 0$$

whose restriction to A is the short exact sequence above.

The composition of the surjection with $u_{A'}$ defines an $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-module homomorphism

$$w_{A'} : (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \to (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$$

whose restriction to A equals $v_A \otimes \mathrm{Id}$. There is an associated map

$$w'_{A'} : (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}} \to (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n}, \ \ w'_{A'}(\mathbf{e}_i \wedge \mathbf{e}_j) = w_{A'}(\mathbf{e}_i)\mathbf{e}_j - w_{A'}(\mathbf{e}_j)\mathbf{e}_i.$$

Since $(\mathcal{F}_{A'}, u_{A'})$ is a pseudo-ideal sheaf, the induced map $u'_{A'}$ equals 0. Therefore the image of $w'_{A'}$ is contained in the kernel of a. Since $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}}$ is a free $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$-module, there is a lifting

$$c : (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}} \to (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}}$$

such that $w'_{A'} = b \circ c$. In particular, the restriction of c to A is an isomorphism. Thus, by Nakayama's lemma, c is surjective. A surjection of free modules of the same finite rank is automatically an isomorphism. Thus there is a presentation

$$(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus \binom{n}{2}} \xrightarrow{w'_{A'}} (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \xrightarrow{a} F_{A'} \to 0.$$

Because both $R_{A'}$ and $R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D$ are A'-flat, there is a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J \otimes_\kappa R_\kappa & \longrightarrow & R_{A'} & \longrightarrow & R_A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J \otimes_\kappa R_\kappa \otimes_{\mathcal{O}_X} \mathcal{O}_D & \longrightarrow & R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D & \longrightarrow & R_A \otimes_{\mathcal{O}_X} \mathcal{O}_D & \longrightarrow & 0 \end{array}$$

where the vertical maps are each surjective. By the snake lemma, the induced map

$$R_{A'}/(J\mathcal{O}_X(-D) \cdot R_{A'}) \to R_A \times_{(R_A \otimes_{\mathcal{O}_X} \mathcal{O}_D)} (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$$

is an isomorphism. Thus, for every integer $i = 1, \dots, n$, there exists an element $v_{(A',i)}$ in $R_{A'}$ whose image in R_A equals $v_{(A,i)}$ and whose image in $R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D$ equals $w_{A'}(\mathbf{e}_i)$. Moreover, the set of all such elements is naturally a torsor for $J \otimes_\kappa (\mathcal{O}_X(-D) \cdot R_\kappa)$. In other words, there is an $R_{A'}$-module homomorphism

$$v_{A'} : R_{A'}^{\oplus n} \to R_{A'}$$

whose restriction to A equals v_A and such that $v_{A'} \otimes \mathrm{Id}$ equals $w_{A'}$.

Since $(v_{(A,1)}, \dots, v_{(A,n)})$ is a regular sequence in R_A, also $(v_{(A',1)}, \dots, v_{(A',n)})$ is a regular sequence in $R_{A'}$. One way to see this is to tensor the Koszul complex $K^\bullet(R_{A'}, v_{A'})$ of $v_{A'}$ with the short exact sequence of A'-modules

$$0 \to J \to A' \to A \to 0.$$

Since the terms in the Koszul complex are free $R_{A'}$-modules, and since $R_{A'}$ is A'-flat, the associated sequence of complexes is exact

$$0 \to J \otimes_A K^\bullet(R_A, v_A) \to K^\bullet(R_{A'}, v_{A'}) \to K^\bullet(R_A, v_A) \to 0.$$

Thus there is a long exact sequence of Koszul cohomology

$$\cdots \to J \otimes_A H^n(K^\bullet(R_A, v_A)) \to H^n(K^\bullet(R_{A'}, v_{A'})) \to H^n(K^\bullet(R_A, v_A)) \to J \otimes_A H^{n+1}(K^\bullet(R_A, v_A)) \to \dots$$

Since v_A is regular, $H^{n-1}(K^\bullet(R_A, v_A))$ is zero, which then implies $H^{n-1}(K^\bullet(R_{A'}, v_{A'}))$ by the long exact sequence above. Thus also $v_{A'}$ is regular. Moreover, this gives a short exact sequence

$$0 \to J \otimes_\kappa (R_A/I_A) \to R'_A/\mathrm{Image}(v_{A'}) \to R_A/I_A \to 0$$

from which it follows that $R_{A'}/\mathrm{Image}(v_{A'})$ is A'-flat. Denote by $I_{A'}$ the image of $v_{A'}$.

Both the pseudo-ideal $I_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D$ and $F_{A'}$ equal the cokernel of $w'_{A'}$. Thus there is a unique isomorphism between them compatible with $w'_{A'}$. Moreover, since the compositions

$$(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \xrightarrow{a} F'_A \xrightarrow{u_{A'}} (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$$

and

$$(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)^{\oplus n} \xrightarrow{v_{A'} \otimes \mathrm{Id}} I'_A \otimes_{\mathcal{O}_X} \mathcal{O}_D \to (R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$$

both equal $w_{A'}$, the isomorphism above is compatible with the maps to $(R_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D)$. Thus it is an isomorphism of pseudo-ideal sheaves. Therefore there exists $I_{A'}$ satisfying all the conditions in the proposition.

For every lift $I_{A'}$ of I_A which is A-flat, the surjection $v_A : R_A^{\oplus n} \to I_A$ lifts to a surjection $R_{A'}^{\oplus n} \to I_{A'}$. Composing this surjection with the injection $I_{A'} \hookrightarrow R_{A'}$, it follows that every lift $I_{A'}$ arises from a lift $v_{A'}$ of v_A. As mentioned previously, the set of lifts $v_{A'}$ whose restriction to A equals v_A and with $v_{A'} \otimes \mathrm{Id}$ equal to $w_{A'}$ is naturally a torsor for

$$J \otimes_\kappa \mathrm{Hom}_{R_{A'}}(R_{A'}^{\oplus n}, \mathcal{O}_X(-D) \cdot R_\kappa).$$

But the translate of a lift by a homomorphism with image in I_κ gives the same ideal $I_{A'}$ (just different surjections from $R_{A'}^{\oplus n}$ to the ideal). Thus, the set of lifts $I_{A'}$ of I_A whose pseudo-ideal sheaf is the pullback of $(\mathcal{F}_{A'}, u_{A'})$ is naturally a torsor for

$$J \otimes_\kappa \mathrm{Hom}_{R_{A'}}(R_{A'}^{\oplus n}, \mathcal{O}_X(-D) \cdot (R_\kappa/I_\kappa)) = J \otimes_\kappa \mathrm{Hom}_{R_\kappa}(I_\kappa, R_\kappa/I_\kappa).$$

(ii) Let Γ be an indexing set and let $(\mathrm{Spec}\, R^\gamma_{A'} \to X_{A'})_{\gamma \in \Gamma}$ be an étale covering such that for every γ, either $I^\gamma_\kappa \subset R^\gamma_\kappa$ is generated by a regular sequence of length n or else equals R^γ_κ. By the hypothesis that C_κ is a regular immersion of codimension n, there exists such a covering.

Suppose first that I^γ_κ equals R^γ_κ. Then also $(F^\gamma_\kappa, u^\gamma_\kappa)$ is isomorphic to $R^\gamma_\kappa \otimes_{\mathcal{O}_X} \mathcal{O}_D \xrightarrow{=} R^\gamma_\kappa \otimes_{\mathcal{O}_X} \mathcal{O}_D$. It is straightforward to see that the only deformations over A' of $R^\gamma_\kappa \xrightarrow{=} R^\gamma_\kappa$, resp. $R^\gamma_\kappa \otimes_{\mathcal{O}_X} \mathcal{O}_D \xrightarrow{=} R^\gamma_\kappa \otimes_{\mathcal{O}_X} \mathcal{O}_D$, as pseudo-ideals are $R^\gamma_{A'} \xrightarrow{=} R^\gamma_{A'}$, resp. $R^\gamma_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D \xrightarrow{=} R^\gamma_{A'} \otimes_{\mathcal{O}_X} \mathcal{O}_D$. Thus there is a lifting $I^\gamma_{A'}$ of I^γ_A, and it is unique.

On the other hand, if I^γ_κ is generated by a regular sequence of length n, by (i) there exist liftings $I^\gamma_{A'}$ and the set of all liftings is a torsor for

$$J \otimes_\kappa \mathcal{O}_X(-D) \cdot \mathrm{Hom}_{R^\gamma_\kappa}(I^\gamma_\kappa, R^\gamma_\kappa/I^\gamma_\kappa).$$

For a collection of liftings $(I^{\gamma}_{A'})_{\gamma\in\Gamma}$, for every $\gamma_1,\gamma_2\in\Gamma$, the basechanges of $I^{\gamma_1}_{A'}$ and $I^{\gamma_2}_{A'}$ to

$$R^{\gamma_1,\gamma_2}_{A'} := R^{\gamma_1}_{A'} \otimes_{\mathcal{O}_X} R^{\gamma_2}_{A'}$$

differ by an element $\omega^{\gamma_1,\gamma_2}$ in

$$J \otimes_\kappa \mathcal{O}_X(-D)\cdot \mathrm{Hom}_{R^{\gamma_1,\gamma_2}_\kappa}(I^{\gamma_1,\gamma_2}_\kappa, R^{\gamma_1,\gamma_2}_\kappa/I^{\gamma}_\kappa).$$

It is straightforward to see that $(\omega^{\gamma_1,\gamma_2})_{\gamma_1,\gamma_2\in\Gamma}$ is a 1-cocycle for

$$\mathcal{O}_X(-D)\cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}^2_\kappa, \mathcal{O}_{C_\kappa})$$

with respect to the given étale covering. Moreover, changing the collection of lifts $(I^{\gamma}_{A'})$ by translating by elements in

$$(J \otimes_\kappa \mathcal{O}_X(-D)\cdot \mathrm{Hom}_{R^{\gamma}_\kappa}(I^{\gamma}_\kappa, R^{\gamma}_\kappa/I^{\gamma}_\kappa))_{\gamma\in\Gamma}$$

precisely changes $(\omega^{\gamma_1,\gamma_2})_{\gamma_1,\gamma_2\in\Gamma}$ by a 1-coboundary. Therefore, the cohomology class

$$\omega \in J\otimes_\kappa H^1(C_\kappa, \mathcal{O}_X(-D)\cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}^2_\kappa, \mathcal{O}_{C_\kappa}))$$

is well-defined and equals 0 if and only if there is a lifting $\mathcal{I}_{A'}$ as in the proposition. And in this case, the set of liftings is a torsor for the set of compatible families $(t^\gamma)_{\gamma\in\Gamma}$ of elements t^γ in

$$J \otimes_\kappa \mathcal{O}_X(-D)\cdot \mathrm{Hom}_{R^{\gamma}_\kappa}(I^{\gamma}_\kappa, R^{\gamma}_\kappa/I^{\gamma}_\kappa),$$

i.e., it is a torsor for the set of elements t in

$$J\otimes_\kappa H^0(C_\kappa, \mathcal{O}_X(-D)\cdot \mathit{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}^2_\kappa, \mathcal{O}_{C_\kappa})).$$

□

4. Deforming comb-like curves

sec-def

Let k be an algebraically closed field. Let B be a smooth, projective, connected curve over k. Let X be a normal, projective, connected k-scheme. And let $\pi : X \to B$ be a surjective k-morphism of relative dimension d. Assume that the geometric generic fiber of π is normal and contains a very free rational curve inside its smooth locus.

lem-comblike

Lemma 4.1. *There is an open subscheme of $Hilb_{X/k}$ precisely parametrizing the comb-like curves.*

Proof. There is an open subscheme of $\mathrm{Hilb}_{X/k}$ parametrizing regular immersions of codimension $d-1$. The degree of such a curve over B is a locally constant integer-valued function. Thus the open subscheme of comb-like curves is precisely the open subset on which the degree equals 1. □

prop-comblike

Proposition 4.2. *Let $s : B \to X$ be a section of π mapping the geometric generic point of B into the very free locus of the geometric generic fiber of π. Let E be an effective Cartier divisor in B, and denote by D the inverse image Cartier divisor $\pi^{-1}(E)$ in X. Let C be a comb-like curve with handle s which is contained in the scheme-theoretic union of $s(B)$ and D. There exists an integer N with the following property. For every comb-like curve C' obtained by attaching at least N very free curves in fibers of π to C at general points of $s(B)$ and with general normal directions to $s(B)$, the divisor restriction map, $\iota_D : Hilb_{X/k} \to Pseudo_{D/k}$, is smooth at $[C']$.*

Proof. Since s maps the geometric generic point of B into the very free locus of π, with the exception of finitely many points, for every k-point b of B, π is smooth at $s(b)$ and for every normal direction to $s(B)$ in X at $s(b)$, there exists a very free curve in $\pi^{-1}(b)$ containing $s(b)$ and having the specified normal direction. So long as the attachment points $s(b)$ is a smooth point of C (which also holds with finitely many exceptions), the curve obtained by attaching this very free curve to C is again comb-like. The final exception is that we only attach very free curves at points b not contained in E.

Let C' be a curve obtained from C by attaching a collection of very free rational curves in fibers of π at points of $s(B)$ which are smooth points of C. Denote by C'' the subcurve of C' consisting of $s(B)$ and all the attached very free curves in fibers of π. Since $C' \to X$ is a regular immersion, the sheaf $\mathcal{I}_{C'/X}/\mathcal{I}^2_{C'/X}$ is a locally free $\mathcal{O}_{C'}$-module, where $\mathcal{I}_{C'/X}$ is the ideal sheaf of C' in X. Denote by $N_{C'/X}$ the dual sheaf,

$$N_{C'/X} := Hom_{\mathcal{O}_{C'}}(\mathcal{I}_{C'/X}/\mathcal{I}^2_{C'/X}, \mathcal{O}_{C'}).$$

The subsheaf $\mathcal{O}_X(-D)\cdot N_{C'/X}$ is supported on C'' since C is contained in the scheme theoretic union of $s(B)$ and D. Moreover, $\mathcal{O}_X(-D)\cdot N_{C'/X}|_{s(B)}$ is obtained from $\mathcal{O}_X(-D)\cdot N_{C/X}|_{s(B)}$ by the construction from [GHS03, Lemma 2.6]. Thus, by the argument from [GHS03, p. 62], if C' is obtained from C by attaching sufficiently many very free curves in fibers of π to general points of $s(B)$ with general normal directions, then $h^1(C'', \mathcal{O}_X(-D)\cdot N_{C'/X})$ equals 0. Thus, by Proposition 3.6, the divisor restriction map is smooth at $[C']$. $\square$

cor-comblike

Corollary 4.3. *With the same hypotheses as in Proposition 4.2, let F be a subdivisor of E, let s_F be a section of π over F, assume that the pseudo-ideal sheaf $\iota_D([C])$ deforms to pseudo-ideal sheaves agreeing with s_F. Then there exists a section s_∞ of π whose restriction to F equals s_F.*

Proof. Since none of the very free curves attached to C intersect D, $\iota_D([C])$ equals $\iota_D([C'])$. Let (T,t) be as in Definition 1.2. Form the pullback morphism

$$\iota_{D,T} : T \times_{\psi D/k, \iota_D} \mathrm{Hilb}_{X/k} \to T.$$

By Proposition 4.2, this morphism is smooth at $(t,[C'])$. By Lemma 4.1, there is an open neighborhood U of $(t,[C'])$ precisely parametrizing pairs for which the closed subscheme is a comb-like curve. The restriction of ι_D to this open subscheme is also smooth at $(t,[C'])$. Thus it is dominant. So there exists a k-point $(t_\infty,[C_\infty])$ in this open subscheme such that $t_\infty \neq t$. Therefore the pseudo-ideal sheaf of C_∞ agrees with s_F. Let s_∞ be the unique section of π such that $s_\infty(B)$ is an irreducible component of C_∞. Then the ideal sheaf of $s_\infty(E_F)$ is a subsheaf of the pseudo-ideal sheaf of C_∞ over E_F. But two ideal sheaves of sections can be contained one in the other if and only if they are equal. Therefore the restriction of s_∞ to E_F is the given section over E_F. In particular, the restriction of s_∞ to F equals s_F.1 $\square$

sec-proofs

5. Proofs of the main theorems

Proof of Theorem 1.6. Let E' be an effective divisor in B containing E and such that for the inverse image Cartier divisor $D = \pi^{-1}(E')$, C is contained in the scheme-theoretic union $s(B)\cup D$. The étale local deformation of C to section curves agreeing with s_E in particularly restricts on D to a deformation of the pseudo-ideal sheaf of C to ideal sheaves of section curves agreeing with s_E. Thus the theorem follows from Corollary 4.3. $\square$

Proof of Proposition 1.9. First we consider the case when the $\mathcal{O}$-morphism

$$f : \mathbb{P}^1_K \to \text{Spec } \mathcal{O} \times_B X$$

does not necessarily extend to all of $\mathbb{P}^1_{\mathcal{O}}$.

claim-Hassett1

Claim 5.1. *There exists an E-conic $\nu : P \to \mathbb{P}^1 \times_k B$ such that the composition*

$$f \circ \nu : \textit{Spec } K \times_B P \to \mathbb{P}^1_K \to \textit{Spec } \mathcal{O} \times_B X$$

extends to an $\mathcal{O}$-morphism

$$f_P : \textit{Spec } \mathcal{O} \times_B P \to \textit{Spec } \mathcal{O} \times_B X.$$

To prove this, first form the closure Σ of the graph of f in $\mathbb{P}^1_{\mathcal{O}} \times_B X$. This gives a possibly singular, 2-dimensional scheme whose projection onto $\mathbb{P}^1_{\mathcal{O}}$ is projective and birational. By resolution of singularities for 2-dimensional schemes, e.g. as in [Lip78], there exists a projective desingularization of Σ. The composition with projection onto $\mathbb{P}^1_{\mathcal{O}}$ is a projective, birational morphism, thus equals the blowing up of $\mathbb{P}^1_{\mathcal{O}}$ along a coherent ideal sheaf $\mathcal{I}_{\mathcal{O}}$ supported on the closed fiber. Since every such ideal sheaf $\mathcal{I}_{\mathcal{O}}$ equals the basechange of a coherent ideal sheaf $\mathcal{I}$ on $\mathbb{P}^1 \times_k B$ supported on $\mathbb{P}^1 \times_k \{b\}$, the blowing up of $\mathbb{P}^1_{\mathcal{O}}$ along $\mathcal{I}_{\mathcal{O}}$ equals Spec $\mathcal{O} \times_B P$ where P is the E-conic obtained by blowing up $\mathbb{P}^1 \times_k B$ along $\mathcal{I}$. This proves Claim 5.1.

claim-Hassett2

Claim 5.2. *Denote the graph of f_P by*

$$\Gamma_{f_P} : \textit{Spec } \mathcal{O} \times_B P \to \textit{Spec } \mathcal{O} \times_B (X \times_B P).$$

Each comb-like curve C in P which equals $s(B)$ over $B - \{b\}$ equals the isomorphic projection of a unique comb-like curve C_Γ in $X \times_B P$ whose basechange to Spec $\mathcal{O} \times_B (X \times_B P)$ equals Γ_{f_P}(Spec $\mathcal{O} \times_B C$).

There exists a nonnegative integer r such that C is contained in the scheme-theoretic union of $s(B)$ and $(r \cdot b) \times_B P$. Denote by Z the scheme-theoretic union of $s(B)$ and $(r \cdot b) \times_B (X \times_B P)$. The pullback of $\mathcal{O}_Z$ to Spec $\mathcal{O} \times_B (X \times_B P)$ equals the scheme-theoretic union of the germ of $s(B)$ and $(r \cdot b) \times_B (X \times_B P)$. The ideal sheaf $\mathcal{I}$ of $\Gamma_{f_P}(\text{Spec } \mathcal{O} \times_B C)$ in Spec $\mathcal{O} \times_B Z$ is a coherent subsheaf supported entirely on $(r \cdot b) \times_B (X \times_B P)$. Thus there exists a unique coherent ideal sheaf $\mathcal{I}_{C_\Gamma}$ in $\mathcal{O}_Z$ supported entirely on $(r \cdot b) \times_B (X \times_B P)$ whose pullback to Spec $\mathcal{O} \times_B (X \times_B P)$ equals $\mathcal{I}$. The corresponding closed subscheme C_Γ of Z is the unique closed subscheme whose basechange to Spec $\mathcal{O} \times_B (X \times_B P)$ equals $\Gamma_{f_P}(\text{Spec } \mathcal{O} \times_B C)$. Moreover, because the projection of $\Gamma_{f_P}(\text{Spec } \mathcal{O} \times_B C)$ into Spec $\mathcal{O} \times_B P$ equals Spec $\mathcal{O} \times_B C$, also the projection of C_Γ into P equals C.

Finally, since Spec $\mathcal{O} \times_B X$ is smooth along S, also Spec $\mathcal{O} \times_B (X \times_B P)$ is smooth along $\Gamma_{f_P}(\text{Spec } \mathcal{O} \times_B P)$. Thus Γ_{f_P} is a regular immersion. Since Spec $\mathcal{O} \times_B C$ is a Cartier divisor in Spec $\mathcal{O} \times_B P$, also $\Gamma_{f_P}(C)$ is regularly immersed in Spec $\mathcal{O} \times_B P$. Thus C_Γ is regularly immersed étale locally, resp. formally locally, near $\{b\} \times_B (X \times_B P)$. Now the property of being a regular immersion is a formal local property since the Koszul cohomology relative to the complete local ring of the images of a minimal set of generators of the stalk of the ideal equals the completion of the Koszul cohomology relative to the local ring of the minimal set of generators. Thus C_Γ is a regular immersion at every point of C_Γ in $\{b\} \times_B (X \times_B P)$. Since also

$X \times_B P$ is smooth along $(s, 0_P)$, C_Γ is regularly immersed at every point of C_Γ, i.e., C_Γ is a comb-like curve in $X \times_B P$. This proves Claim 5.2.

claim-Hassett3

Claim 5.3. *If there exists a comb-like curve C in P with handle 0_P which étale locally deforms to section curves agreeing with the restriction $\infty_P|_E$, then there exists a comb-like curve C in $X \times_B P$ with handle $(s, 0_P)$ which étale locally deforms to section curves agreeing with an E-section (s_E, t_E).*

There is a unique E-section of $X \times_B P$ such that $\Gamma_{f_P} \circ \infty_P$ equals (s_E, t_E). By Claim 5.2, there exists a comb-like curve C_Γ in $X \times_B P$ with handle $(s, 0_P)$ whose basechange to Spec $\mathcal{O} \times_B (X \times_B P)$ equals $\Gamma_{f_P}(\text{Spec } \mathcal{O} \times_B C)$. If Spec $\mathcal{O} \times_B C$ deforms to a family of section curves D_t in Spec $\mathcal{O} \times_B P$ agreeing with $\infty_P|_E$, then $\Gamma_{f_P}(\text{Spec } \mathcal{O} \times_B C)$ deforms to the family of section curves $\Gamma_{f_P}(D_t)$ in Spec $\mathcal{O} \times_B (X \times_B P)$ agreeing with $\Gamma_{f_P} \circ \infty_P = (s_E, t_E)$. This proves Claim 5.3.

claim-Hassett4

Claim 5.4. *There exists a comb-like curve C in P with handle 0_P which Zariski locally deforms to section curves in P agreeing with $\infty_P|_E$.*

On P consider the invertible sheaf $\mathcal{L} = \mathcal{O}_P(\infty_P(B) - 0_P(B) - \pi^*E)$. Since the restriction to the generic fiber of $P \to B$ is a degree 0 invertible sheaf on $\mathbb{P}^1$, it has a 1-dimensional space of global sections. Thus the pushforward of $\mathcal{L}$ to B is a torsion-free, coherent $\mathcal{O}_B$-module of rank 1, i.e., it is an invertible sheaf. Thus there exists an effective divisor Δ in B, not intersecting E such that $\mathcal{O}_P(\infty_P(B) - 0_P(B) + \pi^*\Delta - \pi^*E)$ has a global section. In other words, there is an effective divisor B in P, necessarily vertical, such that

$$\infty_P(B) + \pi^*\Delta = 0_P(B) + \pi^*E + B.$$

Define C to be $0_P(B) + \pi^*E + B$ and let the curves D_t be the members of the pencil spanned by C and $\infty_P(B) + \pi^*\Delta$. Since $\infty_P(B) + \pi^*\Delta$ is a section curve over E, all but finitely many members of this pencil are section curves over E. Since the base locus of the pencil contains $\infty_P(B) \cap \pi^*E$, these section curves agree with $\infty_P(B)$ over E. This proves Claim 5.4, completing the proof of the proposition.

Of course if f extends to a closed immersion

$$f_{\mathcal{O}} : \text{Spec } \mathcal{O} \times_k \mathbb{P}^1 \to \text{Spec } \mathcal{O} \times_B X,$$

then one can take P to be $\mathbb{P}^1 \times_k B$ and one can work with $f_{\mathcal{O}}$ instead of Γ_{f_P}. In this way one produces a comb-like curve in X which étale locally, resp. formally locally, deforms to section curves in X agreeing with s_E. $\square$

Proof of Corollary 1.10. First of all, the results of the first paragraph follow from the results of the second, since every pair of points in a smooth, projective, separably rationally connected variety are connected by a very free rational curve, which may also be taken to be immersed if the dimension is ≥ 3. Next, if there is a very free rational curve in $\pi^{-1}(b)$ containing $s(b)$ and $s_E(b)$, then deformations of the rational curve in fibers of π and intersecting both s and s_E are unobstructed. Thus there exist both étale local and formal local morphisms

$$f_{\mathcal{O}} : \text{Spec } \mathcal{O} \times_k \mathbb{P}^1 \to \text{Spec } \mathcal{O} \times_B X$$

whose closed fiber is the given very free rational curve in $\pi^{-1}(b)$ and whose 0-section, resp. ∞-section, is the basechange of s, resp. agrees with s_E. If the original very free rational curve in $\pi^{-1}(b)$ is an immersion, then also $f_{\mathcal{O}}$ is an immersion. So the corollary follows from Theorem 1.6 and Proposition 1.9. $\square$

Proof of Corollary 1.11. Let U be a dense Zariski open subset of $\mathbb{P}^n_{\text{Spec }\widehat{K}_{B,b}}$ which is isomorphic to a dense Zariski open subset of Spec $\widehat{K}_{B,b} \times_B (X \times_k \mathbb{P}^r_k)$. Possibly after deforming s, we may assume the $\widehat{K}_{B,b}$-point of Spec $\widehat{K}_{B,b} \times_B X$ determined by s is the image of a $\widehat{K}_{B,b}$-point of Spec $\widehat{K}_{B,b} \times_B (X \times_k \mathbb{P}^r_k)$ which is contained in V. Similarly, there exists a formal section of Spec $\widehat{\mathcal{O}}_{B,b} \times_B X$ agreeing with s_E whose $\widehat{K}_{B,b}$-point is the image of a $\widehat{K}_{B,b}$-point of V. The isomorphism of U and V sends these two $\widehat{K}_{B,b}$-points of V two $\widehat{K}_{B,b}$ points of U. Every pair of $\widehat{K}_{B,b}$-points of $\mathbb{P}^n_{\text{Spec }\widehat{K}_{B,b}}$ are contained in a line which is isomorphic to $\mathbb{P}^1_{\text{Spec }\widehat{K}_{B,b}}$. The composition with the isomorphism from U to V determines a $\widehat{K}_{B,b}$-rational map

$$f_K : \mathbb{P}^1_{\text{Spec }\widehat{K}_{B,b}} \dashrightarrow \text{Spec } \widehat{K}_{B,b} \times_B (X \times_k \mathbb{P}^r_k)$$

which extends to all of $\mathbb{P}^1_{\text{Spec }\widehat{K}_{B,b}}$ by the valuative criterion of properness. Now apply Theorem 1.6 and Proposition 1.9. $\square$

Proof of Corollary 1.12. Let b be a point of B for which X_b is a cubic surface with rational double point singularities. The Galois group of $\widehat{K}_{B,b}$ is topologically cyclic. The action on the Néron-Severi lattice of the geometric generic fiber Spec $\overline{\widehat{K}}_{B,b} \times_B X$ sends a topological generator to an element in the automorphism group of the Néron-Severi lattice, namely $\mathcal{W}(E_6)$. The conjugacy class of this element in $\mathcal{W}(E_6)$ is well-defined independent of the choice of topological generator. We call it the *monodromy class*.

The possibilities for the monodromy class are listed in [Man86, Table 1, p. 176]. As explained in [Man86, 31.1(2), p. 174], the cubic surface Spec $\widehat{K}_{B,b} \times_B X$ is minimal if and only if the monodromy class is in one of the first 5 rows of the table, and the minimal model is a Del Pezzo of degree ≥ 5 unless it is one of the first 11 rows. Since every smooth Del Pezzo of degree ≥ 5 is rational, Corollary 1.11 applies unless the monodromy class is in one of the first 11 rows.

claim-cubic1

Claim 5.5. *If the cubic surface X_b has only rational double points, and no greater than* 3 *of these, then the monodromy class is not in one of the first* 11 *rows.*

The 8^{th} column of each row lists the orbit decomposition of the 27 lines. Each orbit specializes to a line in the singular fiber whose multiplicity is at least as large as the size of the orbit. More precisely, the multiplicity of each line in the singular fiber equals the sum of the lengths of the orbits specializing to that line.

Every line in the singular fiber not containing a singular point necessarily has multiplicity 1 since the normal bundle of a line in the nonsingular locus of a cubic surface is isomorphic to the nonspecial invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$. Thus it remains to consider the multiplicities of lines containing a rational double point. For each double point, the length of the scheme of lines in the cubic surface containing that double point is always 6. The multiplicity of a line joining two or more double points equals the sum of the contributions from each double point it contains. Thus, if there are m rational double points, then the sum of the multiplicities of the lines containing at least one double point equals $6m$. Since all other lines have multiplicity 1, the number of orbits of size 1 must be at least $27 - 6m$. Since every orbit type in the first 11 rows begins with 1^e for $e \leq 3$, such a monodromy class may occur only if the cubic surface has 4 double points. This proves Claim 5.5. Finally apply Corollary 1.11 to complete the proof. $\square$

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