CUBIC FOURFOLDS AND SPACES OF RATIONAL CURVES

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Abstract. For a general nonsingular cubic fourfold $X \subset \mathbb{P}^5$ and $e \geq 5$ an odd integer, we show that the space M_e parametrizing rational curves of degree e on X is non-uniruled. For $e \geq 6$ an even integer, we prove that the generic fiber dimension of the maximally rationally connected fibration of M_e is at most one, i.e. passing through a very general point of M_e there is at most one rational curve. For $e < 5$ the spaces M_e are fairly well understood and we review what is known.

1. Introduction

intro

Let k be an algebraically closed field of characteristic 0; unless stated otherwise all schemes will be considered to be of finite type over k. Let X be a nonsingular cubic fourfold in $\mathbb{P}^5_k$. For each integer $e \geq 1$ denote by M_e the variety which parametrizes smooth, geometrically connected curves in X of degree e and arithmetic genus 0, i.e. M_e is the scheme of rational curves of degree e in X. In Section 2 we'll discuss different constructions of this space and how they are related. For the moment all that matters is that M_e is an irreducible variety of dimension $3e+1$, a nontrivial fact discussed in Section 2 as well. The question we consider in this paper is the birational geometry of M_e, specifically the Kodaira dimension of M_e and, in case the Kodaira dimension is negative, the dimension of the general fiber of the maximally rationally connected fibration of M_e (c.f. [18]). This question was originally raised by Joe Harris with regard to the rationality/irrationality of cubic fourfolds. It is a pleasure to acknowledge useful conversations with Joe Harris.

Let $\overline{M}_e$ be a desingularization of a compactification of M_e. We rephrase the question on the dimension of the fibers of the MRC fibration as follows: Given a very general point $p \subset \overline{M}_e$, what is the maximal dimension of a closed subvariety $Z \subset \overline{M}_e$ which contains p and which is rationally connected? Equivalently, if $\overline{M}_e \to Q$ is the MRC fibration in the sense of [18, Def. IV.5.3], what is the difference $\dim(M_e) - \dim(Q)$? For example, if this number is zero then for a very general point $p \in \overline{M}_e$ there is no nonconstant morphism $\mathbb{P}^1 \to \overline{M}_e$ whose image contains p, i.e. $\overline{M}_e$ is not uniruled. We note that the invariant $\dim Z$ is a birational invariant of M_e (in other words it does not matter which choice of desingularized compactification we take).

Discussions with Joe Harris have shown that for small values of e these maximal dimensions can be tabulated as follows:

e	1	2	3	4
$\dim M_e$	4	7	10	13
$\dim Z$	0	3	2	3

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We pause to explain this table: The case of lines is well known, namely M_1 is a 4-dimensional hyperKähler manifold [3, Prop. 1]. In the case of conics, the family of all conics which are residual to a fixed line forms a 3 dimensional rationally connected family Z. In the case of cubic rational curves, one notes that a general cubic rational curve lies on a unique cubic surface and moves in a 2-dimensional linear system on it, so Z has dimension at least 2. A general quartic rational curve lies on a unique cubic threefold, and moves in a 3-dimensional rationally connected family on it (c.f. [11, Theorem 8.2]), so Z has dimension at least 3. This gives a lower bound for the numbers in the bottom row of the diagram, which is easily seen to be the actual dimension of Z when $e = 1$ or 2. For $e = 3$ and $e = 4$, we have not verified these numbers give the actual dimensions, but we would be surprised if they turn out to be larger. We mention a conjecture of Ana-Maria Castravet that for $e = 4$ the actual dimension of Z is precisely 3 and the target of the MRC fibration of $\overline{M}_4$ is birational to the relative intermediate Jacobian of the family of hyperplane sections of X – in other words, this conjecture says that the relative intermediate Jacobian of the family of hyperplane sections of X is not uniruled.

A

Theorem 1.1. *Let $X \subset \mathbb{P}^5$ be a very general cubic fourfold. For every odd degree $e \geq 5$, the variety M_e is non-uniruled. For every even degree $e \geq 6$ the variety M_e has $dim(Z) \leq 1$.*

Actually the method of this paper gives something a little better than Theorem 1.1 as we now explain.

B

Theorem 1.2. *Let $X \subset \mathbb{P}^5$ a smooth cubic hypersurface, and let $\overline{M_e}$ be a non-singular projective model of M_e. There is a canonical section $\omega_e \in H^0(\overline{M}_e, \Omega^2_{\overline{M}_e})$ with the following property:*
(a) In case e is odd, $e \geq 5$. If X is general, and p a general point of $\overline{M}_e$, then ω_e induces a nondegenerate pairing on $T_p(\overline{M}_e)$.
(b) In case e is even, $e \geq 6$. If X is general, and $p \in \overline{M}_e$ a general point, then the linear transformation $T_p(\overline{M}_e) \to T_p^\vee(\overline{M}_e)$ induced by ω_e has a 1-dimensional kernel.

B.5

Corollary 1.3. *If e is odd and at least 5, then the Kodaira dimension $\kappa(M_e) \geq 0$ for X general.*

The corollary follows as the form $\omega_e^{(3e+1)/2}$ is a nonzero section of the canonical line bundle.

In Section 2 we recall the different moduli spaces and how they are related. In Section 4 we give a general method to produce ω_e on the Kontsevich moduli stack $\mathcal{M}_e$ of stable maps for any $e \geq 1$. By Lemma 3.5 this gives a corresponding 2-form ω_e on $\overline{M}_e$. In Section 5 we describe how to compute the associated alternating pairing on Zariski tangent spaces of $\mathcal{M}_e$. In Section 6 we show that this pairing is nondegenerate for a general point of $\mathcal{M}_5$. The case $e = 5$ is particularly nice as almost no explicit calculations are necessary. In Section 7 we prove the nondegeneracy for general odd degree $e \geq 5$. In Section 8 we prove the kernel of the pairing is 1 dimensional in the even degree $e \geq 6$ case. In Section 9 we give a sketch that M_6 is also not uniruled and pose some questions about the spaces M_e.

Finally, Theorem 1.2 implies Theorem 1.1 thanks to the following lemma.

C

Lemma 1.4. *Suppose that M is a smooth, projective scheme, ω is a 2-form on M, and at a general point $p \in \overline{M}_e$ the rank of the 2-form ω is r. Then dim$(Z) \leq$ dim$(M) - r$, i.e., the codimension of the maximal rationally connected subvariety Z passing through a very general point of M is at least r.*

Proof. By [18, Theorem IV.5.8] , if $\dim(Z) = d > 1$, then for a very general point $p \in M$ there is a morphism $g : \mathbb{P}^1 \to M$ whose image passes through p and such that g^*T_M contains a locally free subsheaf $\mathcal{E} \subset g^*T_M$ with $\mathcal{E}$ an ample locally free sheaf of rank d and whose cokernel is a trivial locally free sheaf of rank $n-d$ (actually this result is in the proof of [18, Theorem IV.5.8], not in the statement). But we also have the sheaf map induced by ω: $g^*T_M \to g^*\Omega_M$. Since g^*T_M is semi-positive, the sheaf $g^*\Omega_M$ is seminegative. As there is no nonzero map from an ample locally free sheaf to a seminegative locally free sheaf, we conclude that $\mathcal{E}$ is contained in the kernel of the sheaf map. So $d \leq \dim(M) - r$.

□

2. Discussion of moduli spaces

moduli

In this section we discuss three related functors, each of which gives a compactification of the space of smooth rational curves. The spaces representing these functors are birational, and since we are studying birational properties of these spaces the distinction between them is not crucial to the rest of the paper. But we find it useful to pause, compare these three spaces, and point out what is and is not known about them.

Let $X \subset \mathbb{P}^N$ be a quasi-projective scheme and let M_e denote the scheme which parametrizes families of smooth, proper, geometrically connected curves $C \subset X$ of arithmetic genus 0 and degree e. Even before we try to compactify M_e, there are already several versions of M_e and we concentrate on two of these M_e^h and M_e^c. Here M_e^h is an open subscheme of the Hilbert scheme $\text{Hilb}^{et+1}(X)$ as defined in [10]. And M_e^c is an open subvariety of the Chow variety $\text{Chow}_{1,e}(X)$ defined in [18, Def. I.3.20]. Please note that there is not universal acceptance of the definition of the Chow variety (e.g. there is also the definition in [2]), but we find Kollár's definition best suited to our needs. In particular, we have the following comparison between M_e^h and M_e^c.

HilbChow

Lemma 2.1. *There exists a fundamental class morphism $FC : \left(M_e^h\right)^{sn} \to M_e^c$ where $\left(M_e^h\right)^{sn}$ is the* semi-normalization *of M_e^h as defined in* [18, Def. I.7.2.1]. *The morphism FC is an isomorphism. Therefore the inverse $(FC)^{-1} : M_e^c \to M_e^h$ is the semi-normalization of M_e^h, and in particular it is bijective on points.*

Proof. This follows from [18, Thm. 6.3] and the semi-normal analogue of Zariski's main theorem. □

It can happen that M_e^h is not semi-normal so that M_e^c and M_e^h are not isomorphic, for example whenever M_e^h is non-reduced. A simple example of this is given by any pair (X, L) where $L \subset \mathbb{P}^3$ is a line and $X \subset \mathbb{P}^3$ is a smooth hypersurface of degree $d \geq 4$ which contains L. In this case there is a unique connected component of M_1^h whose reduced scheme consists just of the point $[L] \in M_1^h$, but M_1^h is non-reduced.

For the special case that $X \subset \mathbb{P}^n$ is a smooth cubic hypersurface, which is the case of interest in this paper, we suspect that M_e^h is always semi-normal.

ques-1

Question 2.2. If $X \subset \mathbb{P}^n$ is a smooth cubic hypersurface, is M_e^h semi-normal? Is M_e^h normal?

There are some partial answers. For n arbitrary and $e = 1$, M_1^h is smooth by [5, Thm. 7.8]. For $n = 3$ and e arbitrary, M_e^h is an open subset of a projective space and so it is smooth. For $n = 4$ and $e = 2, 3$, M_e^h is smooth by [12, Lemma 3.2, Lemma 4.6]. For $n = 4$ and e arbitrary, then M_e^h is an irreducible, reduced, local complete intersection scheme by [13]. So, by Serre's criterion, to prove that M_e^h is normal it remains to prove that M_e^h is nonsingular in codimension one. We do not know whether this is true.

In the general case of a projective scheme $X \subset \mathbb{P}^N$, we denote by $\overline{M}_e^h$ the closure of M_e^h in $\mathrm{Hilb}^{et+1}(X)$ and we denote by $\overline{M}_e^c$ the closure of M_e^c in $\mathrm{Chow}_{1,e}(X)$. These are the first two compactifications of M_e which we consider.

The Chow variety and the Hilbert scheme have been studied by algebraic geometers since they were introduced. Many results have been proved, and very readable accounts exist [18, 19]. For instance, it follows from [18, Thm. I.6.3] that the morphism FC extends to a morphism $\mathrm{FC} : (\overline{M}_e^h)^{sn} \to \overline{M}_e^c$. But both $\overline{M}_e^c$ and $\overline{M}_e^h$ have certain drawbacks. For example the morphism $(\mathrm{FC})^{-1}$ does not extend to a regular morphism $\overline{M}_e^c \to \overline{M}_e^h$ (this fails even in the case $X = \mathbb{P}^N$). Moreover, the closed subsets $\overline{M}_e^h \subset \mathrm{Hilb}^{et+1}(X)$ and $\overline{M}_e^c \subset \mathrm{Chow}_{1,e}(X)$ are usually not connected components. Because of this, it is difficult to carry out an infinitesimal analysis of $\overline{M}_e^h$ and $\overline{M}_e^c$ as in [18, Section I.2].

In the case of a projective scheme $X \subset \mathbb{P}^N$ over a field k of characteristic 0, there is a third compactification of M_e which is very useful: the Kontsevich moduli space of stable maps. A prestable map to X of genus g with r marked points and degree e (over a field k) is a triple $(C, (p_1, \ldots, p_r), f)$ where C is a geometrically connected, reduced, at-worst-nodal curve of arithmetic genus g, where $p_1, \ldots, p_r$ is an ordered set of k-rational points in the nonsingular locus of C, and where $f : C \to X$ is a morphism of k-schemes such that the degree of $f^*\mathcal{O}(1)$ is e. The triple is called a *stable map* if there are no infinitesimal automorphisms of the triple. There is a good notion of families of stable maps and morphisms between stable maps, and there is a proper Deligne-Mumford stack over k, $\overline{\mathcal{M}}_{g,n}(X, e)$ parametrizing stable maps of genus g with r marked points and degree e. The coarse moduli space $\overline{M}_{g,n}(X, e)$ of the stack $\overline{\mathcal{M}}_{g,n}(X, e)$ is a projective k-scheme. The reader is referred to [4, 8] for details.

In particular, when $X \subset \mathbb{P}^4$ is a smooth cubic hypersurface, we will denote by $\overline{\mathcal{M}}_e$ the Kontsevich moduli space of stable maps of genus zero with no marked points and of degree e.

HilbKv

Lemma 2.3. *The scheme M_e^h is isomorphic to an open substack of $\overline{\mathcal{M}}_e$.*

Proof. This follows from the definitions of $\overline{\mathcal{M}}_e$ and M_e^h. ∎

Also there is an analogue of the morphism FC, i.e. a 1-morphism FC : $(\overline{\mathcal{M}}_e)^{sn} \to \overline{M}_e^c$. One drawback of $\overline{\mathcal{M}}_e$ as compared to $\overline{M}_e^h$ and $\overline{M}_e^c$ is that it is a stack rather than a scheme, which makes some arguments more technical. On the other hand, the deformation and obstruction theory of $\overline{\mathcal{M}}_e$ and the "boundary" are understood quite well. These are the key components in the proof of the following proposition.

irred

Proposition 2.4 ([14]). *For $n \geq 5$ and $X \subset \mathbb{P}^n$ a general cubic hypersurface, the stack $\overline{\mathcal{M}}_e$ is irreducible and reduced of the expected dimension $(n-2)e + (n-4)$ and has only local complete intersection singularities.*

Proof. First of all, we just point out that the proposition is false for $n = 3$: if $e \geq 3$, then $\overline{\mathcal{M}}_e$ will be disconnected. For $n = 4$, a slight variant of the proposition is true but the proof involves different methods which are developed in [13] (there is another irreducible component corresponding to e-fold covers of lines, but the proposition holds if we replace $\overline{\mathcal{M}}_e$ by the complement of this locus). For $n \geq 6$, the proposition follows from [14, Prop. 7.4]. The only remaining case is $n = 5$ which we now consider.

We prove the proposition by applying [14, Cor. 7.3]. This result reduces the proposition to proving that the condition $\mathcal{B}(X, \tau_1(e), f)$ of [14, Def. 6.1] holds for $e = 1$ and $e = 2$. And this condition has three parts (1), (2) and (3): constancy of the fiber dimension of ev_f, irreducibility of a general fiber of ev_f, and existence of a *free* stable map of degree e.

First we consider $e = 1$. The condition (3) is quite easy to verify: in characteristic zero, for every smooth cubic hypersurface $X \subset \mathbb{P}^n$ (for any $n \geq 4$ in fact), and for every point $p \in X$, there exists a line $L \subset X$ containing p. Also, by [6, Prop. 4.14], for every smooth cubic hypersurface X and a *general* point $p \in X$, every line $L \subset X$ containing p is *free*, i.e. $T_X|_L$ is generated by global sections. Choosing any line L passing through p, we see that condition (3) holds for $e = 1$. Moreover, by [18, Cor. II.3.5.4.2], the evaluation morphism $\text{ev}_f : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is smooth over p. And the fiber F is canonically a complete intersection of hypersurfaces in $\mathbb{P}^{n-1}$ of dimension $n-4$. Whenever $n \geq 5$, this complete intersection is connected (by computing $H^0(F, \mathcal{O}_F)$, for instance). Since F is smooth and connected, it is irreducible. This proves that condition (2) holds for $e = 1$. Finally, if X is a *general* hypersurface, then by [12, Thm. 2.1], condition (1) holds for $e = 1$.

Next we consider $e = 2$. The condition (3) can be checked by considering any double cover of a free line $L \subset X$. To check condition (1) and (2), observe that there is an a priori lower bound on the dimension of every irreducible component of every fiber of $\text{ev}_f : \overline{\mathcal{M}}_{0,1}(X, 2) \to X$, namely the difference of the expected dimension of $\overline{\mathcal{M}}_{0,1}(X, 2)$ and $\dim(X)$, which is 4 (or $2n - 6$ for general $n \geq 3$). To prove (1), it suffices to prove that *every* fiber of ev_f has dimension exactly 4. And to prove (2), we need to prove that some fiber is irreducible and reduced of dimension 4.

Now assume that X contains no linear $\mathbb{P}^2$: this certainly holds for a general cubic hypersurface in $\mathbb{P}^5$. Then every stable map $f : C \to X$ of degree 2 which is not a double cover of a line is an embedded plane conic. And the span of the conic C, say $\Lambda \subset \mathbb{P}^n$, intersects X in a plane cubic curve $C' \subset \Lambda$. Of course $C \subset C'$, and the residual curve is a line $L \subset X$. Conversely, for a general pair of a line $L \subset X$

and a Λ which contains L, the residual to L in $\Lambda \cap X$ is a plane conic. Using this, we see that the set of embedded plane conics in X passing through a general point p, is isomorphic to an open subset of the space of lines $\overline{\mathcal{M}}_1$. This space is smooth of dimension 4 when $n = 5$. So to finish the proof of (1) and (2), it suffices to show that this set is Zariski dense in $\mathrm{ev}_f^{-1}(p)$ for every $p \in X$. In other words, we have to prove for every $p \in X$, that the subset of $\mathrm{ev}_f^{-1}(p)$ consisting of double covers of lines is not dense in any irreducible component of $\mathrm{ev}_f^{-1}(p)$.

When $\mathrm{ev}_f : \overline{\mathcal{M}}_{0,1}(X, 1) \to X$ is flat (which holds for general X as mentioned above), the space of lines in X containing p has dimension 1 (or $n - 4$ for general $n \geq 4$). So the space of double covers of lines whose image contains p has dimension 3 (or $n - 2$ for general $n \geq 4$). Since we have an a priori lower bound of 4 for every fiber of ev_f, we conclude that the subset of double covers of lines is not dense in any irreducible component of ev_f. This finishes the proof of conditions (1) and (2) for $e = 2$ when $X \subset \mathbb{P}^5$ is a general cubic hypersurface. □

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Remark 2.5. We have a few remarks on this proposition.

(1) Even though the proof above only works for a general hypersurface X, we suspect the proposition holds for *every* smooth cubic hypersurface $X \subset \mathbb{P}^n$.
(2) In fact the argument above proves much more than the proposition, namely for every stable genus 0 A-graph τ and every flag f of τ, $\mathcal{B}(X, \tau, f)$ holds. In particular $\overline{\mathcal{M}}(X, \tau)$ is irreducible.

cor-irred

Corollary 2.6. *For $X \subset \mathbb{P}^5$ a general cubic hypersurface, the schemes M_e^c and M_e^h are irreducible and reduced of dimension $3e + 1$. They are birational to each other and to $\overline{\mathcal{M}}_e$.*

trace

3. Trace maps and descent for p-forms

In the next section we define linear maps

$$H^{q+1}\left(X, \Omega^{p+1}\right) \to H^q\left(\overline{\mathcal{M}}_{g,r}(X, e), \Omega^p\right). \tag{1}$$

In particular, when $q = 0$ this gives a method for producing p-forms on the Kontsevich moduli stack. But for our applications, we actually want a p-form on a desingularization of the coarse moduli space of the stack. To accomplish this, we associate *trace maps* to any proper, generically étale morphism of schemes, $f : Y \to Z$, with Z normal:

$$\mathrm{Tr}_f^p : f_*(\Omega_Y^p) \to (\Omega_Y^p)^{\vee\vee}. \tag{2}$$

We construct these trace maps in greater generality than is strictly needed to prove the main theorem. In particular we work over a ground field k, but we assume neither that k is algebraically closed, nor that k is of characteristic 0 (although there is an assumption on the characteristic stated in the proposition). The results in this section are well-known, but we could not find a particularly elementary reference, so we reprove them here.

trace-et

3.1. **Construction for f étale.** In this subsection, the ground field k is arbitrary: it is not necessarily algebraically closed and there is no condition on the characteristic. First we consider the case when f is finite and étale. Let n denote the degree of the morphism f, i.e. $f_*\mathcal{O}_Y$ is locally free of rank n. In this case the pullback morphism on Kähler differentials $(df)^\dagger : f^*\Omega^1_Z \to \Omega^1_Y$ is an isomorphism and for every p this isomorphism induces an isomorphism

$$\alpha_p : f^*\Omega^p_Z \to \Omega^p_Y. \tag{3}$$

Pushing forward, we have canonical isomorphisms

$$f_*\alpha_p : f_*f^*\Omega^p_Z \to f_*\Omega^p_Y. \tag{4}$$

On the other hand, we have a canonical isomorphism

$$\beta_p : \Omega^p_Z \otimes_{\mathcal{O}_Z} f_*\mathcal{O}_Y \to f_*f^*\Omega^p_Z. \tag{5}$$

Since f is finite and étale, in particular it is finite and flat. Recall the usual trace morphism $f_*\mathcal{O}_Y \to \mathcal{O}_Z$ is defined by composing the morphism of $\mathcal{O}_Z$-algebras $f_*\mathcal{O}_Y \to \mathit{Hom}_{\mathcal{O}_Z}(f_*\mathcal{O}_Y, f_*\mathcal{O}_Y)$ with the trace $\mathrm{Tr} : \mathit{Hom}_{\mathcal{O}_Z}(f_*\mathcal{O}_Y, f_*\mathcal{O}_Y) \to \mathcal{O}_Z$. We denote this morphism by $\mathrm{Tr}^0_f : f_*\mathcal{O}_Y \to \mathcal{O}_Z$. We define the $\mathcal{O}_Z$-linear morphism $\mathrm{Tr}^p_f : f_*\Omega^p_Y \to \Omega^p_Z$ to be the unique morphism such that $\mathrm{Tr}^p_f \circ f_*\alpha_p \circ \beta_p$ equals

$$n \cdot \mathrm{Id} \otimes \mathrm{Tr}^0_f : \Omega^p_Z \otimes_{\mathcal{O}_Z} f_*\mathcal{O}_Y \to \Omega^p_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_Z. \tag{6}$$

We summarize this construction in the following lemma.

lem-t-e

Lemma 3.1. *Let $f : Y \to Z$ be a finite étale morphism of k-schemes of degree n. For each integer $p \geq 0$, there exists a unique $\mathcal{O}_Z$-linear morphism $Tr^p_f : f_*\Omega^p_Y \to \Omega^p_Z$ such that $Tr^p_f \circ f_*\alpha_p \circ \beta_p = n \cdot Id \otimes Tr^0_f$. Moreover, we have*

(1) *For any open subscheme $U \subset Z$, for any section $\sigma \in H^0(U, \Omega^p_Z)$, and for any section $\tau \in H^0(f^{-1}(U), \Omega^q_Y)$ we have*

$$Tr^{p+q}_f f_* (f^*\sigma \wedge \tau) = \sigma \wedge Tr^q_f f_*\tau. \tag{7}$$

(2) *For any open subscheme $U \subset Z$ and for any section $\tau \in H^0(f^{-1}(U), \Omega^q_Y)$, we have*

$$Tr^{q+1}_f f_* (d\tau) = d\left(Tr^q_f f_*\tau \right). \tag{8}$$

(3) *If $g : X \to Y$ is also a finite étale morphism, then for every p we have*

$$Tr^p_{f\circ g} = Tr^p_f \circ f_* Tr^p_g. \tag{9}$$

trace-gen

3.2. **Construction for f generically étale.** In this section we do not assume that k is algebraically closed. But there is a condition on the characteristic of k stated in Proposition 3.2 (see also (2) of Remark 3.3). Now we construct Tr^p_f in the general case, under the additional assumption that Z is normal and where the target of Tr^p_f is now $(\Omega^p_Z)^{\vee\vee}$ rather than Ω^p_Z.

Let $i : U \hookrightarrow Z$ be the (dense) open subscheme over which f is finite and étale. Define $j : V \hookrightarrow Y$ to be the open subscheme $V = f^{-1}(U)$. And define $g : V \to U$ to be the restriction of f. By Lemma 3.1, we have morphisms $\mathrm{Tr}^p_g : g_*\Omega^p_V \to \Omega^p_U$. Pushing forward by i gives morphisms of quasi-coherent $\mathcal{O}_Z$-modules

$$i_*\mathrm{Tr}^p_g : i_*g_*\Omega^p_V \to i_*\Omega^p_U. \tag{10}$$

There is a canonical isomorphism $(dj)^\dagger : j^*\Omega^p_Y \to \Omega^p_V$. Pushing forward gives an isomorphism

$$(i \circ g)_*(dj)^\dagger : (i \circ g)_* j^*\Omega^p_Y \to i_* g_* \Omega^p_V. \tag{11}$$

Of course $(i\circ g)_* = (f \circ j)_* = f_* \circ j_*$. And there is a canonical morphism $\gamma_p : \Omega^p_Y \to j_* j^* \Omega^p_Y$. Composing all of these gives an $\mathcal{O}_Z$-linear morphism:

$$i_* \mathrm{Tr}^p_g \circ (i \circ g)_*(dj)^\dagger \circ f_*\gamma_p : f_*\Omega^p_Y \to i_*\Omega^p_U. \tag{12}$$

Finally, there is a canonical isomorphism $i_*(di)^\dagger : i_* i^* \Omega^p_Z \to i_*\Omega^p_U$. We denote by $\epsilon_p : f_*\Omega^p_Y \to i_* i^* \Omega^p_Z$ the unique morphism such that $i_*(di)^\dagger \circ \epsilon_p$ equals the morphism above.

There is a canonical morphism

$$\kappa_p : i_* i^* \Omega^p_Z \to i_* i^* \left(\Omega^p_Z\right)^{\vee\vee}. \tag{13}$$

And there is a injective morphism of $\mathcal{O}_Z$-modules,

$$\lambda_p : \left(\Omega^p_Z\right)^{\vee\vee} \to i_* i^* \left(\Omega^p_Z\right)^{\vee\vee}. \tag{14}$$

lem-t-g

Proposition 3.2. *Suppose that $n!$ is relatively prime to char(k). Let Z be a normal k-scheme and let $f : Y \to Z$ be a proper, generically étale morphism of degree n. With notation as above, for each integer $p \geq 0$, there exists a unique $\mathcal{O}_Z$-linear morphism $Tr^p_f : f_*\Omega^p_Y \to \left(\Omega^p_Z\right)^{\vee\vee}$ such that $\lambda_p \circ Tr^p_f = \kappa_p \circ \epsilon_p$.*

Proof. This statement is clearly Zariski local on Z. Thus we assume that Z is an irreducible, normal affine scheme.

Since $\left(\Omega^p_Z\right)^{\vee\vee}$ is torsion-free, it is clear that if Tr^p_f as above exists, then it is unique. Let $g : \widetilde{Y} \to Y$ denote the morphism which is the disjoint union over all irreducible components $Y_i \subset Y$ dominating Z of the normalization of Y_i. Define $\widetilde{f} : \widetilde{Y} \to Z$ to be $f \circ g$. Notice that $\widetilde{f}$ is also proper and generically étale. And $\widetilde{Y}$ is a normal scheme such that every irreducible component of $\widetilde{Y}$ dominates Z, and $g : g^{-1}(V) \to V$ is an isomorphism.

Consider the morphism $(dg)^\dagger : g^*\Omega^1_Y \to \Omega^1_{\widetilde{Y}}$. For each p, we can form the pth exterior power of this map and then take the adjoint to get a morphism

$$\mu_p : \Omega^p_Y \to g_*\Omega^p_{\widetilde{Y}}. \tag{15}$$

If we prove that $\mathrm{Tr}^p_{\widetilde{f}}$ exists, then it follows that $\mathrm{Tr}^p_{\widetilde{f}} \circ f_*\mu_p$ satisfies the hypothesis of Tr^p_f, so Tr^p_f exists. Therefore we are reduced to proving that $\mathrm{Tr}^p_{\widetilde{f}}$ exists. So, without loss of generality, we now assume that Y is normal and every irreducible component of Y dominates Z.

Define $\iota : W \hookrightarrow Z$ to be the maximal open subscheme such that W is smooth, such that $f^{-1}(W)$ is smooth, and such that $f : f^{-1}(W) \to W$ is finite. Since Y and Z are normal and since f is generically finite, it follows that the complement of W in Z has codimension at least 2. Define $T = f^{-1}(W)$ and define $h : T \to W$ to be the restriction of f. If we prove that Tr^p_h exists, then this will be a morphism

$$\mathrm{Tr}^p_h : \iota^* f_*\Omega^p_Y \to \iota^* \left(\Omega^p_Z\right)^{\vee\vee}. \tag{16}$$

The adjoint of this map will be a morphism

$$f_*\Omega^p_Y \to \iota_*\iota^*\left(\Omega^p_Z\right)^{\vee\vee}. \tag{17}$$

There is a canonical morphism $\left(\Omega^p_Z\right)^{\vee\vee} \to \iota_*\iota^*\left(\Omega^p_Z\right)^{\vee\vee}$. Since Z is normal, since the complement of W has codimension 2, and since $\left(\Omega^p_Z\right)^{\vee\vee}$ is reflexive, this morphism is an isomorphism. So the adjoint above is a morphism which satisfies the condition for Tr^p_f. Therefore to prove that Tr^p_f exists, it suffices to prove that Tr^p_h exists. So, without loss of generality, we now assume that Z is a connected, smooth, affine scheme, $f: Y \to Z$ is finite, and Y is smooth. In particular, f is flat.

Actually, what the argument in the previous paragraph shows is that to prove that Tr^p_h exists, it suffices to prove for each irreducible divisor $D \subset Z$, the image of $\kappa_p \circ \epsilon_p$ is contained in the subsheaf which is the image of $(\Omega^p_Z) \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,D}$. Let us call this condition $(\mathcal{I}_D)$.

By the Noether normalization theorem [7, Thm. 13.3], there exists a flat morphism $\pi: Z \to B$ of relative dimension 1 such that $\pi|_D : D \to B$ is dominant. Up to replacing π by the Stein factorization of π, we may also suppose that π is separably generated. Since f is finite, flat and generically étale, also $\pi \circ f : Y \to B$ is flat of relative dimension 1 and separably generated. By generic smoothness, there exists a dense open subset $B^o \subset B$ such that both π and $\pi \circ f$ are smooth over B^o. Since $\pi|_D : D \to B$ is dominant, we can check condition $(\mathcal{I}_D)$ after localizing over B^o. So, without loss of generality, we assume that π and $\pi \circ f$ are both smooth.

Now choose any closed point $z \in D$. The claim is that the image of $\kappa_p \circ \epsilon_p$ is in the image of $\Omega^p_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,z}$. Since the image of $\Omega^p_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,D}$ is the intersection over all $z \in D$ of the image of $\Omega^p_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,p}$, to prove the proposition it suffices to prove the claim. And the claim may be checked after base change to the formal completion of $\mathcal{O}_{Z,z}$. Thus choose an isomorphism $\widehat{\mathcal{O}_{B,\pi(z)}} \cong k[[b_1,\dots,b_s]]$, and choose an isomorphism $\widehat{\mathcal{O}_{X,z}} \cong k[[b_1,\dots,b_s]][[t]]$. Let $f^{-1}(z) = \{w_1,\dots,w_m\}$. For each $i = 1,\dots,m$, choose an isomorphism $\widehat{\mathcal{O}_{Y,w_i}} \cong k[[b_1,\dots,b_s]][[u_i]]$. For each $i = 1,\dots,m$, there is an induced integral ring extension

$$\phi_i : k[[b_1,\dots,b_s]][[t]] \to k[[b_1,\dots,b_s]][[u]]. \tag{18}$$

In particular, u satisfies a monic polynomial of the form

$$u^{n_i} + \sum_{j=1}^{n_i-1} \nu_{i,n_i-j}(b_1,\dots,b_s,t)u^{n_i-j} + \nu_{i,0}(b_1,\dots,b_s,t). \tag{19}$$

Notice that $n_1 + \cdots + n_m = n$, the degree of f. So each $n_i \le n$. We define a $\widehat{\mathcal{O}_{Z,z}}$-linear morphism

$$\left(\mathrm{Tr}^p_f\right)_i : \Omega^p_Y \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}_{Y,w_i}} \to \Omega^p_Z \otimes_{\mathcal{O}_Z} \widehat{\mathcal{O}_{Z,z}} \tag{20}$$

by sending any element of the form $\rho(b_1,\dots,b_s,u)db_{k_1} \wedge \cdots \wedge db_{k_p}$ to the element $\mathrm{Tr}_\phi(\rho)db_{k_1} \wedge \cdots \wedge db_{k_p}$, and by sending any element of the form

$$\sigma = \sum_{j=0}^{n_i-1} \rho_j(b_1,\dots,b_s,t)u^i du \wedge db_{k_1} \wedge \cdots \wedge db_{k_{p-1}} \tag{21}$$

to the element

$$\left(\mathrm{Tr}_f^p\right)_i \sigma = -\frac{1}{n_i}\rho_{n_i-1} d\nu_{i,0} \wedge db_{k_1} \wedge \cdots \wedge db_{k_{p-1}}. \tag{22}$$

Notice that this is a well-defined morphism because any element has a unique decomposition into terms of the form above.

For each $i = 1, \ldots, m$, denote by pr_i the localization map

$$\mathrm{pr}_i : \Omega_Y^p \otimes_{\mathcal{O}_Z} \widehat{\mathcal{O}_{Z,z}} \to \Omega_Y^p \otimes_{\mathcal{O}_Y} \widehat{\mathcal{O}_{Y,w_i}}. \tag{23}$$

Then we define $\left(\mathrm{Tr}_f^p\right)_z$ to be

$$\sum_{i=1}^m \left(\mathrm{Tr}_f^p\right)_i \circ \mathrm{pr}_i : \Omega_Y^p \otimes_{\mathcal{O}_Z} \widehat{\mathcal{O}_{Z,z}} \to \Omega_Z^p \otimes_{\mathcal{O}_Z} \widehat{\mathcal{O}_{Z,z}}. \tag{24}$$

It is straightforward to compute that $\left(\mathrm{Tr}_f^p\right)_z$ agrees with the base-change of $\kappa_p \circ \epsilon_p$ when we base change to the fraction field of $\widehat{\mathcal{O}_{Z,z}}$. Since the base change of $\kappa_p \circ \epsilon_p$ factors through $\Omega_Z^p \otimes_{\mathcal{O}_Z} \widehat{\mathcal{O}_{Z,z}}$, it follows that $\kappa_p \circ \epsilon_p$ factors through $\Omega_Z^p \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z,z}$, which was to be proved. This completes the proof of the proposition. □

rmk-t

Remark 3.3. We make a few remarks on Proposition 3.2:

(1) Obviously this is not the most general result in this direction. For instance, it is clear that the proof also works if Z satisfies Serre's criterion S_2 and f is étale away from codimension 2.

(2) The condition on the characteristic of k was to insure that each of the *ramification indices* n_i at a general point of an irreducible component of the ramification divisor of f is invertible in k. Clearly the proof works without the condition that $\mathrm{char}(k)$ not divide $n!$ if we know each of the n_i is invertible in k.

(3) For any integer $p \geq 0$, there is a *generic trace map*

$$\left(\mathrm{Tr}_f^{\otimes p}\right)_\eta : f_*\left(\Omega_Y^1\right)^{\otimes p} \to \left(\Omega_Z^1\right)^{\otimes p} \otimes_{\mathcal{O}_Z} K(Z). \tag{25}$$

The proposition essentially proves that when one considers the direct summand corresponding to an exterior power, the generic trace map factors through $(\Omega_Z^p)^{\vee\vee}$. One might hope, more generally, that the generic trace map factors through the reflexive hull of $\left(\Omega_Z^1\right)^{\otimes p}$. This is the case, for instance, when $f : Y \to Z$ is étale away from codimension 2. But typically this is not the case: Consider $f : \mathbb{A}_k^1 \to \mathbb{A}_k^1$ which pulls back a coordinate t on the target to u^2 where u is a coordinate on the domain. Then the generic trace of $du \otimes du$ is $\frac{1}{4t} dt \otimes dt$.

lem-t-g2

Lemma 3.4. *With Tr_f^p defined as in Proposition 3.2, we have:*

(1) *For any open subscheme $U \subset Z$, for any section $\sigma \in H^0(U, \Omega_Z^p)$, and for any section $\tau \in H^0(f^{-1}(U), \Omega_Y^q)$ we have*

$$Tr_f^{p+q} f_*\left(f^*\sigma \wedge \tau\right) = \sigma \wedge Tr_f^q f_*\tau. \tag{26}$$

(2) *For any open subscheme $U \subset Z$ and for any section $\tau \in H^0(f^{-1}(U), \Omega^q_Y)$, we have*

$$Tr^{q+1}_f f_*(d\tau) = d\left(Tr^q_f f_*\tau\right). \tag{27}$$

(3) *If $g : X \to Y$ is also a proper, generically étale morphism, and if Y is also normal, then for every p we have*

$$Tr^p_{f\circ g} = Tr^p_f \circ f_* Tr^p_g. \tag{28}$$

3.3. **Descent for p-forms on a stack.** We apply Proposition 3.2 to prove the following "descent theorem" for p-forms on a stack. The setup is the following. Let k be a field (not necessarily of characteristic 0). Let B be a k-scheme, locally of finite type (the *base* scheme). Let $\overline{\mathcal{M}}$ be a irreducible, generically reduced Deligne-Mumford stack over k which is *tame* (in the sense of [1]) along with a proper 1-morphism $\overline{\mathcal{M}} \to B$. Recall that a stack is tame if for each geometric point, the stabilizer group of the point has order which is prime to the characteristic of the residue field of the point. Denote by $\overline{M}$ the coarse moduli space of $\overline{\mathcal{M}}$ (which exists by [17]).

It is too much to hope that every global section of $\Omega^p_{\overline{\mathcal{M}}}$ is the pullback of a global section of $\Omega^p_{\overline{M}}$. For one thing it can happen that $\Omega^p_{\overline{\mathcal{M}}}$ has torsion sections even though $\Omega^p_{\overline{M}}$ is torsion-free.

A more serious issue is raised by the following example: Suppose $\mathrm{char}(k) \neq 2$ and consider $\mathbb{A}^2_k$ with coordinates x, y. Let Γ be the cyclic group of order 2 and let Γ act on $\mathbb{A}^2_k$ by $x \mapsto -x, y \mapsto -y$. Let $\overline{\mathcal{M}}$ be the quotient stack $[\mathbb{A}^2_k/\Gamma]$. Then the 2-form $dx \wedge dy$ is Γ-invariant and thus gives rise to a global section of $\Omega^2_{\overline{\mathcal{M}}}$. But this 2-form is not the pullback of any global section of $\Omega^2_{\overline{M}}$. In this case the coarse moduli space $\overline{M}$ is a quadric cone in $\mathbb{A}^3$, and there does exist a global section of the reflexive hull $\left(\Omega^2_{\overline{\mathcal{M}}}\right)^{\vee\vee}$ which pulls back to $dx \wedge dy$. More generally, if the coarse moduli space is normal one can get a "descent map" of the form

$$H^0(\overline{\mathcal{M}}, \Omega^p_{\overline{\mathcal{M}}}) \to H^0(\overline{M}, (\Omega^p_{\overline{M}})^{\vee\vee}). \tag{29}$$

But in terms of using p-forms to get a lower bound on the Kodaira dimension, this is useless since the sheaf $\left(\Omega^p_{\overline{M}}\right)^{\vee\vee}$ does not admit pullback maps.

Now suppose that $\widetilde{M}$ is a nonsingular k-scheme along with a finite type morphism $\widetilde{M} \to B$, and a rational transformation $u : \widetilde{M} \to \overline{M}$ commuting with the maps to B (e.g. u could be a desingularization of $\overline{M}$ if it exists). What we really want is a pullback map from p-forms on $\overline{\mathcal{M}}$ to p-forms on $\widetilde{M}$.

map

Proposition 3.5. *For each integer $p \geq 0$, consider the maximal torsion-free quotient $\Omega^p_{\overline{\mathcal{M}}}/\langle torsion\rangle$ of $\Omega^p_{\overline{\mathcal{M}}}$. There exists a map of $H^0(B, \mathcal{O}_B)$-modules*

$$\alpha : H^0(\overline{\mathcal{M}}, \Omega^p_{\overline{\mathcal{M}}}/\langle torsion\rangle) \longrightarrow H^0(\widetilde{M}, \Omega^p_{\widetilde{M}}) \tag{30}$$

with the following property: Suppose that $V \subset \overline{\mathcal{M}}$ and $U \subset \widetilde{M}$ are open and that there is a morphism $\pi : V \to U$ expressing the birational correspondence between $\overline{\mathcal{M}}$ and $\widetilde{M}$. Then $\pi^*\big(\alpha(\eta)|_U\big) = \eta|_V$ for any global section η of the torsion-free quotient of $\Omega^p_{\overline{\mathcal{M}}}$.

Proof. Since $\widetilde{M}$ is smooth, and since $\overline{M}$ is proper over B, there exists an open subset $W \subset \widetilde{M}$ whose complement has codimension at least 2 and on which u is regular. Since $\widetilde{M}$ is smooth and since the complement of W has codimension 2, the pullback map

$$H^0(\widetilde{M}, \Omega^p_{\widetilde{M}}) \to H^0(W, \Omega^p_W) \tag{31}$$

is an isomorphism. Therefore it suffices to prove the proposition with W in place of $\widetilde{M}$. So, without loss of generality, we assume that $u : \widetilde{M} \to \overline{M}$ is a regular morphism.

Now let $\pi : \widetilde{\mathcal{M}} \to \widetilde{M}$ be the normalization of the unique irreducible component of $\widetilde{M} \times_{\overline{M}} \overline{\mathcal{M}}$ dominating $\widetilde{M}$. There is a pullback morphism

$$H^0(\overline{\mathcal{M}}, \Omega^p_{\overline{\mathcal{M}}}) \to H^0(\widetilde{\mathcal{M}}, \Omega^p_{\widetilde{\mathcal{M}}}). \tag{32}$$

If there exists a morphism β from $H^0(\widetilde{\mathcal{M}}, \Omega^p_{\widetilde{\mathcal{M}}})$ to $H^0(\widetilde{M}, \Omega^p_{\widetilde{M}})$ with the property in the proposition, then we can compose β with the pullback morphism above to get a morphism α with the property in the proposition. Therefore it suffices to construct β. So, without loss of generality, suppose that $u : \widetilde{M} \to \overline{M}$ is the identity map.

Now the existence of α with the property in the proposition clearly may be checked after étale base change on $\widetilde{M}$: the property of the proposition guarantees that such α will be unique and will satisfy the usual étale descent condition. By [1, Lemma 2.2.3], there exists an étale covering $\left\{X_i \to \widetilde{M}\right\}$ such that each base change stack $\widetilde{\mathcal{M}} \times_{\widetilde{M}} X_i \to X_i$ is a finite group quotient $[U_i/\Gamma_i]$ where U_i is a scheme which is finite over X_i and each Γ is a finite group acting on U_i by X_i-morphisms. Since $\overline{\mathcal{M}}$ is a *tame* stack, the order of Γ is relatively prime to the characteristic of the ground field k. Without loss of generality, we now assume that $\widetilde{\mathcal{M}}$ is a quotient stack $[U/\Gamma]$ where U is a scheme along with a finite morphism $f : U \to \widetilde{M}$ and Γ is a finite group acting on U by $\widetilde{M}$-morphisms and such that the order of Γ is prime to the characteristic.

First of all, the ramification index of f at each codimension one component $D \subset U$ of the ramification divisor equals the index of the stabilizer subgroup (in Γ) of a generic point of U considered as a subgroup of the stabilizer of a generic point of D. Thus the ramification index divides the order of Γ, and so is relatively prime to the characteristic of the ground field. Similarly, the degree n of the finite morphism f is relatively prime to the characteristic (being the index of the stabilizer of a generic point in all of Γ). Therefore $U \to \widetilde{M}$ satisfies the hypotheses of Proposition 3.2 (see (2) of Remark 3.3). So there exists a trace map

$$\mathrm{Tr}^p_f : f_*\Omega^p_U \to \Omega^p_{\widetilde{M}}. \tag{33}$$

Observe that this map necessarily annihilates f_* of the torsion subsheaf of Ω^p_U, since $\Omega^p_{\widetilde{M}}$ is torsion-free. And since f_* preserves exactness (being a finite morphism), it follows that Tr^p_f factors through f_* of the torsion-free quotient of Ω^p_U. Now a global section η of $\Omega^p_{\widetilde{\mathcal{M}}}$ is precisely a global section η of Ω^p_U which is Γ-invariant. We define α to be the restriction of $\frac{1}{n}\mathrm{Tr}^p_f$ to the subspace of Γ-invariant global sections of the torsion-free quotient of Ω^p_U (recalling that n is invertible in the ground field).

It remains to verify the property of the proposition. Suppose that η is a Γ-invariant global section of the torsion-free quotient of Ω^p_U. Consider $\mathrm{Tr}^p_f(\eta)$. Since f is generically étale, it follows by étale descent that there exists a dense open subset of $\widetilde{M}$ over which η equals the pullback of a p-form τ on $\widetilde{M}$. By (1) of Lemma 3.4, it follows that $\mathrm{Tr}^p_f(\eta) = n\tau$ when restricted to this open set, i.e. $\alpha(\eta) = \tau$ when restricted to this open set. So $f^*\alpha(\eta)$ agrees with η over a dense open subset. Therefore $\eta = f^*\alpha(\eta)$.

□

4. Construction of the 2-form

form

In this section we use an algebraic analogue of "integrating along fibers" to construct a 2-form on the space $\overline{\mathcal{M}}_e$ associated to a smooth cubic hypersurface $X \subset \mathbb{P}^4$. To do this we use the universal curve $p : \mathcal{C} \to \overline{\mathcal{M}}_e$ together with the universal morphism $f : \mathcal{C} \to X$. The cohomology group $H^1(X, \Omega^3_X)$ is 1-dimensional (we review this in Subsection 5.1 below). Choose once and for all a fixed nonzero element η in this space. By pulling back via f we obtain $f^*\eta \in H^1(\mathcal{C}, \Omega^1_{\mathcal{C}})$.

Now we will put ourselves in a slightly more general context. Suppose that $\overline{\mathcal{M}}$ is a finite type Deligne-Mumford stack over k and $p : \mathcal{C} \to \overline{\mathcal{M}}$ is a representable 1-morphism of Deligne-Mumford stacks which is proper and flat of relative dimension 1, such that every geometric fiber of p is a reduced, at-worst-nodal curve, i.e. $p : \mathcal{C} \to \overline{\mathcal{M}}$ is a semi-stable family of curves. There is a canonical morphism from the sheaf of relative Kähler differentials to the dualizing sheaf $\Omega^1_p \to \omega_p$, which is an isomorphism on the open substack $U \subset \mathcal{C}$ which is the smooth locus of p. Using this isomorphism, we obtain for each i a morphism on U:

$$\phi_{U,i} \colon \Omega^{i+1}_{\mathcal{C}}|_U \to \left(\Omega^{i+1}_{\mathcal{C}}/p^*\Omega^{i+1}_{\overline{\mathcal{M}}}\right)|_U \cong p^*\Omega^i_{\overline{\mathcal{M}}} \otimes \omega_p|_U. \tag{34}$$

Note that this map has the property that for every section $\alpha \in \Omega^i_{\overline{\mathcal{M}}}$ and $\beta \in \Omega^j_{\mathcal{C}}$, we have $\phi_{U,i+j}(p^*\alpha \wedge \beta) = p^*\alpha \wedge \phi_{U,j}(\beta)$.

claim-1

Lemma 4.1. *For each i there exists a unique morphism $\phi_i : \Omega^{i+1}_{\mathcal{C}} \to p^*\Omega^i_{\overline{\mathcal{M}}} \otimes \omega_p$ such that $\phi_i|_U = \phi_{U,i}$ and such that for every section $\alpha \in \Omega^i_{\overline{\mathcal{M}}}$ and $\beta \in \Omega^j_{\mathcal{C}}$, we have $\phi_{i+j}(p^*\alpha \wedge \beta) = p^*\alpha \wedge \phi_j(\beta)$.*

Proof. First of all, if such ϕ_i exists, then by construction it annihilates $p^*\Omega^{i+1}_{\overline{\mathcal{M}}}$, i.e. it factors through the quotient. The quotient has a canonical subsheaf isomorphic to $p^*\Omega^i_{\overline{\mathcal{M}}} \otimes \Omega^1_p$ with an obvious map to $p^*\Omega^i_{\overline{\mathcal{M}}} \otimes \omega_p$. The main issue is to prove that this map extends to the entire quotient. There is a secondary issue of uniqueness, but the cokernel of $p^*\Omega^i_{\overline{\mathcal{M}}} \otimes \Omega^1_p$ is a sheaf which is torsion on all fibers, whereas the sheaf $p^*\Omega^i_{\overline{\mathcal{M}}} \otimes \omega_p$ is torsion-free on fibers. So it is clear that there is no nonzero map from the cokernel to $p^*\Omega^i_{\overline{\mathcal{M}}} \otimes \omega_p$. Moreover, the extension problem can be phrased as the vanishing of a section of a sheaf Ext, and this vanishing can be checked after passing to the completion of the local ring at each geometric closed point of $\mathcal{C}$.

Since we can check the property formally locally, without loss of generality we assume that $\overline{\mathcal{M}}$ is a scheme. Let $z \in \mathcal{C}$ be a closed point. Denoting $A = \widehat{\mathcal{O}_{\overline{\mathcal{M}},p(z)}}$, we can find an isomorphism

$$B = \widehat{\mathcal{O}_{\mathcal{C},z}} \cong A[[x,y]]/\langle xy - a \rangle. \tag{35}$$

for some element $a \in A$. Now by Remark 4.2, it follows that the base change of $\phi_{U,i}$ does extend to a map $\phi_i \otimes_{\mathcal{O}_\mathbb{C}} B$ as required, i.e. the element of the sheaf Ext vanishes when we base change to B. This proves the existence of ϕ_i as in the lemma. $\square$

In particular, back in the context that X is a smooth cubic threefold and $\overline{\mathcal{M}}_e$ is the stack of genus 0 stable maps of degree e, this gives a map of sheaves

$$\phi_2 : \Omega^3_{\mathcal{C}} \longrightarrow p^*(\Omega^2_{\overline{\mathcal{M}}_e}) \otimes \omega_{\mathcal{C}/\overline{\mathcal{M}}_e}. \tag{36}$$

We compose the maps

$$H^1(\mathcal{C}, \Omega^3_{\mathcal{C}}) \to H^1(\mathcal{C}, p^*\Omega^2_{\overline{\mathcal{M}}_e} \otimes \omega_{\mathcal{C}/\overline{\mathcal{M}}_e}) \to H^0\Big(\overline{\mathcal{M}}_e, R^1p_*\big(p^*\Omega^2_{\overline{\mathcal{M}}_e} \otimes \omega_{\mathcal{C}/\overline{\mathcal{M}}_e}\big)\Big). \tag{37}$$

The image of $f^*\eta$ under this map is an element in the last group which is equal to $H^0(\overline{\mathcal{M}}_e, \Omega^2_{\overline{\mathcal{M}}_e} \otimes R^1p_*\omega_{\mathcal{C}/\overline{\mathcal{M}}_e})$. Finally, we apply the trace map

$$R^1p_*\omega_{\mathcal{C}/\overline{\mathcal{M}}_e} \to \mathcal{O}_{\overline{\mathcal{M}}_e}.$$

The result is the 2-form ω_e which we will study.

algebra

Remark 4.2. Let A be a ring and let $B = A[x,y]/(xy-a)$ for some $a \in A$. Consider the canonical exact sequence

$$0 \to \Omega^1_A \otimes B \to \Omega^1_B \to \Omega^1_{B/A} \to 0.$$

Exactness on the left follows as B is a complete intersection flat over A whose cotangent complex $L_{B/A}$ is quasi-isomorphic to $\Omega^1_{B/A}$. Moreover, the relative dualizing sheaf is the determinant of $L_{B/A}$ (which is perfect of amplitude $[-1,0]$). So, the relative dualizing module $\omega_{B/A}$ is free with generator

$$\theta = \frac{\mathrm{d}x \wedge \mathrm{d}y}{xy-a}.$$

and there is a canonical B-module homomorphism

$$\Omega^1_{B/A} \longrightarrow \omega_{B/A}$$

which is determined by the rules $\mathrm{d}x \mapsto x\theta$ and $\mathrm{d}y \mapsto -y\theta$. From this we will define maps

$$\Omega^i_B \to \Omega^{i-1}_A \otimes_A \omega_{B/A}.$$

Namely, any element in Ω^i_B can be written as a B-linear combination of forms of the type η, $\eta \wedge \mathrm{d}x$, $\eta \wedge \mathrm{d}y$ and $\eta \wedge \mathrm{d}x \wedge \mathrm{d}y$, where η is in Ω^j_A, with $j = i$, $i-1$, or $i-2$. We claim there exists a map as above such that

$$\eta \mapsto 0, \quad \eta \wedge \mathrm{d}x \mapsto \eta \otimes x\theta, \quad \eta \wedge \mathrm{d}y \mapsto -\eta \otimes y\theta, \quad \eta \wedge \mathrm{d}x \wedge \mathrm{d}y \mapsto -\eta \wedge \mathrm{d}a \otimes \theta.$$

The reader easily verifies that this is well defined (the main concern being that forms of the type $\eta \wedge (y\mathrm{d}x + x\mathrm{d}y - \mathrm{d}a)$ and $\eta \wedge (y\mathrm{d}x + x\mathrm{d}y - \mathrm{d}a) \wedge \mathrm{d}x$ get mapped to zero).

any

Remark 4.3. Note that the same construction gives maps $H^{b+1}(X, \Omega^{a+1}) \to H^b(\mathcal{M}, \Omega^a)$ for any variety X (not necessarily proper or smooth), and any Kontsevich moduli space of maps into X (not necessarily genus 0). Actually, there is a corresponding "integration along fibers" map on de Rham cohomology and on Betti-cohomology (in case the ground field is $\mathbb{C}$), and presumably on any reasonable cohomology theory. This is nothing new, but since we have to compute explicitly the corresponding pairing below, we thought we should explain.

5. Explicit description of the 2-form

compute

In the last section we gave a general argument which associates to a variety X and a Kontsevich moduli space of maps $\mathcal{M}$ into X certain linear maps $H^{b+1}(X, \Omega_X^{a+1}) \to H^b(\mathcal{M}, \Omega^a_{\mathcal{M}})$. The case we are interested in is $b = 0$, so that elements of the target are actually sections of the sheaf $\Omega^q_{\mathcal{M}}$. In particular, we can consider the fiber of such a section at a geometric point $z \in \mathcal{M}$, and try to describe this section (with respect to a basis of the Zariski tangent sheaf) in terms of the local geometry of the parametrized curve $C_z \subset X$, i.e. we can try to make the construction of the last section *explicit*. In this section we will make this very explicit in the special case of genus 0 maps to the smooth locus of a cubic threefold.

GRC

5.1. **Explicit description of $H^1(X, \Omega_X^3)$.** First we recall a very small part of the Griffiths residue calculus [9, Section 8]. Let $X \subset \mathbb{P}^n$ be a hypersurface of degree d, and let $U \subset X$ be the smooth locus. We have the cotangent sequence:

$$0 \longrightarrow \mathcal{O}_U(-d) \longrightarrow \Omega^1_{\mathbb{P}^n}|_U \longrightarrow \Omega^1_U \longrightarrow 0 \tag{38}$$

Taking the exterior power of this sequence, and twisting by $\mathcal{O}_X(d)|_U$, we have an exact sequence:

$$0 \longrightarrow \Omega_U^{n-2} \longrightarrow \Omega^{n-1}_{\mathbb{P}^n}|_U \otimes \mathcal{O}_U(d) \longrightarrow \Omega^n_{\mathbb{P}^n}|_U \otimes \mathcal{O}_U(2d) \longrightarrow 0 \tag{39}$$

One can also get this by taking the dual of the first exact sequence and twisting by $\Omega^n_{\mathbb{P}^n}|_U \otimes \mathcal{O}_U(d)$. At any rate, the connecting homomorphism in cohomology gives a map

$$H^0(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(2d)) \to H^1(U, \Omega_U^{n-2}). \tag{40}$$

In the special case of a cubic fourfold, we get an exact sequence: *eqn-2*

$$0 \longrightarrow \Omega^3_U \longrightarrow \Omega^4_{\mathbb{P}^5}|_U \otimes \mathcal{O}_U(3) \longrightarrow \Omega^5_{\mathbb{P}^5}|_U \otimes \mathcal{O}_U(6) \longrightarrow 0. \tag{41}$$

Of course we have $\Omega^5_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}^5}(2 \cdot 3) \cong \mathcal{O}_{\mathbb{P}^5}$. Notice that if $U = X$, this map is surjective. We choose some nonzero element in $H^0(\mathbb{P}^5, \Omega^5_{\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}^5}(6))$, and we define η to be the image of this element in $H^1(U, \Omega^3_U)$.

exp

5.2. **The explicit description.** Let $f : C \to X$ be a point of M_e. Assume that $\mathbb{P}^1 = C$ is smooth and that f is a regular embedding into the smooth locus $U \subset X$. Consider the sequence of vector bundles over C given by the normal bundle $N_{C/X}$ of C in X mapping to the normal bundle $N_{C/\mathbb{P}^5}$ of C in $\mathbb{P}^5$: eqn-1

$$0 \longrightarrow N_{C/X} \longrightarrow N_{C/\mathbb{P}^5} \longrightarrow f^*N_{X/\mathbb{P}^5} \longrightarrow 0. \tag{42}$$

Of course $N_{X/\mathbb{P}^5} \cong \mathcal{O}_X(3)$, so that $f^*N_{X/\mathbb{P}^5} \cong \mathcal{O}_{\mathbb{P}^1}(3e)$ where we use the notation $\mathcal{O}_{\mathbb{P}^1}(a)$ to indicate any invertible sheaf of degree a on $\mathbb{P}^1 = C$. In particular, observe that $\bigwedge^3 N_{C/X} = \mathcal{O}_{\mathbb{P}^1}(3e-2)$ and that $\bigwedge^4 N_{C/\mathbb{P}^5} = \mathcal{O}_{\mathbb{P}^1}(6e-2)$. The Zariski tangent space $T_{[f]}(M_e)$, which is the same thing as the dual vector space of the fiber $\Omega^1_{M_e}|_{[f]}$, is given by the space of global sections $H^0(C, N_{C/X})$ (c.f. [18, Theorem I.2.8]). So the fiber $\Omega^2_{M_e}|_{[f]}$ is just the vector space dual of $\bigwedge^2 H^0(C, N_{C/X})$. And the 2-form ω_e gives a procedure to associate to any two sections of $N_{C/X}$ a complex number.

Next consider the exact sequence *eqn-3*

$$0 \longrightarrow \bigwedge^3 N_{C/X} \otimes \mathcal{O}_{\mathbb{P}^1}(-3e) \longrightarrow \bigwedge^3 N_{C/\mathbb{P}^5} \otimes \mathcal{O}_{\mathbb{P}^1}(-3e) \longrightarrow \bigwedge^2 N_{C/X} \longrightarrow 0. \tag{43}$$

This sequence is obtained from Equation 42 by taking exterior powers and twisting by $\mathcal{O}_{\mathbb{P}^1}(-3e)$. In any case, the sheaf on the left is $\mathcal{O}_{\mathbb{P}^1}(-2)$ by what was said above. Choose an isomorphism $H^1(C, \mathcal{O}_{\mathbb{P}^1}(-2)) = \mathbb{C}$, and let

$$\delta : H^0(C, \bigwedge^2 N_{C/X}) \to H^1(C, \bigwedge^3 N_{C/X} \otimes \mathcal{O}_{\mathbb{P}^1}(-3)) = H^1\left(C, \mathcal{O}(-2)\right) = \mathbb{C}$$

be the boundary map on cohomology coming from the exact sequence above. This is another procedure which associates to any two sections of $N_{C/X}$ a complex number. In the following theorem we prove that the two procedures agree. The best argument for this is the usual: What else could it be? The actual proof is even more annoying.

Theorem 5.1. *Up to a nonzero scalar factor the pairing associated to ω_e on $T_{[f]}(\mathcal{M}_e) = H^0\left(C, N_{C/X}\right)$ is equal to the pairing $(s_1, s_2) \mapsto \delta(s_1 \wedge s_2)$.*

Proof. Observe that the construction of Section 4 is compatible with arbitrary base change of the stack $\mathcal{M}$. To prove the theorem, we will base change to a certain Artin local ring $Z = \mathrm{Spec} A$ over which we have the universal first order deformation of $C \subset X$, say $\mathbb{C} \subset Z \times X$. The construction of Section 4 instructs us to restrict the exact sequence from Equation 41 to $\mathbb{C}$ and then push the sequence out by the map

$$f^*\left(\Omega^3_X\right) \to \Omega^3_{\mathbb{C}} \to p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathbb{C}/Z}. \tag{44}$$

Then we are instructed to take cohomology of the resulting sequence to obtain the 2-form ω_e. By a diagram chase, we see that the resulting sequence is simply the "Serre dual" of the sequence from Equation 43 from which the theorem follows.

First we compute the universal first order deformation of $C \subset X$. By Serre duality the vector space $V = H^1\left(C, I/I^2 \otimes \omega_C\right)$ is dual to $H^0\left(C, N_{C/X}\right)$. Here I is the ideal sheaf of C in X. Consider the local Artin ring $A = k \oplus V$, where V is an ideal of square zero. Set $Z = \mathrm{Spec}\, A$. Over Z we have the universal first order deformation $\mathcal{C} \to Z$ of C. Let $s_1, \ldots, s_A$ be an ordered basis for $H^0\left(C, N_{C/X}\right)$ and let $t_1, \ldots, t_A$ in V be the dual ordered basis. We think of the elements $s_1, \ldots, s_A$ as $\mathcal{O}_C$-linear maps $I/I^2 \to \mathcal{O}_C$. Affine locally on X at a point of C suppose that I is generated by g_1, g_2, g_3. Then the ideal of $\mathcal{C}$ is locally generated by the equations

$$\widetilde{g}_j := g_j + \sum_{i=1}^{A} t_i \cdot s_i(f_j), \quad j = 1, 2, 3 \tag{45}$$

$$\widetilde{g}_j \in \mathcal{O}_X[t_1, \ldots, t_A]/\langle t_i t_{i'}, t_i g_j, g_j g_{j'} | i, i' = 1, \ldots, A, j, j' = 1, 2, 3\rangle. \tag{46}$$

Denote by $p\colon \mathcal{C} \to Z$ and $\widetilde{f}\colon \mathcal{C} \to X$ the two projections.

To prove the theorem, we will compute the 2-form on Z obtained from the construction of Section 4 applied to $(p\colon \mathcal{C} \to Z, \widetilde{f}\colon \mathcal{C} \to X)$. This is not as crazy as it sounds, namely $\Omega^2_{A/k} \otimes_A k = \wedge^2 V$ so this computation will provide us with the information we want.

To compute $\widetilde{f}^*\eta$, we form the pullback by $\widetilde{f}^*$ of the exact sequence from Equation 41. Considered as an element of the Yoneda-Ext group $\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \Omega^3_{\mathcal{C}})$, the element $\widetilde{f}^*\eta$ is simply the push out of this exact sequence by the canonical map $\widetilde{f}^*\left(\Omega^3_X\right) \to \Omega^3_{\mathcal{C}}$. According to Section 4, we now take the $f^*\eta$ under the map

$$\mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, \Omega^3_{\mathcal{C}}) \to \mathrm{Ext}^1_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}}, p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathcal{C}/Z}) \tag{47}$$

In terms of Yoneda-Ext, this means that we take an additional push our of our exact sequence by $\Omega^3_{\mathcal{C}} \to p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathcal{C}/Z}$. So, in terms of Yoneda-Ext, our exact sequence is obtained as the push out of the pullback of Equation 41 by the map $\widetilde{f}^*\Omega^3_X \to p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathcal{C}/Z}$.

Of course we really only need to have this exact sequence on the closed fiber, so we restrict the push out exact sequence to the closed fiber. In particular, we have that the restriction to the closed fiber of $p^*\left(\Omega^2_Z\right) \otimes \omega_{\mathcal{C}/Z}$ is just $\bigwedge^2 V \otimes_k \Omega^1_C$. Next we give an explicit local description of the map

$$\psi\colon \Omega^3_X|_C \to \bigwedge^2 V \otimes_k \Omega^1_C. \tag{48}$$

Let t be a regular function on X which restricts to a local coordinate on C. We can write any local 3-form on X as an $\mathcal{O}_X$-linear combination of the forms $\epsilon_{jj'} = \mathrm{d}f_j \wedge \mathrm{d}f_{j'} \wedge \mathrm{d}t, 1 \le j < j' \le 3$ and the form $\mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}f_3$, so it suffices to evaluate ψ on these 3-forms. The result is

$$\psi(\eta_{jj'}) = \sum_{i,i'=1}^{A} s_i(f_j)s_{i'}(f_{j'})t_i \wedge t_{i'} \otimes dt,\ 1 \le j < j' \le 3, \tag{49}$$

$$\psi(df_1 \wedge df_2 \wedge df_3) = 0. \tag{50}$$

Of course there is more "global" way of thinking about ψ. The exact sequence *eqn-4*

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1_X|_C \longrightarrow \Omega^1_C \longrightarrow 0, \tag{51}$$

determines a canonical map $\alpha : \Omega^3_X|_C \to \bigwedge^2 I/I^2 \otimes_{\mathcal{O}_C} \Omega^1_C$. And there is a map of $\mathcal{O}_C$-modules $\beta : I/I^2 \to V \otimes_k \mathcal{O}_C$ defined as the transpose of the map $H^0\left(C, N_{C/X}\right) \otimes_k \mathcal{O}_C \to N_{C/X}$. The global description of ψ is as the composition of α with $\bigwedge^2 \beta \otimes \mathrm{Id}_{\Omega^1_C}$.

Just as the exact sequence in Equation 51 induces the map α, also the exact sequence *eqn-5*

$$0 \longrightarrow \tilde{I}/\tilde{I}^2 \longrightarrow \Omega^1_{\mathbb{P}^5}|_C \longrightarrow \Omega^1_C \longrightarrow 0. \tag{52}$$

induces a map $\alpha' : \Omega^4_{\mathbb{P}^5}|_C \to \bigwedge^3 \widetilde{I}/\widetilde{I}^2 \otimes \Omega^1_C$ where $\widetilde{I}$ is the ideal sheaf of C in $\mathbb{P}^5$. By adjunction, we have isomorphisms $\Omega^5_{\mathbb{P}^5}|_C \otimes \mathcal{O}_C(3e) \cong \Omega^4_X|_C$ and $\Omega^4_X|_C \cong \bigwedge^3 I/I^2 \otimes \Omega^1_C$. Combining these adjunction isomorphisms gives an isomorphism

$$\alpha''\colon \Omega^5_{\mathbb{P}^5}|_C \otimes \mathcal{O}_C(6e) \to \bigwedge^3 I/I^2 \otimes \mathcal{O}_C(3e) \otimes \Omega^1_C. \tag{53}$$

Of course both terms in this map are isomorphic to $\mathcal{O}_C$. Choosing such isomorphisms, α'' is just an isomorphism of $\mathcal{O}_C$ to itself.

We leave it to the reader to verify that the following diagram commutes: *eqn-6*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^3_X|_C & \longrightarrow & \Omega^4_{\mathbb{P}^5}|_C(3e) & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & \bigwedge^2 I/I^2 \otimes \Omega^1_C & \longrightarrow & \bigwedge^3 \tilde{I}/\tilde{I}^2(3e) \otimes \Omega^1_C & \longrightarrow & \mathcal{O}_C & \longrightarrow & 0. \end{array} \tag{54}$$

The top exact sequence is just the restriction to C of Equation 41, and the bottom exact sequence is the dual of Equation 43 tensored with Ω^1_C. More canonically, the last term in the top sequence is $\Omega^5_{\mathbb{P}^5}|_C(6e)$ and the last term in the bottom sequence is $\bigwedge^3 I/I^2(3e) \otimes \Omega^1_C$. But we choose isomorphisms of these sheaves with $\mathcal{O}_C$ as described in the last paragraph.

The conclusion is that the extension of $\mathcal{O}_C$ by $\wedge^2 V \otimes_k \Omega^1_C$ obtained from $\widetilde{f}^*\eta$ is precisely the Serre dual exact sequence of Equation 43 used to define the coboundary map δ. Hence the coboundary map on cohomology $H^0(C, \mathcal{O}_C) \to H^1(C, \bigwedge^2 I/I^2 \otimes \Omega^1_C)$ is the dual of δ. This equality implies the result of Theorem 5.1. ☐

6. Proof of Theorem 1.2: degree five case

The strategy of the proof of Theorem 1.2 is the following. Form the $\mathbb{P}^{55}$ parametrizing all cubic hypersurfaces in $\mathbb{P}^5$, and let $U_e \to \mathbb{P}^{55}$ be the parameter space for pairs $([X],[C])$ where $C \subset X$ is a smooth curve of degree e such that X is smooth along C and such that $H^1(C, N_{C/X})$ is zero (i.e. $C \subset X$ is *unobstructed*). This last condition guarantees that $U_e \to \mathbb{P}^{55}$ is a smooth morphism. Also recall from Proposition 2.4 that the general fiber of $U_e \to \mathbb{P}^{55}$ is irreducible. In particular, U_e is also irreducible.

We can perform a relative version of the construction of Section 4 to obtain a 2-form ω_e as a section of $\Omega^2_{U_e/\mathbb{P}^{55}}$ whose restriction to any fiber is the 2-form of the fiber. The rank of ω_e on fibers is lower semicontinuous on U_e, so to prove that the rank of ω_e is as expected for a general pair $([X],[C])$, it suffices to find a single pair $([X],[C]) \in U_e$ where the rank of ω_e is as expected.

Suppose that we have an exact sequence of the form

$$0 \longrightarrow \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \mathcal{O}(a_3) \longrightarrow N_{C/\mathbb{P}^5} \longrightarrow \mathcal{O}(3e) \longrightarrow 0$$

with $a_1 + a_2 + a_3 = 3e - 2$. In other words $N_{C/X} = \bigoplus \mathcal{O}(a_i)$. The extension class of this sequence is an element ψ of $H^1(\mathbb{P}^1, \mathcal{O}(a_1 - 3e) \oplus \mathcal{O}(a_2 - 3e) \oplus \mathcal{O}(a_3 - 3e))$. If we write $\mathbb{P}^1 = \mathrm{Proj}\,(S)$, where $S = \mathbb{C}[X_0, X_1]$, then we have, using Serre duality, that $\psi = \psi_1 \oplus \psi_2 \oplus \psi_3$ with $\psi_i \in \mathrm{Hom}\,(S_{3e-a_i-2}, \mathbb{C})$. If we write elements of $H^0(C, N_{C/X})$ in the form (g_1, g_2, g_3) where each $g_i \in H^0(C, \mathcal{O}(a_i))$, then the reader verifies readily that in this case the pairing takes the following form

$$\left\langle \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} \right\rangle = \psi_3(g_1h_2 - g_2h_1) + \psi_2(g_1h_3 - g_3h_1) + \psi_1(g_2h_3 - g_3h_2).$$

In order to compute the pairing for a given curve we have to find the linear functionals ψ_1, ψ_2, ψ_3 above. For large e this reduces to a rather involved computation which is straightforward, but tedious. We will present this computation later, but first we show that in the special case $e = 5$ there is an elegant solution (which hopefully will motivate the reader to "trudge through" the computations of the next two sections).

Theorem 6.1. *Suppose that $f : C \to X$ is a general quintic rational curve on a general cubic fourfold X. Then $N_{C/X} = \mathcal{O}(4) \oplus \mathcal{O}(4) \oplus \mathcal{O}(5)$ and the extension class ψ of the sequence $0 \to N_{C/X} \to N_{C/\mathbb{P}^5} \to \mathcal{O}(15) \to 0$ is a general point of the space* $\mathrm{Hom}\,(S_9 \oplus S_9 \oplus S_8, \mathbb{C})$.

Proof. To prove this we argue as follows. Fix a rational normal curve $C \subset \mathbb{P}^5$ of degree 5. It is easy to see that its normal bundle $N_{C/\mathbb{P}^5}$ is $\mathcal{O}(7)^{\oplus 4}$. Thus any (not necessarily nonsingular) cubic fourfold X containing C determines a homomorphism of $\mathcal{O}_C$-modules

$$\varphi_X : \mathcal{O}(7)^{\oplus 4} \to \mathcal{O}(15).$$

Note that $\varphi_X = 0$ if and only if X is singular along C, which happens if and if the defining equation of X is a section of $I^2(3)$. We leave it to the reader to compute the following dimensions:

$$\dim H^0(\mathbb{P}^5, I(3)) = 40, \quad \dim H^0(\mathbb{P}^5, I^2(3)) = 4, \quad \dim \operatorname{Hom}_C(\mathcal{O}(7)^4, \mathcal{O}(15)) = 36.$$

Thus the rule $X \mapsto \varphi_X$ is onto. Hence we can obtain the general exact sequence of the form $0 \to \operatorname{Ker}(\alpha) \to \mathcal{O}(7)^4 - \alpha \to \mathcal{O}(15) \to 0$ as the normal bundle sequence for general (nonsingular) X. The theorem follows. $\square$

To finish we choose ψ_i as follows:

$$\psi_1(\sum_{i=0}^{9} a_i X_0^{9-i} X_1^i) = \sum_{i=0}^{9} \nu_i a_i, \quad \psi_1(\sum_{i=0}^{9} a_i X_0^{9-i} X_1^i) = \sum_{i=0}^{9} \mu_i a_i,$$

and

$$\psi_1(\sum_{i=0}^{8} a_i X_0^{8-i} X_1^i) = \sum_{i=0}^{8} \lambda_i a_i.$$

Here we choose ν_i, μ_i and λ_i general. The matrix of the pairing with respect to the obvious basis of $H^0(\mathbb{P}^1, \mathcal{O}(4) \oplus \mathcal{O}(4) \oplus \mathcal{O}(5))$. Here is the result:

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \mu_0 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\
0 & 0 & 0 & 0 & 0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 \\
0 & 0 & 0 & 0 & 0 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 \\
0 & 0 & 0 & 0 & 0 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \mu_3 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 \\
0 & 0 & 0 & 0 & 0 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 & \mu_4 & \mu_5 & \mu_6 & \mu_7 & \mu_8 & \mu_9 \\
-\lambda_0 & -\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_4 & 0 & 0 & 0 & 0 & 0 & \nu_0 & \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 \\
-\lambda_1 & -\lambda_2 & -\lambda_3 & -\lambda_4 & -\lambda_5 & 0 & 0 & 0 & 0 & 0 & \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 \\
-\lambda_2 & -\lambda_3 & -\lambda_4 & -\lambda_5 & -\lambda_6 & 0 & 0 & 0 & 0 & 0 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 \\
-\lambda_3 & -\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 & 0 & 0 & 0 & 0 & 0 & \nu_3 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & \nu_8 \\
-\lambda_4 & -\lambda_5 & -\lambda_6 & -\lambda_7 & -\lambda_8 & 0 & 0 & 0 & 0 & 0 & \nu_4 & \nu_5 & \nu_6 & \nu_7 & \nu_8 & \nu_9 \\
-\mu_0 & -\mu_1 & -\mu_2 & -\mu_3 & -\mu_4 & -\nu_0 & -\nu_1 & -\nu_2 & -\nu_3 & -\nu_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_1 & -\mu_2 & -\mu_3 & -\mu_4 & -\mu_5 & -\nu_1 & -\nu_2 & -\nu_3 & -\nu_4 & -\nu_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_2 & -\mu_3 & -\mu_4 & -\mu_5 & -\mu_6 & -\nu_2 & -\nu_3 & -\nu_4 & -\nu_5 & -\nu_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_3 & -\mu_4 & -\mu_5 & -\mu_6 & -\mu_7 & -\nu_3 & -\nu_4 & -\nu_5 & -\nu_6 & -\nu_7 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_4 & -\mu_5 & -\mu_6 & -\mu_7 & -\mu_8 & -\nu_4 & -\nu_5 & -\nu_6 & -\nu_7 & -\nu_8 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\mu_5 & -\mu_6 & -\mu_7 & -\mu_8 & -\mu_9 & -\nu_5 & -\nu_6 & -\nu_7 & -\nu_8 & -\nu_9 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}$$

Finally, to end the proof of Theorem 1.2 in the case $e = 5$, we show that the determinant of this matrix is nonzero. This we achieve by specializing as follows $\lambda_0 = 1, \lambda_1 = 2, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = 1, \lambda_5 = 1, \lambda_6 = -1, \lambda_7 = -4, \lambda_8 = 2, \mu_0 = 1, \mu_1 = 2, \mu_2 = -1, \mu_3 = 2, \mu_4 = 5, \mu_5 = -1, \mu_6 = 13, \mu_7 = -1, \mu_8 = 1, \mu_9 = 1, \nu_0 = 1, \nu_1 = 2, \nu_2 = 3, \nu_3 = 5, \nu_4 = 4, \nu_5 = -5, \nu_6 = -6, \nu_7 = -7, \nu_8 = -5, \nu_9 = 1$ and computing the determinant. The result of the computation is 445717799641 which is not zero as desired.

7. Proof of Theorem 1.2: the odd degree case odd

In the previous section we saw an elegant proof of Theorem 1.2 in the case that $e = 5$. What made the proof so short and non-computational is that in this case the extension class ψ can be chosen to be general by our parameter count. The analogous parameter count breaks down as the degree e becomes larger – the dimension of the relevant Ext space grows faster than the dimension of the space U_e. Instead we shall work with a specific pair $([X],[C]) \in U_e$ where we can prove that the rank of ω_e is as expected and where $H^1(C, N_{C/X})$ is zero. We warn the reader now that X will not be smooth! But X will be smooth on an open set which contains C, and this is all that matters.

odd-1

7.1. **Computation of $N_{C/\mathbb{P}^5}$.** Write $e = 2r+1$ where $r \geq 2$. We begin by specifying C and computing $N_{C/\mathbb{P}^5}$. As in the last section, choose homogeneous coordinates X_0, X_1 on $\mathbb{P}^1$. Choose homogeneous coordinates $Y_0, Y_1, Y_2, Y_3, Y_4, Y_5$ on $\mathbb{P}^5$. Consider the map $f : \mathbb{P}^1 \to \mathbb{P}^5$ given by

$$f([X_0 : X_1]) = [X_0^{2r+1} : X_0^{2r}X_1 : X_0^{r+1}X_1^r : X_0^r X_1^{r+1} : X_0 X_1^{2r} : X_1^{2r+1}].$$

This is a monomial embedding of $\mathbb{P}^1$ which is as "balanced" as possible. To compute the normal bundle of C in $\mathbb{P}^5$, we use the Euler sequences for $T_{\mathbb{P}^1}$ and $T_{\mathbb{P}^5}$. There is a map between these Euler sequences induced by f and the important term is

$$\widetilde{df} : \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to f^*\left(\mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 6}\right) = \mathcal{O}_{\mathbb{P}^1}(2r+1)^{\oplus 6}$$

which is given by the matrix

$$\widetilde{df} = \begin{bmatrix} (2r+1)X_0^{2r} & 0 \\ 2rX_0^{2r-1}X_1 & X_0^{2r} \\ (r+1)X_0^r X_1^r & rX_0^{r+1}X_1^{r-1} \\ rX_0^{r-1}X_1^{r+1} & (r+1)X_0^r X_1^r \\ X_1^{2r} & 2rX_0X_1^{2r-1} \\ 0 & (2r+1)X_1^{2r} \end{bmatrix} \tag{55}$$

First of all observe that this matrix does have rank 2 at every point. This proves that f separates tangent vectors; injectivity of f follows from the fact that $[Y_0 : Y_1]$ and $[Y_1 : Y_2]$ are local inverses of f. Moreover the normal bundle of C in $\mathbb{P}^5$ is just the cokernel of $\widetilde{df}$. To compute this, consider the sheaf morphism $T : \mathcal{O}_{\mathbb{P}^1}(2r+1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(3r+1)^{\oplus 4}$ given by the matrix

$$\begin{bmatrix} (r-1)X_1^r & -rX_0X_1^{r-1} & X_0^r & 0 & 0 & 0 \\ 0 & X_1^r & -rX_0^{r-1}X_1 & (r-1)X_0^r & 0 & 0 \\ 0 & 0 & (r-1)X_1^r & -rX_0X_1^{r-1} & X_0^r & 0 \\ 0 & 0 & 0 & X_1^r & -rX_0^{r-1}X_1 & (r-1)X_0^r \end{bmatrix} \tag{56}$$

It is straightforward to verify that $T \circ \widetilde{df}$ is zero and that T has rank 4 everywhere. Thus T gives an isomorphism of $N_{C/\mathbb{P}^5}$ with $\mathcal{O}_{\mathbb{P}^1}(3r+1)^{\oplus 4}$, and we shall take this isomorphism to be an identification of locally free sheaves.

odd-2

7.2. **Computation of $N_{C/X}$.** Next we specify X and compute the normal bundle $N_{C/X}$. Observe that the quadric equations $Q_a = Y_1Y_4 - Y_0Y_5$ and $Q_b = Y_2Y_3 - Y_0Y_5$ both vanish on the image of f. Let L_a and L_b be any linear homogeneous polynomials in $Y_0, \dots, Y_5$ which are linearly independent and consider the homogeneous cubic polynomial $F = L_aQ_a + L_bQ_b$ (later we will specialize to the case that L_a and L_b are general linear homogeneous polynomials in Y_0 and Y_5 alone). For our purposes it is convenient to make a "change of variables" and define $M = L_a + L_b$ and $N = L_a + rL_b$ (here we are using that $r \neq 1$ to see that L_a and L_b are uniquely determined by M and N). Consider $X = \{[Y_0 : \cdots : Y_5] \in \mathbb{P}^5 | F(Y_0, \dots, Y_5) = 0\}$. Let us point out to the reader that X will be singular along the common zero locus of L_a, L_b, Q_a and Q_b – which will typically be a geometrically connected degree 4 curve of arithmetic genus 1.

To determine whether X is smooth along the image of f, we need to compute the pullback by f of the "gradient vector" $[\frac{\partial F}{\partial Y_i}]_{i=0,\dots,5}$. If we define $\widetilde{L}_a = f^*L_a, \widetilde{L_b} = f^*L_b, \widetilde{M} = f^*M$ and $\widetilde{N} = f^*N$ considered as sections of $H^0(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^5}(1)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(5))$, then the pullback of the gradient vector of F is the sheaf morphism $U : \mathcal{O}_{\mathbb{P}^1}(2r+1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(6r+3)$ given by

$$\left[\begin{array}{cccccc} -X_1^{2r+1}(\widetilde{L}_a + \widetilde{L}_b) & X_0X_1^{2r}\widetilde{L}_a & X_0^rX_1^{r+1}\widetilde{L}_b & X_0^{r+1}X_1\widetilde{L}_b & X_0^{2r}X_1\widetilde{L}_a & -X_0^{2r+1}(\widetilde{L}_a + \widetilde{L}_b) \end{array} \right]. \tag{57}$$

One readily verifies that if $\widetilde{L}_a$ and $\widetilde{L}_b$ have no common zeroes and if $\widetilde{L}_a + \widetilde{L}_b$ is nonzero at the points $[1:0]$ and $[0:1]$, then this matrix is everywhere nonzero, i.e. X is smooth along C. From now on we assume this is the case; in other words, $\widetilde{M}$ and $\widetilde{N}$ are linearly independent and $\widetilde{M}$ is not zero at the points $[0:1]$ and $[1:0]$. The matrix U factors as $U = S \circ T$ where $S : \mathcal{O}_{\mathbb{P}^1}(3r+1)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(6r+3)$ is given by the matrix

$$S = \frac{-1}{r-1} \left[\begin{array}{cccc} X_1^{r+1}\widetilde{M} & X_0X_1^r\widetilde{N} & X_0^rX_1\widetilde{N} & X_0^{r+1}\widetilde{M} \end{array} \right]. \tag{58}$$

The normal bundle of C in X, $N_{C/X}$ is just the kernel of the sheaf morphism S. To describe this map, we write out

$$\begin{cases} M = c_0Y_0 + c_1Y_1 + c_2Y_2 + c_3Y_3 + c_4Y_4 + c_5Y_5, \\ N = d_0Y_0 + d_1Y_1 + d_2Y_2 + d_3Y_3 + d_4Y_4 + d_5Y_5 \end{cases} \tag{59}$$

Then we have

$$\begin{cases} \widetilde{M} = c_0X_0^{2r+1} + c_1X_0^{2r}X_1 + c_2X_0^{r+1}X_1^r + c_3X_0^rX_1^{r+1} + c_4X_0X_1^{2r} + c_5X_1^{2r+1}, \\ \widetilde{N} = d_0X_0^{2r+1} + d_1X_0^{2r}X_1 + d_2X_0^{r+1}X_1^r + d_3X_0^rX_1^{r+1} + d_4X_0X_1^{2r} + d_5X_1^{2r+1} \end{cases} \tag{60}$$

We make the definitions

$$\begin{cases} n = d_4X_0^2X_1^r + d_5X_0X_1^{r+1}, \\ n' = d_0X_0^{r+1}X_1 + d_1X_0^rX_1^2 + d_2X_0X_1^{r+1} + d_3X_1^{r+2}, \\ m = c_4X_0X_1^{r+1} + c_5X_1^{r+2}, \\ m' = c_0X_0^{r+2} + c_1X_0^{r+1}X_1 + c_2X_0^2X_1^r + c_3X_0X_1^{r+1} \end{cases} \tag{61}$$

In other words, $X_0X_1N = X_1^{r+1}n + X_0^{r+1}n'$ and $M = X_1^{r-1}m + X_0^{r-1}m'$. Then consider the sheaf morphism $R : \mathcal{O}_{\mathbb{P}^1}(2r) \oplus \mathcal{O}_{\mathbb{P}^1}(2r+2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r-1) \to \mathcal{O}_{\mathbb{P}^1}(3r+1)^{\oplus 4}$

given by

$$R = \begin{bmatrix} X_0^{r+1} & 0 & n \\ 0 & X_0^{r-1} & -m \\ 0 & -X_1^{r-1} & -m' \\ -X_1^{r+1} & 0 & n' \end{bmatrix} \tag{62}$$

One readily verifies that $S \circ R$ is zero. The matrix R has rank three generically, namely, it has rank three at $[0:1]$ and $[1:0]$ by our hypothesis that $\widetilde{M}$ is nonvanishing at those points. By degree considerations, it follows that R has rank 3 everywhere and gives an isomorphism of $\mathcal{O}_{\mathbb{P}^1}(2r) \oplus \mathcal{O}_{\mathbb{P}^1}(2r+2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r-1)$ with the kernel of S, i.e. $N_{C/X}$. Observe, in particular, that $H^1(\mathbb{P}^1, N_{C/X})$ is trivial, so $([X],[C])$ is a point of U_e. We illustrate the situation with the following diagram:

$$\begin{array}{ccccc} & & \mathcal{O}(1)^2 & & \\ & & \downarrow{\scriptstyle \widetilde{df}} & & \\ & & \mathcal{O}(2r+1)^6 & \xrightarrow{=} & \mathcal{O}(2r+1)^6 \\ & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle U} \\ \mathcal{O}(2r) \oplus \mathcal{O}(2r+2) \oplus \mathcal{O}(2r-1) & \xrightarrow{R} & \mathcal{O}(3r+1)^4 & \xrightarrow{S} & \mathcal{O}(6r+3) \end{array}$$

odd-3

7.3. **Initial description of the pairing.** In this subsection we begin to describe the skew-symmetric bilinear pairing on $H^0(C, N_{C/X})$ induced by ω_e. We will complete the description in the next subsection. Let's introduce a little notation. We will usually refer to elements in $H^0(\mathbb{P}^1, N_{C/X})$ by (g_1, g_2, g_3) and also $g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3$ where $\mathbf{e}_i$ is the ith column of the matrix R and where $g_1 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r)), g_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r+2))$ and $g_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-1))$.

Now by Theorem 5.1, to compute the bilinear pairing ω_e on $H^0(\mathbb{P}^1, N_{C/X})$ it is equivalent (up to a nonzero scaling) to compute the boundary map

$$\delta : H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X}) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)).$$

Notice that the next term in the long exact sequence of cohomology is $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r))^{\oplus 4}$, which is trivial. Therefore the connecting homomorphism is the cokernel of the map on global sections

$$R^\dagger : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r))^{\oplus 4} \to H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X}) = H^0(\mathbb{P}^1, \mathcal{O}(4r+1) \oplus \mathcal{O}(4r-1) \oplus \mathcal{O}(4r+2))$$

determined by the sheaf morphism $R^\dagger : \mathcal{O}_{\mathbb{P}^1}(3r)^{\oplus 4} \to \bigwedge^2 N_{C/X}$ which is adjoint to R. (The adjoint $R^\dagger = \mathrm{diag}(1, -1, 1) \circ R^t$, where R^t is the transpose of R.) If we use as "ordered basis" for $\bigwedge^2 N_{C/X}$ the elements $\mathbf{e}_2 \wedge \mathbf{e}_3$, $\mathbf{e}_1 \wedge \mathbf{e}_3$ and $\mathbf{e}_1 \wedge \mathbf{e}_2$, then the matrix of $R^\dagger$ is simply

$$\begin{bmatrix} X_0^{r+1} & 0 & 0 & -X_1^{r+1} \\ 0 & -X_0^{r-1} & X_1^{r-1} & 0 \\ n & -m & -m' & n' \end{bmatrix} \tag{63}$$

So, in other words, the pairing ω_e is just given by

$$[(g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3), (h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3)] = (g_1h_2 - g_2h_1)\mathbf{e}_1 \wedge \mathbf{e}_2 \\ + (g_1h_3 - g_3h_1)\mathbf{e}_1 \wedge \mathbf{e}_3 + (g_2h_3 - g_3h_2)\mathbf{e}_2 \wedge \mathbf{e}_3 \bmod \mathrm{Im}(R^\dagger).$$

odd-4

7.4. **The image of the map $R^\dagger$.** In order to have an explicit formula for the pairing $[\cdot,\cdot]$ above, we need to determine the image of $R^\dagger$. We begin by determining the intersection of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r+2))\mathbf{e}_1 \wedge \mathbf{e}_2$ with the image of $R^\dagger$. First of all, a global section of $\mathcal{O}_{\mathbb{P}^1}(3r)^{\oplus 4}$ is mapped under $R^\dagger$ into $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r+2))\mathbf{e}_1 \wedge \mathbf{e}_2$ iff it is of the form

$$v = \begin{bmatrix} X_1^{r+1}p \\ -X_1^{r-1}q \\ -X_0^{r-1}q \\ X_0^{r+1}p \end{bmatrix} \tag{64}$$

for some $p \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-1))$ and $q \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r+1))$. And the image of this element is just

$$R^\dagger(v) = (X_0X_1\widetilde{N}p + \widetilde{M}q)\mathbf{e}_1 \wedge \mathbf{e}_2. \tag{65}$$

At this point we make our last simplification. We will assume that $c_1 = c_2 = c_3 = c_4 = 0$ and $d_1 = d_2 = d_3 = d_4 = 0$, in other words L_a and L_b are both linear combinations of Y_0 and Y_5 which are linearly independent and such that c_0, c_5, d_0 and d_5 are all nonzero. Now consider just those q such that $q = X_0X_1q'$ for some $q' \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-1))$. Then we have that $R^\dagger(v)$ is simply $X_0X_1(\widetilde{N}p + \widetilde{M}q')$. Since $\widetilde{M}$ and $\widetilde{N}$ are linearly independent elements in the span of X_0^{2r+1} and X_1^{2r+1}, as we allow p and q' to vary the expression $R^\dagger(v)$ varies over the whole linear span of

$$X_0^{4r+1}X_1, \ldots, X_0^{2r+2}X_1^{2r}, X_0^{2r}X_1^{2r+2}, \ldots, X_0X_1^{4r+1}.$$

Notice that $X_0^{4r+2}, X_0^{2r+1}X_1^{2r+1}$ and X_1^{4r+2} are missing. But taking $q = X_0^{2r+1}$ and $q = X_1^{2r+1}$ does give us $c_0X_0^{4r+2} + c_5X_0^{2r+1}X_1^{2r+1}$ and $c_0X_0^{2r+1}X_1^{2r+1} + c_5X_1^{4r+2}$. Thus we have that intersection of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r+2))\mathbf{e}_1 \wedge \mathbf{e}_2$ with the image of $R^\dagger$ is precisely the subspace with basis

$$c_0X_0^{4r+2} + c_5X_0^{2r+1}X_1^{2r+1}, X_0^{4r+1}X_1, X_0^{4r}X_1^2, \ldots, X_0^{2r+2}X_1^{2r}, X_0^{2r}X_1^{2r+2}, \ldots \\ X_0X_1^{4r+1}, c_0X_0^{2r+1}X_1^{2r+1} + c_5X_1^{4r+2}.$$

Now we introduce some more notation. For each pair of nonnegative integers (i,j), let $\alpha_{i,j} : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i+j)) \to \mathbb{C}$ be the linear functional such that for any homogeneous polynomial g of degree d we have

$$g(X_0, X_1) = \sum_{i+j=d} \alpha_{i,j}(g)X_0^iX_1^j, \tag{66}$$

i.e. $\alpha_{i,j}(g)$ is just the coefficient of $X_0^iX_1^j$ in g. Then the linear functional $c_5^2\alpha_{4r+2,0} - c_0c_5\alpha_{2r+1,2r+1} + c_0^2\alpha_{0,4r+2}$ is a nonzero linear functional on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r+2))$ whose kernel is precisely the intersection with the image of $R^\dagger$.

Also, we can use the first two rows of $R^\dagger$ to represent every element in $H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X})$ as being congruent to some element in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r+2))\mathbf{e}_1 \wedge \mathbf{e}_2$ modulo the image of $R^\dagger$. Carrying this out we see that, up to a nonzero scalar, the pairing $[\cdot,\cdot]$ is uniquely determined to be

$$[(g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3), (h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3)] = \\ (c_5^2\alpha_{4r+2,0} - c_0c_5\alpha_{2r+1,2r+2} + c_0^2\alpha_{0,4r+2})(g_1h_2 - g_2h_1) + \\ c_0c_5(c_5\alpha_{3r,r-1} - c_0\alpha_{r-1,3r})(g_1h_3 - g_3h_1) + \\ c_0c_5(d_5\alpha_{3r+1,r} - d_0\alpha_{r,3r+1})(g_2h_3 - g_3h_2).$$

odd-5

7.5. **Diagonalizing the pairing.** The antisymmetric bilinear map $[\cdot,\cdot]$ gives a linear transformation $\widetilde{\omega}_e : H^0(\mathbb{P}^1, N_{C/X}) \to H^0(\mathbb{P}^1, N_{C/X})^\vee$ and we want to determine the kernel of this linear transformation. To do this, first we will "diagonalize" the pair $(H^0(\mathbb{P}^1, N_{C/X}), [\cdot,\cdot])$, i.e. we will find a direct sum decomposition

$$H^0(\mathbb{P}^1, N_{C/X}) = \bigoplus_{i=0}^{r-2} E_i \oplus E_{r-1} \oplus E_r$$

such that for each $i \neq j$, E_i and E_j are mutually orthogonal subspaces with respect to $[\cdot,\cdot]$. To show that $[\cdot,\cdot]$ has trivial kernel, it suffices then to show that the restriction of the pairing to each space E_i has trivial kernel. And this we will do by computing the determinant of the matrix of $[\cdot,\cdot]$ with respect to a suitable basis.

For $i = 0, \ldots, r-2$, consider the subspace $E_i \subset H^0(\mathbb{P}^1, N_{C/X})$ generated by

$$\begin{cases} \mathbf{v}_{i,1} &= X_0^{r+1+i} X_1^{r-1-i} \mathbf{e}_1 \\ \mathbf{v}_{i,2} &= X_0^{r-i} X_1^{r+2+i} \mathbf{e}_2 \\ \mathbf{v}_{i,3} &= X_0^{2r-1-i} X_1^{i} \mathbf{e}_3 \\ \mathbf{v}_{i,4} &= X_0^{i} X_1^{2r-1-i} \mathbf{e}_3 \\ \mathbf{v}_{i,5} &= X_0^{r+2+i} X_1^{r-i} \mathbf{e}_2 \\ \mathbf{v}_{i,6} &= X_0^{r-1-i} X_1^{r+1+i} \mathbf{e}_1 \end{cases}$$

For $i = r-1$ consider the subspace $E_{r-1} \subset H^0(\mathbb{P}^1, N_{C/X})$ generated by

$$\begin{cases} \mathbf{v}_{r-1,1} &= X_0^{2r} \mathbf{e}_1 \\ \mathbf{v}_{r-1,2} &= X_0^{2r+2} \mathbf{e}_2 \\ \mathbf{v}_{r-1,3} &= X_0^{2r+1} X_1 \mathbf{e}_2 \\ \mathbf{v}_{r-1,4} &= X_0^{r} X_1^{r+1} \mathbf{e}_3 \\ \mathbf{v}_{r-1,5} &= X_0^{r+1} X_1^{r} \mathbf{e}_3 \\ \mathbf{v}_{r-1,6} &= X_0 X_1^{2r+1} \mathbf{e}_2 \\ \mathbf{v}_{r-1,7} &= X_1^{2r+2} \mathbf{e}_2 \\ \mathbf{v}_{r-1,8} &= X_1^{2r} \mathbf{e}_1 \end{cases}$$

Finally, for $i = r$ we consider the subspace $E_r \subset H^0(\mathbb{P}^1, N_{C/X})$ generated by

$$\begin{cases} \mathbf{v}_{r,1} &= X_0^{r} X_1^{r} \mathbf{e}_1 \\ \mathbf{v}_{r,2} &= X_0^{r+1} X_1^{r+1} \mathbf{e}_2 \end{cases}$$

First of all, observe that each of these generating sets is simply a sub-basis of the standard monomial basis of $H^0(\mathbb{P}^1, N_{C/X})$ which is

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r))\mathbf{e}_1 \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r+2))\mathbf{e}_2 \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-1))\mathbf{e}_3.$$

It is very easy to check that every monomial basis vector is in precisely one of the subspaces E_i, and thus these spaces give a direct sum decomposition of $H^0(\mathbb{P}^1, N_{C/X})$. Just as a consistency check, observe that for $i = 0, \ldots, r-2$ we have $\dim(E_i) = 6$, $\dim(E_{r-1}) = 8$ and $\dim(E_r) = 2$. So the total dimension is $6(r-1)+8+2 = 6r+4$ which is $(2r+1)+(2r+3)+2r$, i.e.

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r))\mathbf{e}_1 + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r+2))\mathbf{e}_2 + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-1))\mathbf{e}_3.$$

Checking that the spaces E_i are pairwise orthogonal with respect to $[\cdot,\cdot]$ is straightforward, but tedious. One way to think of it is to consider the graph whose vertices are the standard monomial basis vectors of $H^0(\mathbb{P}^1, N_{C/X})$, and where there is an edge between two such basis vectors iff the pairing is nonzero for this pair. Thus there is never an edge between $g_1\mathbf{e}_1$ and $h_1\mathbf{e}_1$, nor between $g_2\mathbf{e}_2$ and $h_2\mathbf{e}_2$, nor between $g_3\mathbf{e}_3$ and $h_3\mathbf{e}_3$. There is an edge between $g_1\mathbf{e}_1$ and $h_2\mathbf{e}_2$ iff $g_1h_2 = X_0^{4r+2}, X_0^{2r+1}X_1^{2r+1}$ or X_1^{4r+2}. There is an edge between $g_1\mathbf{e}_1$ and $h_3\mathbf{e}_3$ iff $g_1h_3 = X_0^{3r}X_1^{r-1}$ or $X_0^{r-1}X_1^{3r}$. And there is an edge between $g_2\mathbf{e}_2$ and $h_3\mathbf{e}_3$ iff $g_2h_3 = X_0^{3r+1}X_1^{r}$ or $X_0^{r}X_1^{3r+1}$. In particular, it is easy to see that the valence of $X_0^{2r}\mathbf{e}_1$, $X_1^{2r}\mathbf{e}_1$, $X_0^rX_1^{r-1}\mathbf{e}_3$ and $X_0^{r-1}X_1^r\mathbf{e}_3$ is three, the valence of $X_0^rX_1^r\mathbf{e}_1$ and $X_0^{r+1}X_1^{r+1}\mathbf{e}_2$ is one, and every other vertex has valence two. Moreover, there is an obvious symmetry in the graph obtained by permuting the variables X_0 and X_1. Using this, it is straightforward to take each of the vectors $\mathbf{v}_{i,1}$ and compute that the maximal connected subgraph containing this vertex gives the generating set for E_i (this is an exercise left to the reader). Thus the E_i are pairwise orthogonal.

odd-6

7.6. **Computing the determinants.** In this last subsection, we compute the determinant of $\widetilde{\omega}_e$ restricted to each of the direct summands. We show that each determinant is nonzero, which proves that the pairing is non-degenerate. For $i = 0, \ldots, r-2$, we can form the matrix of $\widetilde{\omega}_e : E_i \to E_i^\vee$ with respect to the ordered basis $\mathbf{v}_{i,1}, \ldots, \mathbf{v}_{i,6}$ and the dual ordered basis of $E_i^\vee$. This is straightforward to compute and turns out to be:

$$A_i = \begin{bmatrix} 0 & c_0c_5 & -c_0c_5^2 & 0 & 0 & 0 \\ -c_0c_5 & 0 & 0 & c_0c_5d_0 & 0 & 0 \\ c_0c_5^2 & 0 & 0 & 0 & c_0c_5d_5 & 0 \\ 0 & -c_0c_5d_0 & 0 & 0 & 0 & -c_0^2c_5 \\ 0 & 0 & -c_0c_5d_5 & 0 & 0 & -c_0c_5 \\ 0 & 0 & 0 & c_0^2c_5 & c_0c_5 & 0 \end{bmatrix} \tag{67}$$

It is straightforward to compute that the Pfaffian of this matrix is $\mathrm{Pfaff}(A_i) = c_0^3c_5^3(c_0d_5 - c_5d_0)$ and thus the determinant is $\mathrm{Det}(A_i) = c_0^6c_5^6(c_0d_5 - c_5d_0)^2$. Since we are assuming that c_0, c_5 are nonzero and that (c_0, c_5) is linearly independent from (d_0, d_5), this determinant is nonzero.

For $i = r-1$, we can form the matrix of $\widetilde{\omega}_e : E_{r-1} \to E_{r-1}^\vee$ with respect to the ordered basis $\mathbf{v}_{r-1,1}, \ldots, \mathbf{v}_{r-1,8}$ and the dual ordered basis of $E_{r-1}^\vee$. This is straightforward to compute and turns out to be:

$$A_{r-1} = \begin{bmatrix} 0 & -c_5^2 & 0 & -c_0c_5^2 & 0 & c_0c_5 & 0 & 0 \\ c_5^2 & 0 & 0 & 0 & -c_0c_5d_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_0c_5d_5 & 0 & 0 & 0 & -c_0c_5 \\ c_0c_5^2 & 0 & c_0c_5d_5 & 0 & 0 & 0 & -c_0c_5d_0 & 0 \\ 0 & c_0c_5d_5 & 0 & 0 & 0 & -c_0c_5d_0 & 0 & -c_0^2c_5 \\ -c_0c_5 & 0 & 0 & 0 & c_0c_5d_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_0c_5d_0 & 0 & 0 & 0 & c_0^2 \\ 0 & 0 & c_0c_5 & 0 & c_0^2c_5 & 0 & -c_0^2 & 0 \end{bmatrix} \tag{68}$$

It is straightforward to compute that the Pfaffian of this matrix is $\mathrm{Pfaff}(A_{r-1}) = c_0^3c_5^3(c_0d_5 - c_5d_0)^2$ and thus the determinant is $\mathrm{Det}(A_{r-1}) = c_0^6c_5^6(c_0d_5 - c_5d_0)^4$. Since we are assuming that c_0, c_5 are nonzero and that (c_0, c_5) is linearly independent from (d_0, d_5), this determinant is nonzero.

For $i = r$, we can form the matrix of $\widetilde{\omega}_e : E_r \to E_r^\vee$ with respect to the ordered basis $\mathbf{v}_{r,1}, \mathbf{v}_{r,2}$ and the dual ordered basis of $E_r^\vee$. This is straightforward to compute and turns out to be:

$$A_r = \begin{bmatrix} 0 & c_0c_5 \\ -c_0c_5 & 0 \end{bmatrix} \tag{69}$$

Visibly the Pfaffian of this matrix is c_0c_5 and the determinant is $c_0^2c_5^2$. Since we are assuming that c_0, c_5 are nonzero, this determinant is nonzero.

Since each of the determinants above are nonzero $\widetilde{\omega}_e$ has maximal rank on each of these subspaces. Since the subspaces are pairwise orthogonal, we conclude that $\widetilde{\omega}_e$ has maximal rank on all of $H^0(\mathbb{P}^1, N_{C/X})$, i.e. the kernel of $\widetilde{\omega}_e$ is zero. This proves Theorem 1.2 in case $e = 2r+1$ is an odd integer with $e \geq 5$.

even

8. Proof of Theorem 1.2: the even degree case

In the last section we saw the proof of Theorem 1.2 in the odd degree case. In this section we shall prove Theorem 1.2 in the even degree case. The proof will be exactly analogous to the last case and, if anything, simpler than that case. As in the last section, for each even degree $e \geq 6$, we shall find a specific pair $([X],[C])$ where $C \subset \mathbb{P}^5$ is an embedded rational curve of degree e and where $X \subset \mathbb{P}^5$ is a cubic hypersurface containing C and such that C is disjoint from the singular locus of X. For our special pair, we will prove that the rank of ω_e is as expected and $H^1(C, N_{C/X})$ will be zero.

even-1

8.1. **Computation of $N_{C/\mathbb{P}^5}$.** Write $e = 2r$ where $r \geq 3$ is some integer. We begin by specifying C and computing $N_{C/\mathbb{P}^5}$. Choose homogeneous coordinates X_0, X_1 on $\mathbb{P}^1$ and as before choose homogeneous coordinates $Y_0, Y_1, Y_2, Y_3, Y_4, Y_5$ on $\mathbb{P}^5$. Consider the map $f : \mathbb{P}^1 \to \mathbb{P}^5$ given by

$$f([X_0 : X_1]) = [X_0^{2r} : X_0^{2r-1}X_1 : X_0^{r+1}X_1^{r-1} : X_0^{r-1}X_1^{r+1} : X_0X_1^{2r-1} : X_1^{2r}].$$

This is a monomial embedding of $\mathbb{P}^1$ which is as "balanced" as possible. To compute the normal bundle of C in $\mathbb{P}^5$, we use the Euler sequences for $T_{\mathbb{P}^1}$ and $T_{\mathbb{P}^5}$. There is a map between these Euler sequences induced by f and the important term is

$$\widetilde{df} : \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to f^*\left(\mathcal{O}_{\mathbb{P}^5}(1)^{\oplus 6}\right) = \mathcal{O}_{\mathbb{P}^1}(2r)^{\oplus 6}$$

which is given by the matrix

$$\widetilde{df} = \begin{bmatrix} 2rX_0^{2r-1} & 0 \\ (2r-1)X_0^{2r-2}X_1 & X_0^{2r-1} \\ (r+1)X_0^rX_1^{r-1} & (r-1)X_0^{r+1}X_1^{r-2} \\ (r-1)X_0^{r-2}X_1^{r+1} & (r+1)X_0^{r-1}X_1^r \\ X_1^{2r-1} & (2r-1)X_0X_1^{2r-2} \\ 0 & 2rX_1^{2r-1} \end{bmatrix} \tag{70}$$

To see that f is an embedding, notice that $[Y_0 : Y_1]$ and $[Y_4 : Y_5]$ give local inverses of f. Moreover the normal bundle of C in $\mathbb{P}^5$ is just the cokernel of $\widetilde{df}$. To compute this, consider the sheaf morphism $T : \mathcal{O}_{\mathbb{P}^1}(2r)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(3r-1) \oplus \mathcal{O}_{\mathbb{P}^1}(3r) \oplus \mathcal{O}_{\mathbb{P}^1}(3r) \oplus \mathcal{O}_{\mathbb{P}^1}(3r-1)$ given by the matrix

$$\begin{bmatrix} (r-2)X_1^{r-1} & -(r-1)X_0X_1^{r-1} & X_0^{r-1} & 0 & 0 & 0 \\ 0 & 2X_1^r & -rX_0^{r-2}X_1^2 & (r-2)X_0^r & 0 & 0 \\ 0 & 0 & (r-2)X)1^r & -rX_0^2X_1^{r-2} & 2X_0^r & 0 \\ 0 & 0 & 0 & X_1^{r-1} & -(r-1)X_0^{r-2}X_1 & (r-2)X_0^{r-1} \end{bmatrix} \tag{71}$$

It is straightforward to verify that $T \circ \widetilde{df}$ is zero and that T has rank 4 everywhere. Thus T gives an isomorphism of $N_{C/\mathbb{P}^5}$ with $\mathcal{O}_{\mathbb{P}^1}(3r-1) \oplus \mathcal{O}_{\mathbb{P}^1}(3r) \oplus \mathcal{O}_{\mathbb{P}^1}(3r) \oplus \mathcal{O}_{\mathbb{P}^1}(3r-1)$, and we shall take this isomorphism to be an identification of locally free sheaves.

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8.2. **Computation of $N_{C/X}$.** Next we specify X and compute the normal bundle $N_{C/X}$. Observe that the quadric equations $Q_a = Y_1Y_4 - Y_0Y_5$ and $Q_b = Y_2Y_3 - Y_0Y_5$ both vanish on the image of f. Let L_a and L_b be any linear homogeneous polynomials in $Y_0, \dots, Y_5$ and consider the homogeneous cubic polynomial $F = L_aQ_a + L_bQ_b$ (later we will specialize to the case that L_a and L_b are general linear homogeneous polynomials in Y_0 and Y_5 alone). For our purposes it is convenine to make the "change of variables" $M = L_a + L_b$ and $N = L_a + (r-1)L_b$. Consider $X = \{[Y_0 : \cdots : Y_5] \in \mathbb{P}^5 | F(Y_0, \dots, Y_5) = 0\}$. Let us point out to the reader that X will be singular along the common zero locus of L_a, L_b, Q_a and Q_b – which will typically be a geometrically connected degree 4 curve of arithmetic genus 1.

To determine whether X is smooth along the image of f, we need to compute the pullback by f of the "gradient vector" $[\frac{\partial F}{\partial Y_i}]_{i=0,\dots,5}$. If we define $\widetilde{L}_a = f^*L_a, \widetilde{L_b} = f^*L_b, \widetilde{M} = f^*M$ and $\widetilde{N} = f^*N$ considered as sections of $H^0(\mathbb{P}^1, f^*\mathcal{O}_{\mathbb{P}^5}(1)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(5))$, then the pullback of the gradient vector of F is the sheaf morphism $U : \mathcal{O}_{\mathbb{P}^1}(2r)^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^1}(6r)$ given by

$$\begin{bmatrix} -X_1^{2r}(\widetilde{L}_a + \widetilde{L}_b) & X_0X_1^{2r-1}\widetilde{L}_a & X_0^{r-1}X_1^{r+1}\widetilde{L}_b & X_0^{r+1}X_1^{r-1}\widetilde{L}_b & X_0^{2r-1}X_1\widetilde{L}_a & -X_0^{2r}(\widetilde{L}_a + \widetilde{L}_b) \end{bmatrix}. \tag{72}$$

One readily verifies that if $\widetilde{L}_a$ and $\widetilde{L}_b$ have no common zeroes and if $\widetilde{L}_a + \widetilde{L}_b$ is nonzero at the points $[1:0]$ and $[0:1]$, then this matrix is everywhere nonzero, i.e. X is smooth along C. Moreover this matrix factors as $U = S \circ T$ where $S : N_{C/\mathbb{P}^5} \to \mathcal{O}_{\mathbb{P}^1}(6r)$ is given by the matrix

$$S = \frac{-1}{2(r-2)} \begin{bmatrix} 2X_1^{r+1}\widetilde{M} & X_0X_1^{r-1}\widetilde{N} & X_0^{r-1}X_1\widetilde{N} & 2X_0^{r+1}\widetilde{M} \end{bmatrix}. \tag{73}$$

The normal bundle of C in X, $N_{C/X}$ is just the kernel of the sheaf morphism S. To describe this map, we write out

$$\begin{cases} M = c_0Y_0 + c_1Y_1 + c_2Y_2 + c_3Y_3 + c_4Y_4 + c_5Y_5, \\ N = d_0Y_0 + d_1Y_1 + d_2Y_2 + d_3Y_3 + d_4Y_4 + d_5Y_5 \end{cases} \tag{74}$$

Then we have

$$\begin{cases} \widetilde{M} = c_0X_0^{2r} + c_1X_0^{2r-1}X_1 + c_2X_0^{r+1}X_1^{r-1} + c_3X_0^{r-1}X_1^{r+1} + c_4X_0X_1^{2r-1} + c_5X_1^{2r}, \\ \widetilde{N} = d_0X_0^{2r} + d_1X_0^{2r-1}X_1 + d_2X_0^{r+1}X_1^{r-2} + d_3X_0^{r-1}X_1^{r+1} + d_4X_0X_1^{2r-1} + d_5X_1^{2r} \end{cases} \tag{75}$$

We make the definitions

$$\begin{cases} n = d_3X_0X_1^r + d_4X_0^2X_1^{r-1} + d_5X_0X_1^r, \\ n' = d_0X_0^rX_1 + d_1X_0^{r-1}X_1^2 + d_2X_0X_1^r, \\ m = 2c_4X_0X_1^{r+1} + 2c_5X_1^{r+2}, \\ m' = 2c_0X_0^{r+2} + 2c_1X_0^{r+1}X_1 + 2c_2X_0^3X_1^{r-1} + 2c_3X_0X_1^{r+1} \end{cases} \tag{76}$$

In other words, $X_0X_1\widetilde{N} = X_1^{r+1}n + X_0^{r+1}n'$ and $2\widetilde{M} = X_1^{r-2}m + X_0^{r-2}m'$. Then consider the sheaf morphism $R : \mathcal{O}_{\mathbb{P}^1}(2r-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r+2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r-2) \to N_{C/\mathbb{P}^5}$

given by

$$R = \begin{bmatrix} X_0^{r+1} & 0 & n \\ 0 & X_0^{r-2} & -m \\ 0 & -X_1^{r-2} & -m' \\ -X_1^{r+1} & 0 & n' \end{bmatrix} \tag{77}$$

One readily verifies that $S \circ R$ is zero and R has rank three generically (in particular is has rank three at $[0:1]$ and $[1:0]$ by our hypothesis that $\widetilde{M}$ is nonvanishing at those points). By degree considerations, it follows that R has rank 3 everywhere and gives an isomorphism of $\mathcal{O}_{\mathbb{P}^1}(2r-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2r) \oplus \mathcal{O}_{\mathbb{P}^1}(2r-2)$ with the kernel of S, i.e. $N_{C/X}$. Observe, in particular, that $H^1(\mathbb{P}^1, N_{C/X})$ is trivial so $([X],[C])$ is in U_e.

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8.3. **Initial description of the pairing.** In this subsection we begin to describe the skew-symmetric bilinear pairing on $H^0(C, N_{C/X})$ induced by ω_e. We will complete the description in the next section. Let's introduce a little notation. We will usually refer to elements in $H^0(\mathbb{P}^1, N_{C/X})$ by (g_1, g_2, g_3) and also $g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3$ where $\mathbf{e}_i$ is the ith column of the matrix R and where $g_1 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-2)), g_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r+2))$ and $g_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-2))$.

Now by Theorem 5.1, to compute the bilinear pairing ω_e on $H^0(\mathbb{P}^1, N_{C/X})$ it is equivalent (up to a nonzero scaling) to compute the boundary map

$$\delta : H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X}) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)).$$

Notice that the next term in the long exact sequence of cohomology is $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r-1))^{\oplus 2} \oplus H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r-2))^{\oplus 2}$, which is trivial. Therefore the connecting homomorphism is the cokernel of the map on global sections

$$\begin{aligned} R^\dagger : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3r-1) \oplus \mathcal{O}_{\mathbb{P}^1}(3r-2) \oplus \mathcal{O}_{\mathbb{P}^1}(3r-2)) \oplus \mathcal{O}_{\mathbb{P}^1}(3r-1)) \\ \longrightarrow H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X}) \end{aligned}$$

determined by the sheaf morphism $R^\dagger : N^\vee_{C\mathbb{P}^5} \otimes \bigwedge^3 N_{C/X} \to \bigwedge^2 N_{C/X}$ which is adjoint to R. If we use as "ordered basis" for $\bigwedge^2 N_{C/X}$ the elements $\mathbf{e}_2 \wedge \mathbf{e}_3$, $\mathbf{e}_1 \wedge \mathbf{e}_3$ and $\mathbf{e}_1 \wedge \mathbf{e}_2$, then the matrix of $R^\dagger$ is simply

$$\begin{bmatrix} X_0^{r+1} & 0 & 0 & -X_1^{r+1} \\ 0 & -X_0^{r-2} & X_1^{r-2} & 0 \\ n & -m & -m' & n' \end{bmatrix} \tag{78}$$

So, in other words, the pairing ω_e is just given by

$$\begin{aligned} [(g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3), (h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3)] = (g_1h_2 - g_2h_1)\mathbf{e}_1 \wedge \mathbf{e}_2 \\ +(g_1h_3 - g_3h_1)\mathbf{e}_1 \wedge \mathbf{e}_3 + (g_2h_3 - g_3h_2)\mathbf{e}_2 \wedge \mathbf{e}_3 \quad (\text{modulo} R^\dagger). \end{aligned}$$

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8.4. **The image of the map $R^\dagger$.** In order to have an explicit formula for the pairing $[\cdot,\cdot]$ above, we need to determine the image of $R^\dagger$. We begin by determining the intersection of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r))\mathbf{e}_1 \wedge \mathbf{e}_2$ with the image of $R^\dagger$. First of all, a global section of $\mathcal{O}_{\mathbb{P}^1}(3r-1) \oplus \mathcal{O}_{\mathbb{P}^1}(3r-2) \oplus \mathcal{O}_{\mathbb{P}^1}(3r-2) \oplus \mathcal{O}_{\mathbb{P}^1}(3r-1)$ is mapped under $R^\dagger$ into $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r))\mathbf{e}_1 \wedge \mathbf{e}_2$ iff it is of the form

$$v = \begin{bmatrix} X_1^{r+1}p \\ -X_1^{r-2}q \\ -X_0^{r-2}q \\ X_0^{r+1}p \end{bmatrix} \tag{79}$$

for some $p \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-2))$ and $q \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r))$. And the image of this element is just

$$R^\dagger(v) = (-X_0X_1\widetilde{N}p + 2\widetilde{M}q)\mathbf{e}_1 \wedge \mathbf{e}_2. \tag{80}$$

At this point we make our last simplification. We will assume that $c_1 = c_2 = c_3 = c_4 = 0$ and $d_1 = d_2 = d_3 = d_4 = 0$, in other words L_a and L_b are both linear combinations of Y_0 and Y_5 which are linearly independent and such that c_0, c_5, d_0 and d_5 are all nonzero. Now consider just those q such that $q = X_0X_1q'$ for some $q' \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-2))$. Then we have that $R^\dagger(v)$ is simply $X_0X_1(\widetilde{N}p + 2\widetilde{M}q)$. Since $\widetilde{M}$ and $\widetilde{N}$ are linearly independent elements in the span of X_0^{2r} and X_1^{2r}, as we allow p and q' to vary, $R^\dagger(v)$ varies over the whole linear span of

$$X_0^{4r-1}X_1, \dots, X_0^{2r+1}X_1^{2r-1}, X_0^{2r-1}X_1^{2r+1}, \dots, X_0X_1^{4r-1}.$$

Notice that $X_0^{4r}, X_0^{2r}X_1^{2r}$ and X_1^{4r} are missing. But taking $q = X_0^{2r}$ and $q = X_1^{2r}$ does give us $c_0X_0^{4r} + c_5X_0^{2r}X_1^{2r}$ and $c_0X_0^{2r}X_1^{2r} + c_5X_1^{4r}$. Thus we have that the intersection of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r))\mathbf{e}_1 \wedge \mathbf{e}_2$ with the image of $R^\dagger$ is precisely the subspace with basis

$$c_0X_0^{4r} + c_5X_0^{2r}X_1^{2r}, X_0^{4r-1}X_1, X_0^{4r-2}X_1^2, \dots, X_0^{2r+1}X_1^{2r-1}, X_0^{2r-1}X_1^{2r+1}, \dots \\ X_0X_1^{4r-1}, c_0X_0^{2r}X_1^{2r} + c_5X_1^{4r}.$$

Now we introduce some more notation. For each pair of nonnegative integers (i, j), let $\alpha_{i,j} : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i+j)) \to \mathbb{C}$ be the linear functional such that for any homogeneous polynomial g of degree d we have

$$g(X_0, X_1) = \sum_{i+j=d} \alpha_{i,j}(g) X_0^i X_1^j, \tag{81}$$

i.e. $\alpha_{i,j}(g)$ is just the coefficient of $X_0^iX_1^j$ in g. Then the linear functional $c_5^2\alpha_{4r,0} - c_0c_5\alpha_{2r,2r} + c_0^2\alpha_{0,4r}$ is a nonzero linear functional on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r))$ whose kernel is precisely the intersection with the image of $R^\dagger$.

Also, we can use the first two rows of $R^\dagger$ to represent every element in $H^0(\mathbb{P}^1, \bigwedge^2 N_{C/X})$ as being congruent to some element in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4r))\mathbf{e}_1 \wedge \mathbf{e}_2$ modulo the image of $R^\dagger$. Carrying this out we see that, up to a nonzero scalar, the pairing $[\cdot, \cdot]$ is uniquely determined to be

$$\begin{aligned} &[(g_1\mathbf{e}_1 + g_2\mathbf{e}_2 + g_3\mathbf{e}_3), (h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3)] = \\ &(c_5^2\alpha_{4r,0} - c_0c_5\alpha_{2r,2r} + c_0^2\alpha_{0,4r})(g_1h_2 - g_2h_1) + \\ &2c_0c_5(c_5\alpha_{3r-2,r-2} - c_0\alpha_{r-2,3r-2})(g_1h_3 - g_3h_1) + \\ &c_0c_5(d_5\alpha_{3r,r} - d_0\alpha_{r,3r})(g_2h_3 - g_3h_2). \end{aligned}$$

8.5. **Diagonalizing the pairing.** The antisymmetric bilinear map $[\cdot,\cdot]$ gives a linear transformation $\widetilde{\omega}_e : H^0(\mathbb{P}^1, N_{C/X}) \to H^0(\mathbb{P}^1, N_{C/X})^\vee$ and we want to determine the kernel of this linear transformation. To do this, first we will "diagonalize" the pair $(H^0(\mathbb{P}^1, N_{C/X}), [\cdot,\cdot])$, i.e. we will find a direct sum decomposition

$$H^0(\mathbb{P}^1, N_{C/X}) = \bigoplus_{i=0}^{r-3} E_i \oplus E_{r-2} \oplus E_{r-1} \oplus E_r$$

such that for each $i \neq j$, E_i and E_j are mutually orthogonal subspaces with respect to $[\cdot,\cdot]$. We will see that there is a vector $\mathbf{w}$ in E_r in the kernel of $\widetilde{\omega}_e$. On the quotient vector space $H^0(\mathbb{P}^1, N_{C/X})/\mathbb{C}\{\mathbf{w}\}$, we have an induced alternating bilinear form $[\cdot,\cdot]'$ and an induced direct sum decomposition $\bigoplus_{i=0}^r E_i'$ by pairwise orthogonal subspaces. To show that $[\cdot,\cdot]'$ has trivial kernel, it suffices then to show that the restriction of the form to each space E_i' has trivial kernel. And this we will do by computing the determinant of the matrix of $[\cdot,\cdot]'$ with respect to a suitable basis.

For $i = 0, \dots, r-3$, consider the subspace $E_i \subset H^0(\mathbb{P}^1, N_{C/X})$ generated by

$$\begin{cases} \mathbf{v}_{i,1} &= X_0^{r+i} X_1^{r-2-i} \mathbf{e}_1 \\ \mathbf{v}_{i,2} &= X_0^{r-i} X_1^{r+2+i} \mathbf{e}_2 \\ \mathbf{v}_{i,3} &= X_0^{2r-2-i} X_1^{i} \mathbf{e}_3 \\ \mathbf{v}_{i,4} &= X_0^{i} X_1^{2r-2-i} \mathbf{e}_3 \\ \mathbf{v}_{i,5} &= X_0^{r+2+i} X_1^{r-i} \mathbf{e}_2 \\ \mathbf{v}_{i,6} &= X_0^{r-2-i} X_1^{r+i} \mathbf{e}_1 \end{cases}$$

For $i = r-2$, consider the subspace $E_{r-2} \subset H^0(\mathbb{P}^1, N_{C/X})$ generated by

$$\begin{cases} \mathbf{v}_{r-2,1} &= X_0^{2r-2} \mathbf{e}_1 \\ \mathbf{v}_{r-2,2} &= X_0^{2r+2} \mathbf{e}_2 \\ \mathbf{v}_{r-2,3} &= X_0^{r} X_1^{r-2} \mathbf{e}_3 \\ \mathbf{v}_{r-2,4} &= X_0^{2} X_1^{2r} \mathbf{e}_2 \\ \mathbf{v}_{r-2,5} &= X_0^{2r} X_1^{2} \mathbf{e}_2 \\ \mathbf{v}_{r-2,6} &= X_0^{r-2} X_1^{r} \mathbf{e}_3 \\ \mathbf{v}_{r-2,7} &= X_1^{2r+2} \mathbf{e}_2 \\ \mathbf{v}_{r-2,8} &= X_1^{2r-2} \mathbf{e}_1 \end{cases}$$

For $i = r-1$, consider the subspace $E_{r-1} \subset H^0(\mathbb{P}^1, N_{C/X})$ generated by

$$\begin{cases} \mathbf{v}_{r-1,1} &= X_0^{r-1} X_1^{r-1} \mathbf{e}_1 \\ \mathbf{v}_{r-1,2} &= X_0^{r+1} X_1^{r+1} \mathbf{e}_2 \end{cases}$$

For $i = r$, consider the subspace $E_r \subset H^0(\mathbb{P}^1, N_{C/X})$ generated by

$$\begin{cases} \mathbf{v}_{r,1} &= X_0^{2r+1} X_1 \mathbf{e}_2 \\ \mathbf{v}_{r,2} &= X_0^{r-1} X_1^{r-1} \mathbf{e}_3 \\ \mathbf{v}_{r,3} &= X_0 X_1^{2r+1} \mathbf{e}_2 \end{cases}$$

First of all, observe that each of these generating sets is simply a subbasis of the standard monomial basis of $H^0(\mathbb{P}^1, N_{C/X})$ which is

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-2))\mathbf{e}_1 \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r+2))\mathbf{e}_2 \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-2))\mathbf{e}_3.$$

It is very easy to check that every monomial basis vector is in precisely one of the subspaces E_i, and thus these spaces give a direct sum decomposition of $H^0(\mathbb{P}^1, N_{C/X})$.

Just as a consistency check, observe that for $i = 0, \ldots, r-3$ we have $\dim(E_i) = 6$, $\dim(E_{r-2}) = 8$, $\dim(E_{r-1}) = 2$, and $\dim(E_r) = 3$. So the total dimension is $6(r-2) + 8 + 2 + 3 = 6r + 1$ which is $(2r-1) + (2r+3) + (2r-1)$, i.e.

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-2))\mathbf{e}_1 + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r+2))\mathbf{e}_2 + \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2r-2))\mathbf{e}_3.$$

Checking that the spaces E_i are pairwise orthogonal with respect to $[\cdot, \cdot]$ is straightforward, but tedious. One way to think of it is to consider the graph whose vertices are the standard monomial basis vectors of $H^0(\mathbb{P}^1, N_{C/X})$, and where there is an edge between two such basis vectors iff the pairing is nonzero for this pair. Thus there is never an edge between $g_1\mathbf{e}_1$ and $h_1\mathbf{e}_1$, nor between $g_2\mathbf{e}_2$ and $h_2\mathbf{e}_2$, nor between $g_3\mathbf{e}_3$ and $h_3\mathbf{e}_3$. There is an edge between $g_1\mathbf{e}_1$ and $h_2\mathbf{e}_2$ iff $g_1h_2 = X_0^{4r}, X_0^{2r}X_1^{2r}$ or X_1^{4r}. There is an edge between $g_1\mathbf{e}_1$ and $h_3\mathbf{e}_3$ iff $g_1h_3 = X_0^{3r-2}X_1^{r-2}$ or $X_0^{r-2}X_1^{3r-2}$. And there is an edge between $g_2\mathbf{e}_2$ and $h_3\mathbf{e}_3$ iff $g_2h_3 = X_0^{3r}X_1^r$ ro $X_0^rX_1^{3r}$. In particular, it is easy to see that the valence of $X_0^{2r-2}\mathbf{e}_1$, $X_1^{2r-2}\mathbf{e}_1$, $X_0^rX_1^{r-2}\mathbf{e}_3$ and $X_0^{r-2}X_1^r\mathbf{e}_3$ is three, the valence of $X_0^{r-1}X_1^{r-1}\mathbf{e}_1, X_0^{r+1}X_1^{r+1}\mathbf{e}_2, X_0^{2r+1}X_1\mathbf{e}_2$ and $X_0X_1^{2r+2}\mathbf{e}_2$ is one, and every other vertex has valence two. Moreover, there is an obvious symmetry in the graph obtained by permuting the variables X_0 and X_1. Using this, it is straightforward to take each of the vectors $\mathbf{v}_{i,1}$ and compute that the maximal connected subgraph containing this vertex gives the generating set for E_i (this is an exercise left to the reader). Thus the E_i are pairwise orthogonal.

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8.6. **Computing the determinants.** In this last subsection, we compute the matrix and determinant of $\widetilde{\omega}_e$ restricted to each of the direct summands. Using this computation, we identify the kernel of $\widetilde{\omega}_e$. For $i = 0, \ldots, r-3$, we can form the matrix of $\widetilde{\omega}_e : E_i \to E_i^\vee$ with respect to the ordered basis $\mathbf{v}_{i,1}, \ldots, \mathbf{v}_{i,6}$ and the dual ordered basis of $E_i^\vee$. This is straightforward to compute and turns out to be:

$$A_i = \begin{bmatrix} 0 & c_0c_5 & -2c_0c_5^2 & 0 & 0 & 0 \\ -c_0c_5 & 0 & 0 & c_0c_5d_0 & 0 & 0 \\ 2c_0c_5^2 & 0 & 0 & 0 & c_0c_5d_5 & 0 \\ 0 & -c_0c_5d_0 & 0 & 0 & 0 & -2c_0^2c_5 \\ 0 & 0 & -c_0c_5d_5 & 0 & 0 & -c_0c_5 \\ 0 & 0 & 0 & 2c_0^2c_5 & c_0c_5 & 0 \end{bmatrix} \tag{82}$$

It is straightforward to compute that the Pfaffian of this matrix is $\mathrm{Pfaff}(A_i) = 2c_0^3c_5^3(c_0d_5 - c_5d_0)$ and thus the determinant is $\mathrm{Det}(A_i) = 4c_0^6c_5^6(c_0d_5 - c_5d_0)^2$. Since we are assuming that c_0, c_5 are nonzero and that (c_0, c_5) is linearly independent from (d_0, d_5), this determinant is nonzero.

For $i = r-2$, we can form the matrix of $\widetilde{\omega}_e : E_{r-2} \to E_{r-2}^\vee$ with respect to the ordered basis $\mathbf{v}_{r-2,1}, \ldots, \mathbf{v}_{r-2,8}$ and the dual ordered basis of $E_{r-2}^\vee$. This is straightforward to compute and turns out to be:

$$A_{r-2} = \begin{bmatrix} 0 & -c_5^2 & -2c_0c_5^2 & c_0c_5 & 0 & 0 & 0 & 0 \\ c_5^2 & 0 & 0 & 0 & 0 & -c_0c_5d_5 & 0 & 0 \\ 2c_0c_5^2 & 0 & 0 & 0 & c_0c_5d_5 & 0 & -c_0c_5d_0 & 0 \\ -c_0c_5 & 0 & 0 & 0 & 0 & c_0c_5d_0 & 0 & 0 \\ 0 & 0 & -c_0c_5d_5 & 0 & 0 & 0 & 0 & -c_0c_5 \\ 0 & c_0c_5d_5 & 0 & -c_0c_5d_0 & 0 & 0 & 0 & -2c_0^2c_5 \\ 0 & 0 & c_0c_5d_0 & 0 & 0 & 0 & 0 & c_0^2 \\ 0 & 0 & 0 & 0 & c_0c_5 & 2c_0^2c_5 & -c_0^2 & 0 \end{bmatrix} \tag{83}$$

It is straightforward to compute that the Pfaffian of this matrix is $\text{Pfaff}(A_{r-2}) = c_0^3c_5^3(c_0d_5 - c_5d_0)^2$ and thus the determinant is $\text{Det}(A_{r-2}) = c_0^6c_5^6(c_0d_5 - c_5d_0)^4$. Since we are assuming that c_0, c_5 are nonzero and that (c_0, c_5) is linearly independent from (d_0, d_5), this determinant is nonzero.

For $i = r-1$, we can form the matrix of $\widetilde{\omega}_e : E_{r-1} \to E_{r-1}^\vee$ with respect to the ordered basis $\mathbf{v}_{r-1,1}, \mathbf{v}_{r-1,2}$ and the dual ordered basis of $E_r^\vee$. This is straightforward to compute and turns out to be:

$$A_{r-1} = \begin{bmatrix} 0 & c_0c_5 \\ -c_0c_5 & 0 \end{bmatrix} \tag{84}$$

Visibly the Pfaffian of this matrix is c_0c_5 and the determinant is $c_0^2c_5^2$. Since we are assuming that c_0, c_5 are nonzero, this determinant is nonzero.

For $i = r$, we can form the matrix of $\widetilde{\omega}_e : E_r \to E_r^\vee$ with respect to the ordered basis $\mathbf{v}_{r,1}, \mathbf{v}_{r,2}, \mathbf{v}_{r,3}$ and the dual ordered basis of $E_r^\vee$. This is straightforward to compute and turns out to be:

$$A_r = \begin{bmatrix} 0 & -c_0c_5d_5 & 0 \\ c_0c_5d_5 & 0 & -c_0c_5d_0 \\ 0 & c_0c_5d_0 & 0 \end{bmatrix} \tag{85}$$

This matrix is singular and the kernel contains the vector $\mathbf{w} = d_0\mathbf{v}_{r,1} + d_5\mathbf{v}_{r,3}$, i.e. $(d_0X_0^{2r} + d_5X_1^{2r})X_0X_1\mathbf{e}_2$. So this vector is in the kernel of $\widetilde{\omega}_e$. Consider the quotient vector space $V' = H^0(\mathbb{P}^1, N_{C/X})/\mathbb{C}\{\mathbf{w}\}$. There is an induced alternating bilinear pairing $\widetilde{\omega}_e'$ on V'. Since $w' \in E_r$, there is an induced direct sum decomposition $V' = \bigoplus_{i=0}^r E_i'$ by pairwise orthogonal subspaces where for $i = 0, \ldots, r-1$ the quotient map $E_i \to E_i'$ is an isomorphism. And E_r' has as basis the images of the vectors $\mathbf{v}_{r,1}, \mathbf{v}_{r,2}$ provided $d_5 \neq 0$ and has as basis the images of the vectors $\mathbf{v}_{r,2}\mathbf{v}_{r,3}$ provided $d_0 \neq 0$.

In case $d_5 \neq 0$ we can form the matrix of $\widetilde{\omega}_e' : E_r' \to (E_r')^\vee$ with respect to the ordered basis $\mathbf{v}_{r,1}', \mathbf{v}_{r,2}'$ and the dual ordered basis of $(E_r')^\vee$. This turns out to be:

$$A_r' = \begin{bmatrix} 0 & -c_0c_5d_5 \\ c_0c_5d_5 & 0 \end{bmatrix} \tag{86}$$

Visibly the Pfaffian of this matrix is $c_0c_5d_5$ and the determinant is $c_0^2c_5^2d_5^2$. Since we are assuming that c_0, c_5, d_5 are nonzero, this determinant is nonzero.

The other case is that $d_0 \neq 0$. In this case we can form the matrix of $\widetilde{\omega}_e' : E_r' \to (E_r')^\vee$ with respect to the ordered basis $\mathbf{v}_{r,2}', \mathbf{v}_{r,3}'$ and the dual ordered basis of

$(E'_r)^\vee$. This turns out to be:

$$A'_r = \begin{bmatrix} 0 & -c_0c_5d_0 \\ c_0c_5d_0 & 0 \end{bmatrix} \tag{87}$$

Visibly the Pfaffian of this matrix is $c_0c_5d_0$ and the determinant is $c_0^2c_5^2d_0^2$. Since we are assuming that c_0, c_5, d_0 are nonzero, this determinant is nonzero. Thus we conclude in both cases that the form $\widetilde{\omega}'_e$ on E'_r is nondegenerate. Combined with the computations from above, we conclude that the kernel of $\widetilde{\omega}_e$ is precisely the span $\mathbb{C}\{(d_0X_0^{2r} + d_5X_1^{2r})X_0X_1\mathbf{e}_2\}$. So the kernel of $\widetilde{\omega}_e$ is one-dimensional. This proves Theorem 1.2 in case $e = 2r$ is an even integer with $e \geq 6$.

9. Comments and Questions

concl

Let us mention a few generalizations of Theorem 1.2. The same method and the same special pair $f : \mathbb{P}^1 \to \mathbb{P}^5$ and $X \subset \mathbb{P}^5$ together with the points $[0:1], [1:0] \in \mathbb{P}^1$ can be used to show the following

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Theorem 9.1. *Let $X \subset \mathbb{P}^5$ a smooth cubic hypersurface, let $\overline{\mathcal{M}}_{0,n}(X, e)$ denote the Kontsevich moduli space of pointed stable maps to X of arithmetic genus 0 and degree e, and let $\overline{M}_{e,n}$ be a nonsingular projective model of the coarse moduli space. There is a canonical section $\omega_e \in H^0(\overline{M}_{e,n}, \Omega^2_{\overline{M}_{e,n}})$ with the following property:*

(a) In case $n = 1$, e is odd, $e \geq 5$. If X is general, and p a general point of $\overline{M}_{e,1}$, the restriction of ω_e to the tangent space at p of the fiber of the evaluation map $ev : \overline{M}_{e,1} \to X$ has a 1-dimensional kernel.

(b) In case $n = 1$, e is even, $e \geq 6$. If X is general, and p a general point of $\overline{M}_{e,1}$, the restriction of ω_e to the tangent space at p of the fiber of the evaluation map $ev : \overline{M}_{e,1} \to X$ is nondegenerate. Therefore the general fiber of ev has Kodaira dimension ≥ 0 and, in particular, is non-uniruled.

(c) In case $n = 2$, e is odd, $e \geq 5$. If X is general and p a general point of $\overline{M}_{e,2}$, the restriction of ω_e to the tangent space at p of the fiber of the evaluation map $(ev_1, ev_2) : \overline{M}_{e,2} \to X \times X$ is nondegenerate. Therefore the general fiber of (ev_1, ev_2) has Kodaira dimension ≥ 0 and, in particular, is non-uniruled.

(d) In case $n = 2$, e is even, $e \geq 6$. If X is general and p a general point of $\overline{M}_{e,2}$ the restriction of ω_e to the tangent space at p of the fiber of the evaluation map $(ev_1, ev_2) : \overline{M}_{e,2} \to X \times X$ has a 1-dimensional kernel.

Proof. We will sketch the proof, but leave most of the details to the reader. The technique is almost identical to the proof of Theorem 1.2 and is roughly as follows: For parts (a) and (b), one considers the special pairs $([X], [C])$ used in Section 7 and Section 8 respectively, except one also specifies d_0, d_5 are both nonzero. For the marked point on C, one uses either $f([0:1])$ (or $f([1:0])$). Then the tangent space to the fiber of the evaluation map is identified with the sections of $H^0(\mathbb{P}^1, N_{C/X})$ which vanish at $[0:1]$ (or $[1:0]$). And the form ω_e on this subspace is just the form computed in Section 8 and Section 7. In particular, since the space of sections vanishing at $[0:1]$ is generated by standard monomial basis vectors of $H^0(\mathbb{P}^1, N_{C/X})$, our direct sum decomposition into pairwise orthogonal subspaces yields a direct sum decomposition of the space of sections vanishing at $[0:1]$.

In the odd case, one can simply identify the kernel – which is generated by $c_5\mathbf{v}_{0,2} + \mathbf{v}_{0,3} + d_5\mathbf{v}_{0,6}$. It is straightforward to check that the form on the quotient space is nondegenerate. In the even case, actually the kernel is nontrivial – it is generated by $d_0\mathbf{v}_{r,1} + d_5\mathbf{v}_{r,3}$ and $\mathbf{v}_{r-2,1} + 2c_5\mathbf{v}_{r-2,3} - d_5\mathbf{v}_{r-2,6}$. However, under a nontrivial first-order deformation of the pointed curve which doesn't change the map $f : \mathbb{P}^1 \to X$, but only moves the point $[0 : 1]$ on $\mathbb{P}^1$, the kernel becomes trivial (this is a simple deformation theory exercise).

Parts (c) and (d) are the same. In the odd case, the kernel is trivial. In the even case, the kernel is generated by $d_0\mathbf{v}_{r,1} + d_5\mathbf{v}_{r,3}$ (no deformation theory is needed). □

Of course, a natural question at this point is the following.

lastq

Question 9.2. What can we say about the Kodaira dimension/uniruledness when the form ω_e does have a kernel? For example, when e is even $e \geq 6$, is M_e uniruled?

We are convinced that in these cases M_e is non-uniruled, but we don't have a proof when e is even and $e \geq 8$. However, in the special case $e = 6$, we can give an answer based on an *ad hoc* analysis.

lastp

Proposition 9.3. *Let $X \subset \mathbb{P}^5$ be a general cubic fourfold. Then M_6 is non-uniruled. More precisely, there exists a rational transformation $f : M_6 \to Hilb_X^{6t}$ whose general fiber is a genus 1 curve which is a leaf of the distribution $Ker(\omega_e)$.*

We only give a very rough sketch of the proof. First we will give a rapid overview of the proof that M_6 is non-uniruled and then fill in some of the details. The method of proof is very similar to that used in [14], but instead of using residual curves in an intersection of X with a cubic scroll, we will use residual curves in an intersection of X with a quartic scroll. For a general nondegenerate, rational, degree 6 curve $C \subset \mathbb{P}^5$, there is a unique quartic scroll $\Sigma \subset \mathbb{P}^5$ which contains C. If X is general, then X contains no quartic scrolls (although special smooth cubic fourfolds can contain a quartic scroll [15, Section 4.1.3]). The intersection $\Sigma \cap X$ is a degree 12 curve in Σ which is a local complete intersection (in particular it is Gorenstein) and contains C as a subcurve of degree 6. Using Gorenstein liaison, the residual curve C' to C in Σ is a degree 6 curve of arithmetic genus 1, which will be a smooth, connected curve for C general. Thus we have a rational transformation from M_6 to the open subset U of the Chow variety/Hilbert scheme parametrizing degree 6 curves in X of arithmetic genus 1 by $[C] \mapsto [C']$. It is not hard to show that the fiber of this rational transformation containing $[C]$ is actually isomorphic to $\text{Pic}^2(C')$, i.e. it is a connected, smooth curve of genus 1 (actually it will only be a dense open subset since we are working on the non-complete variety M_6).

Of course on M_6 we have the 2-form ω_6 constructed in Section 4. Now on U we can define a 2-form by the same process as in Section 4 corresponding to the family of degree 6 curves of arithmetic genus 1. On the domain of definition of the rational transformation $M_6 \to U$, we can form the pullback of the 2-form on U; let us call this pullback 2-form ω'. Also over a dense open set of M_6, the curve $\Sigma \cap X$ is a connected, reduced at-worst-nodal curve and we can again use the technique from Section 4 to construct a 2-form ω'' corresponding to this family of curves. The relation between all these forms is $\omega_6 + \omega' = \omega''$ on the open, dense locus where all three are defined.

On the other hand, we have a *unirational* space $W \subset \mathrm{Hilb}^{(2t+1)(t+1)}(\mathbb{P}^5)$ parametrizing all smooth, nondegenerate quartic scrolls in $\Sigma \subset \mathbb{P}^5$ (in fact this is a homogeneous space for PGL_6 since any two such scrolls are projectively equivalent). Over a dense open subset of W we can construct a 2-form as in Section 4 corresponding to the family of curves whose fiber over $[\Sigma]$ is $\Sigma \cap X$. And ω'' is the pullback of this 2-form by the obvious rational transformation $M_6 \to W$. But since W is unirational, it does not support any nonzero 2-form. In other words, $\omega'' = 0$. So we have $\omega_6 = -\omega'$. In particular, the kernel of ω_6 coincides with the kernel of ω'. Since ω' is a pullback by the rational transformation $M_6 \to U$, in particular the tangent space of the fiber of this rational transformation is contained in ω_6. We know the fiber is one-dimensional. But by Theorem 1.2, we also know that the kernel of ω_6 is one-dimensional. Thus we conclude that the kernel of ω_6 at a general point of M_6 is precisely the tangent space to the fiber of $M_6 \to U$. In other words, the foliation determined by the kernel of ω_6 is algebraically integrable on a dense (Zariski) open subset of M_6, the leaf space is (birationally) an open subset U of the Hilbert scheme of smooth, degree 6 curves in X of genus 1, and the projection to the leaf space is (birationally) the rational transformation $M_6 \to U$.

From this it follows that U has Kodaira dimension ≥ 0, in particular it is non uniruled. As we have seen, $M_6 \to U$ is (birationally) a fibration whose fibers are smooth curves of genus 1. Since both the target U of $M_6 \to U$ and the fibers are non-uniruled, we conclude that M_6 is also non-uniruled. Moreover, it seems certain that one can use [16] to show that M_6 has Kodaira dimension ≥ 0.

There are lots of missing details in this argument: Why is the fiber of $M_6 \to U$ at $[C]$ isomorphic to $\mathrm{Pic}^2(C')$? Why does a general nondegenerate, rational degree 6 curve lie on a unique quartic scroll (or any quartic scroll for that matter)? How does the construction of Section 4 behave with respect to Gorenstein liaison, i.e. what is the justfication of the identity $\omega_6 + \omega' = \omega''$? What is the rigorous argument that ω'' is zero – to conclude a 2-form on a unirational variety is zero, we must show that it extends to a regular 2-form on some nonsingular compactification of that unirational variety and W was not compact?

Let's briefly deal with these issues in reverse order. First of all, choose any nonsingular compactification $\overline{W}$ of W. The association $[\Sigma] \mapsto \Sigma \cap X \subset X$ defines a rational map from $\overline{W}$ to the coarse moduli space $\overline{M}_{10,0}(X, 12)$ of the Kontsevich moduli stack $\overline{\mathcal{M}}_{10,0}(X, 12)$ parametrizing stable maps to X from a connected at-worst-nodal curve of arithmetic genus 10 and degree 12. By Section 4 and Lemma 3.5, we can construct a 2-form on $\overline{M}_{10,0}(X, 12)$ corresponding to the universal family of stable maps over $\overline{\mathcal{M}}_{10,0}(X, 12)$. The 2-form on W is just the pullback of the 2-form on $\overline{M}_{10,0}(X, 12)$. Since $\overline{W}$ is smooth and $\overline{M}_{10,0}(X, 12)$ is proper, the rational map $\overline{W} \to \overline{M}_{10,0}(X, 12)$ is defined on an open set whose complement has codimension at least 2. Therefore the pullback of this 2-form extends to a regular 2-form on all of $\overline{W}$, which shows that this 2-form is identically zero since $\overline{W}$ is unirational.

The Gorenstein liaison property is the following: Suppose B is a scheme, $p : \mathcal{C} \to B$ and $f : \mathcal{C} \to X$ is a flat family of at-worst-nodal curves over B together with a family of maps to X, and suppose that $\mathcal{C}_1 \subset \mathcal{C}$ is a codimension 0 subscheme which is itself Gorenstein and such that $p_1 : \mathcal{C}_1 \to B$ is a flat family of at-worst-nodal curves over B. It follows from a standard argument (c.f. [12, Corollary 2.7]) that the residual scheme $\mathcal{C}_2 \subset \mathcal{C}$ to $\mathcal{C}_1 \subset \mathcal{C}$ is also a flat family of at-worst-nodal curves over B. By Section 4, we can form the 2-forms ω, ω_1 and ω_2 on B corresponding to $\mathcal{C}$, to $\mathcal{C}_1$ and to $\mathcal{C}_2$ respectively. The liaison property is that $\omega_1 + \omega_2 = \omega$. Going through the construction in Section 4, this follows easily from the analogous result for the trace maps $Rp_*\omega_{\mathcal{C}/B} \to \mathcal{O}_B$, $Rp_*\omega_{\mathcal{C}_1/B} \to \mathcal{O}_B$ and $Rp_*\omega_{\mathcal{C}_2/B} \to \mathcal{O}_B$.

Finally, why is it true that every smooth curve $C \subset \mathbb{P}^5$ of degree 6 which is nondegenerate lies on a unique quartic scroll Σ, and why is Σ nonsingular? This must have been known classically, and is undoubtedly already somewhere in the literature. Since the proof is so easy, we will simply rederive the result here. First of all, the result is suggested by a parameter count. A quartic scroll $\Sigma \subset \mathbb{P}^5$ is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by the complete linear system of $\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$, so the space of smooth quartic scrolls in $\mathbb{P}^5$ has dimension $\dim(\mathrm{PGL}_6) - \dim(\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)) = 35 - 6 = 29$. The degree 6 rational curves on Σ are simply the images of curves in the projective linear system of $\mathcal{O}_{\mathbb{P}^1}(4) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$, which has dimension 9. So we obtain a 38-dimensional family of pairs $([\Sigma], [C])$ where $\Sigma \subset \mathbb{P}^5$ is a smooth quartic scroll and $C \subset \Sigma$ is a smooth degree 6 rational curve. On the other hand, the dimension of the space of all smooth, degree 6 rational curves in $\mathbb{P}^5$ is easily computed to be $-K_{\mathbb{P}^5}.[C] + (\dim(\mathbb{P}^5) - 3) = (6).(6) + (5-3) = 38$. For a pair $([\Sigma], [C])$ it is easy to see that the set of all 4-secant 2-planes to C exactly sweep out the cubic Segré threefold $Y \subset \mathbb{P}^5$ associated to Σ, i.e. the Segré threefold swept out by 2-planes which intersect Σ in a conic curve. The Segré threefold Y is the image of an embedding $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^5$ by the complete linear system of $\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. The quartic scroll $\Sigma \subset Y$ is the image under this embedding of a subvariety $D \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1$ where $D \subset \mathbb{P}^2$ is a smooth conic. But then the curve $C \subset Y$ is the image a curve in $\mathbb{P}^2 \times \mathbb{P}^1$, which we will also call C, such that the projection map $\pi_1 : C \to \mathbb{P}^2$ is a 2-to-1 map to the conic D. Thus we can reconstruct D and so also Σ just from the pair $C \subset Y$. But as Y is the union of all 4-secant 2-planes to C, we can also reconstruct Y just from C. Therefore we can uniquely reconstruct Σ from C which shows that there is exactly one quartic scroll Σ giving rise to C. So the map from our 38-dimensional variety of pairs $([\Sigma], [C])$ to the (open subscheme of the) 38-dimensional Hilbert scheme parametrizing smooth, nondegenerate, rational degree 6 curves $C \subset \mathbb{P}^5$ is an injective map and therefore dominates the Hilbert scheme. So for a general, nondegenerate, rational degree 6 curve $C \subset \mathbb{P}^5$, we conclude there is a unique smooth quartic scroll Σ containing C.

Finally, why is the fiber of $M_6 \to U$ at a point $[C]$ isomorphic to $\mathrm{Pic}^2(C')$ where $C \cup C' = \Sigma \cap X$? The image is just $[C']$, and the fiber over $[C']$ is the set of all rational degree 6 curves such that C is residual to C' in an intersection $\Sigma \cap X$ where Σ is a quartic scroll containing C'. Although it is a bit tricky to understand quartic scrolls containing a degree 6 rational curve, it is quite a bit simpler to understand quartic scrolls containing a degree 6 curve C' of arithmetic genus 1. For each scroll Σ such that $C' \subset \Sigma$, we have that Σ is the image of an embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ by the complete linear system of $\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. The curve C' is the image of a curve in the complete linear system of $\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(2)$. In particular, the pullback for $\mathcal{O}_{\mathbb{P}^1}(1)$ by the second projection $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1$ gives an invertible sheaf which restricts on C' to an element $\mathcal{L} \in \mathrm{Pic}^2(C')$. Given $\mathcal{L}$, we can uniquely recover Σ as follows: for each divisor $D \subset C'$ in the complete linear system form the line $\mathrm{span}(D)$ in $\mathbb{P}^5$. Then Σ is the union of the lines $\mathrm{span}(D)$ as D varies among all divisors in the complete linear system of $\mathcal{L}$. This establishes a one-to-one correspondence between the quartic scrolls in $\mathbb{P}^5$ which contain C' and the invertible sheaves $\mathcal{L} \in \mathrm{Pic}^2(C')$.

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