Rational Curves On Hypersurfaces In Projective n-Space

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Abstract

We prove that for suitable dimension n and degree d, a *general* complex hypersurface $X \subset \mathbb{P}^n$ of degree d has the property that for each integer e the space $R_e(X)$ of degree e rational curves on X is an integral, local complete intersection scheme of dimension $(n+1-d)e+(n-4)$.

We also prove that for *any* smooth cubic hypersurface $X \subset \mathbb{P}^4$, for each integer e the space $R_e(X)$ is an integral, local complete intersection scheme of dimension $2e$.

The techniques used in the proof include:

(1) Classical results about lines on hypersurfaces including a new result about flatness of the projection map from the space of *pointed lines*.

(2) The Kontsevich moduli space of stable maps, $\overline{\mathcal{M}}_{0,r}(X,e)$. In particular we use the deformation theory of stable maps, the decomposition of $\overline{\mathcal{M}}_{0,r}(X,e)$ described in [Behrend-Manin96], and the fact that the coarse moduli space is a projective scheme.

(3) A version of Mori's bend-and-break lemma.

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Contents

CHAPTER 1

Rational Curves on Hypersurfaces

1.1. Introduction

All schemes we consider will be $\mathbb{C}$-schemes and all morphisms will be morphisms of $\mathbb{C}$-schemes. All (absolute) products will be over $\mathbb{C}$.

For a projective scheme X over $\mathbb{C}$ along with an ample line bundle L we define $R_e(X)$ to be the open subscheme of the Hilbert scheme $\mathrm{Hilb}^{et+1}(X/k)$ which parametrizes smooth rational curves of degree e lying in X.

THEOREM 1. *Let $n \geq 6$ be an integer and let d be an integer such that $1 \leq d \leq \frac{n+1}{2}$. For a general hypersurface $X \subset \mathbb{P}^n$ of degree d and for every integer $e \geq 1$, the scheme $R_e(X)$ is an integral local complete intersection scheme of dimension $(n+1-d)e + (n-4)$.*

The idea of the proof is as follows. There is an embedding of $R_e(X)$ into the smooth scheme $R_e(\mathbb{P}^n)$. Denote by $\pi : U_e(\mathbb{P}^n) \to R_e(\mathbb{P}^n)$ the universal family of rational curves in $\mathbb{P}^n$ and by $\rho : U_e(\mathbb{P}^n) \to \mathbb{P}^n$ the *evaluation* morphism. Then $R_e(X)$ is the scheme of zeroes of a section of the locally free sheaf $\pi_*\rho^*\mathcal{O}_{\mathbb{P}^n}(d)$. Thus to prove that $R_e(X)$ is a local complete intersection scheme, it suffices to prove that the codimension of $R_e(X)$ in $R_e(\mathbb{P}^n)$ equals the rank of $\pi_*\rho^*\mathcal{O}_{\mathbb{P}^n}(d)$.

The remainder of the proof is a "deformation and specialization" argument: we embed the non-proper scheme $R_e(X)$ as an open subscheme of a proper scheme which is still modular, i.e. we choose a "modular compactification". Then we show that any point in $R_e(X)$ specializes to a point in the "boundary" of the compactification. We use deformation theory to study the irreducible components of the boundary of the compactification. In particular we show that a general point of each irreducible component of the boundary is a unibranch point of the compactification whose local ring is reduced and has the expected dimension. This reduces the proof to a combinatorial argument.

1.2. Detailed Summary

In the next few paragraphs we will give a detailed summary of the proof. Our compactification consists of the embedding of $R_e(X)$ as an open subscheme in the Kontsevich moduli space $\overline{\mathrm{M}}_{0,0}(X,e)$ parametrizing stable maps to X. We partition $\overline{\mathrm{M}}_{0,0}(X,e)$ into locally closed subsets according to the "combinatorial type" of the stable map. In particular, the image of $R_e(X)$ is a dense open subset of a component of this partition. We identify certain *basic components* as those components of the partition parametrizing stable maps such that each irreducible component of the domain curve is mapped to a line in X. We prove a new result about lines on X. We define the incidence correspondence of *pointed lines* in X:

$$F_{0,1}(X) = \{(p,l)|p \text{ a point}, l \text{ a line}, p \in l \subset X\}.$$

We prove that for a general hypersurface $X \subset \mathbb{P}^n$ of degree $d \leq n-3$, the projection morphism $F_{0,1}(X) \to X$ is flat of relative dimension $n-d-1$ From this it easily follows that each basic component B is an integral scheme whose general point is a unibranch point of $\overline{\mathrm{M}}_{0,0}(X,e)$ at which $\overline{\mathrm{M}}_{0,0}(X,e)$ is reduced of dimension $(n+1-d)e+(n-4)$. Thus for each basic component B there is a unique irreducible component $M(B)$ of $\overline{\mathrm{M}}_{0,0}(X,e)$ which contains B, and $M(B)$ is reduced and has dimension $(n+1-d)e+(n-4)$.

Next we consider *almost basic components*: the components of our partition which parametrize stable maps such that one irreducible component of the domain maps to a conic in X, and every other irreducible component is mapped to a line in X. These components arise when we consider smoothings of a node on one of the basic stable maps. By showing the almost basic components are irreducible, we prove that all of the irreducible components $M(B)$ for the different basic components B are actually equal. So we produce an irreducible component M of $\overline{\mathrm{M}}_{0,0}(X,e)$ with the property that if N is any irreducible component of $\overline{\mathrm{M}}_{0,0}(X,e)$ which contains a basic component B, then $N = M$.

Finally we are reduced to proving that every irreducible component N of $\overline{\mathrm{M}}_{0,0}(X,e)$ contains a basic component B. To do this we identify a family of *nice, simple components* of our partition. This family contains the component corresponding to $R_e(X)$ and all the basic components. Using the bend-and-break lemma of Mori we prove the following theorem: for any nice, simple component D which isn't basic, and for each irreducible component D_i of this component, there is a nice, simple component E and an irreducible component E_j of E which is contained in $\overline{D_i}$ such that E_j has codimension 1 in the closure of D_i and such that the expected dimension of E_j is one less than the expected dimension of D_i. We conclude by induction on the expected dimension that D_i has the expected dimension and contains a basic component in its closure. Thus every irreducible component of $\overline{\mathrm{M}}_{0,0}(X,e)$ contains a basic component in its closure, i.e. M is the unique irreducible component of $\overline{\mathrm{M}}_{0,0}(X,e)$. This shows that M is an integral scheme of dimension $(n+1-d)e+(n-4)$.

1.3. Notation

In order to reduce possible confusion in performing constructions on $\mathbb{P}^n$, we shall sometimes choose a k-vector space V of dimension $n+1$ and work with the projective space $\mathbb{P}V$ instead of $\mathbb{P}^n$ (for example, this convention allows us to distinguish a projective space $\mathbb{P}V$ from its dual projective space $\mathbb{P}V^\vee$).

If X is a scheme, if E is a locally free sheaf on X of rank m, and if r is an integer $0 \leq r \leq m$, we shall denote by $\mathrm{G}_X(r,E)$ the Grassmannian bundle over X parametrizing rank r subspaces of the fibers of E. Given a vector space V over $\mathbb{C}$, we will also denote $\mathrm{G}_{\mathrm{Spec}\,\mathbb{C}}(r,V)$ by $\mathrm{G}(r,V)$.

Our notation regarding stacks is taken from [LM-B]. In particular we follow the practice of treating stacks as fibered categories for which there is always some distinguished functor of base change. Recall that a scheme X also determines a Yoneda functor from the category of schemes to the category of sets as well as a groupoid in stacks (which is just the Yoneda functor after we identify the category of sets with the category of small discrete groupoids). We will denote by X the scheme of X as well as the Yoneda functor and stack determined by X.

While we are discussing stacks, we shall need the following simple fact about stacks.

LEMMA 2. *Let S be a scheme and let $f : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of S-stacks (i.e. stacks in groupoids over S). Suppose that $\mathcal{Y}$ is an algebraic stack (resp. Deligne-Mumford stack). Suppose also that for every affine S-scheme U and every 1-morphism $g : U \to \mathcal{Y}$ the fiber product $U \times_{g,\mathcal{Y},f} \mathcal{X}$ is an algebraic stack (resp. Deligne-Mumford stack). Then $\mathcal{X}$ is an algebraic stack (resp. Deligne-Mumford stack).*

PROOF. We need to prove that the diagonal morphism

$$\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$

is representable, separated and quasi-compact and we need to produce an affine S-scheme X and a 1-morphism $X \to \mathcal{X}$ which is surjective and smooth (resp. surjective and étale).

Since $\mathcal{Y}$ is an algebraic stack (resp. Deligne-Mumford stack), we can find an affine S-scheme Y and a surjective, smooth (resp. surjective, étale) 1-morphism $p : Y \to \mathcal{Y}$. The base-change morphism $p' : Y \times_{p,\mathcal{Y},f} \mathcal{X} \to \mathcal{X}$ is also surjective and smooth (resp. surjective and étale). Moreover the fiber product $Y \times_{p,\mathcal{Y},f} \mathcal{X}$ is an algebraic stack (resp. Deligne-Mumford stack). Therefore there exists an affine S-scheme X and a surjective, smooth (resp. surjective, étale) 1-morphism $q : X \to Y \times_{p,\mathcal{Y},f} \mathcal{X}$. Since the composition of surjective, smooth (resp. surjective, étale) morphisms is again surjective and smooth (resp. surjective and étale), we conclude that $p' \circ q : X \to \mathcal{X}$ is surjective and smooth (resp. surjective and étale).

The diagonal morphism $\Delta_{\mathcal{X}}$ factors as a composition:

$$\mathcal{X} \xrightarrow{\Delta_f} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \xrightarrow{i} \mathcal{X} \times_S \mathcal{X}.$$

Also we have a 2-Cartesian diagram:

$$\begin{array}{ccc} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} & \xrightarrow{i} & \mathcal{X} \times_S \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle f \times f.} \\ \mathcal{Y} & \xrightarrow{\Delta_{\mathcal{Y}}} & \mathcal{Y} \times_S \mathcal{Y} \end{array}$$

Since $\Delta_{\mathcal{Y}}$ is representable, separated and quasi-compact, we conclude that i is representable, separated and quasi-compact by base-change. So we are reduced to proving that Δ_f is representable, separated and quasi-compact.

Suppose that T is an affine scheme and $g : T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is a 1-morphism. Define $h : T \to \mathcal{Y}$ to be the composition of g with either of the two canonical morphisms

$$c : \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} :\to \mathcal{Y}.$$

Define $\mathcal{X}_T = T \times_{h,\mathcal{Y},f} \mathcal{X}$. Then $\mathcal{X}_T$ is an algebraic stack (resp. Deligne-Mumford stack) so that the diagonal $\Delta_{\mathcal{X}_T} : \mathcal{X}_T \to \mathcal{X}_T \times_T \mathcal{X}_T$ is representable, separated and quasi-compact. Notice that $\mathcal{X}_T \times_T \mathcal{X}_T$ is canonically isomorphic to $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \times_{c,\mathcal{Y},h} T$. Therefore the pair $g : T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and $\mathrm{id}_T : T \to T$ induces a 1-morphism $k : T \to \mathcal{X}_T \times_T \mathcal{X}_T$. And we have a 2-Cartesian diagram:

$$\begin{array}{ccc} \mathcal{X} \times_{\Delta, \mathcal{X}\times_{\mathcal{Y}}\mathcal{X}, g} T & \xrightarrow{\mathrm{pr}_2} & T \\ \downarrow & & \downarrow k \\ \mathcal{X} \times_{f, \mathcal{Y}, h} T & \xrightarrow{\Delta\times \mathrm{id}_T} & (\mathcal{X}\times_{\mathcal{Y}} \mathcal{X}) \times_{c, \mathcal{Y}, h} T \end{array} .$$

By base-change pr_2 is representable, separated and quasi-compact (i.e. $\mathcal{X}\times_{\Delta,\mathcal{X}\times_{\mathcal{Y}}\mathcal{X},f} T$ is an algebraic space which is separated and quasi-compact). This proves that $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable, separated and quasi-compact. Therefore $\mathcal{X}$ is an algebraic stack (resp. Deligne-Mumford stack). $\square$

1.4. Lines on Hypersurfaces

The projective variety $\mathbb{P}\mathrm{Sym}^d V^\vee$ can be thought of as parametrizing homogeneous polynomials Φ of degree d on V, and can also be thought of as parametrizing degree d hypersurfaces X in $\mathbb{P}V$ via the association

$$[\Phi] \leftrightarrow X = \{[v] \in \mathbb{P}V | \Phi(v) = 0\}.$$

We will often talk of the hypersurface $X \subset \mathbb{P}V$ as being a point of $\mathbb{P}\mathrm{Sym}^d V^\vee$.

Recall our notation that $\mathrm{G}(2, V)$ is the Grassmannian parametrizing 2 dimensional subspaces of V, or equivalently lines $L \subset \mathbb{P}V$. Given a degree d hypersurface $X \subset \mathbb{P}V$ we denote by $F_1(X)$ the subscheme of $\mathrm{G}(2, V)$ which parametrizes lines in X, i.e. $F_1(X)$ is the *Fano scheme* of lines in X. We will also denote by $\mathrm{G}((1, 2), V)$ the projective bundle over $\mathrm{G}(2, V)$ associated to the universal rank 2 locally free subsheaf S of $V \otimes_{\mathbb{C}} \mathcal{O}$. We can think of this as the space of partial flags $V_1 \subset V_2 \subset V$ with $\dim(V_1) = 1,\ \dim(V_2) = 2$. Alternatively we can think of this as the space of pointed lines in $\mathbb{P}^n$, i.e.

$$\mathrm{G}((1, 2), V) = \{(p, L) \in \mathbb{P}V \times \mathrm{G}(2, V) | p \in L\}.$$

We denote the canonical projection morphisms by

$$\pi_1 : \mathrm{G}((1, 2), V) \to \mathbb{P}V,\ \pi_1(p, L) = p$$

$$\pi_2 : \mathrm{G}((1, 2), V) \to \mathrm{G}(2, V),\ \pi_2(p, L) = L.$$

We denote by $F_{0,1}(X)$ the inverse image of $F_1(X)$ under π_1, i.e. $F_{0,1}(X)$ is the space of pointed lines in X

$$F_{0,1}(X) = \{(p, L) \in \mathbb{P}V \times \mathrm{G}(2, V) | p \in L \subset X\}.$$

We shall prove the following theorem:

THEOREM 3. *Let d be an integer with $1 \leq d \leq n-3$. For a general hypersurface $X \in \mathbb{P}Sym^d V^\vee$ the morphism $F_{0,1}(X) \to X$ is flat of relative dimension $n-d-1$.*

Let us denote the universal family of degree d hypersurfaces in $\mathbb{P}V$ as:

$$\mathcal{X} \subset \mathbb{P}\mathrm{Sym}^d V^\vee \times \mathbb{P}V$$

Let us denote by

$$F_1(\mathcal{X}) \subset \mathbb{P}\mathrm{Sym}^d V^\vee \times \mathrm{G}(2,V)$$

the family of Fano schemes $F_1(X)$ of the fibers X of the universal family $\mathcal{X} \to \mathbb{P}\mathrm{Sym}^d V^\vee$. Let us denote by

$$F_{0,1}(\mathcal{X}) \subset \mathbb{P}\mathrm{Sym}^d V^\vee \times \mathrm{G}((1,2),V)$$

the family of schemes $F_{0,1}(X)$ of the fibers X of $\mathcal{X} \to \mathbb{P}\mathrm{Sym}^d V^\vee$. Let us denote by $\mathcal{O}_F$ the pushforward to $\mathbb{P}\mathrm{Sym}^d V^\vee \times \mathrm{G}((1,2),V)$ of the structure sheaf of $F_{0,1}(\mathcal{X})$. Let $\mathcal{O}$ simply denote the structure sheaf of $\mathbb{P}\mathrm{Sym}^d V^\vee \times \mathrm{G}((1,2),V)$. Let S_2 denote the pullback to $\mathbb{P}\mathrm{Sym}^d V^\vee \times \mathrm{G}((1,2),V)$ of the universal rank 2 subbundle of $V \otimes_{\mathbb{C}} \mathcal{O}$. And let $\mathcal{O}(-1)$ denote the pullback to $\mathbb{P}\mathrm{Sym}^d V^\vee \times \mathrm{G}((1,2),V)$ of the universal rank 1 subbundle of $\mathrm{Sym}^d V^\vee$ on $\mathbb{P}\mathrm{Sym}^d V^\vee$. Having set our notation, notice that there is a partial resolution of coherent sheaves:

$$\mathrm{Sym}^d(S_2) \otimes_{\mathcal{O}} \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_F \longrightarrow 0.$$

where the map

$$\mathrm{Sym}^d(S_2) \otimes_{\mathcal{O}} \mathcal{O}(-1) \longrightarrow \mathcal{O}$$

is adjoint to the unique map of coherent sheaves $\mathcal{O}(-1) \to \mathrm{Sym}^d(S_2)^\vee$ obtained by composing the canonical maps

$$\mathcal{O}(-1) \longrightarrow \mathrm{Sym}^d V^\vee \otimes_{\mathbb{C}} \mathcal{O} \longrightarrow \mathrm{Sym}^d(S_2)^\vee.$$

By the Hauptidealsatz every fiber of $\pi_1 : F_{0,1}(\mathcal{X}) \to \mathcal{X}$ has dimension at least $n-d-1$. It follows by upper semicontinuity of the fiber dimension that the subset

$$\mathcal{U} = \{p \in \mathcal{X} | \dim(\pi_1^{-1}(p)) \leq n-d-1\}$$

is open. A priori $\mathcal{U}$ might contain points for which the fiber is empty (this is why $\mathcal{U}$ is defined by an inequality instead of an equality), but in fact π_1 is surjective. We can form the Koszul complex associated to the partial resolution above. It is a result of commutative algebra (theorem 17.4 (iii)(4) of [Matsumura86]) that over $\mathcal{U}$ the Koszul complex is exact. In particular the Hilbert polynomials of all fibers over $\mathcal{U}$ are equal. It follows that π_1 is flat over $\mathcal{U}$ (theorem II.9.9 of [Hartshorne77]). Let $\mathcal{Y}$ denote the reduced, closed subscheme of $\mathcal{X}$ which is the complement of $\mathcal{U}$. Then theorem 3 is equivalent to the statement that $\mathcal{Y}$ does not dominate $\mathbb{P}\mathrm{Sym}^d V^\vee$. Denote by e the codimension of $\mathcal{Y}$ in $\mathcal{X}$. Since the fiber dimension of $\mathcal{X} \to \mathbb{P}\mathrm{Sym}^d V^\vee$ is $n-1$, to prove that $\mathcal{Y}$ fails to dominate $\mathbb{P}\mathrm{Sym}^d V^\vee$, it suffices to prove that $e > n-1$.

There is another way of thinking of the scheme $\mathcal{X}$. On $\mathbb{P}V$ let $F^{1,d}$ be the locally free subsheaf of $\mathrm{Sym}^d V^\vee \times_{\mathbb{C}} \mathcal{O}$ which is the image of the multiplication map $\mathrm{Sym}^{d-1} V^\vee \otimes_{\mathbb{C}} Q^\vee \to \mathrm{Sym}^d V^\vee \times_{\mathbb{C}} \mathcal{O}$. Then the morphism $pr_2 : \mathcal{X} \to \mathbb{P}V$ is isomorphic to the projective bundle associated to $F^{1,d}$. To prove that $e > n-1$ it suffices to prove that for each point $x \in \mathbb{P}V$, the intersection $\mathcal{Y} \cap pr_2^{-1}(W) \cap pr_1^{-1}(x)$ has codimension greater than $n-1$ in $pr_1^{-1}(x)$. This is what we shall prove.

On $\mathbb{P}V$ let Q denote the locally free quotient sheaf of $V \otimes_{\mathbb{C}} \mathcal{O}$ by $\mathcal{O}_{\mathbb{P}V}(-1)$. The dual injection $Q^\vee \hookrightarrow V^\vee \otimes_{\mathbb{C}} \mathcal{O}$ can be considered as a filtration of $V^\vee \otimes_{\mathbb{C}} \mathcal{O}$. The dth symmetric product of this filtration is a filtration of $\mathrm{Sym}^d V^\vee \otimes_{\mathbb{C}} \mathcal{O}$:

$$\mathrm{Sym}^d V^\vee \otimes_{\mathbb{C}} \mathcal{O} = F^{0,d} \supset F^{1,d} \supset \cdots \supset F^{d,d} \supset F^{d+1,d} = 0.$$

Here $F^{i,d}$ is the locally free subsheaf of $\mathrm{Sym}^d V^\vee \otimes_{\mathbb{C}} \mathcal{O}$ which is the image of the multiplication map $\mathrm{Sym}^{d-i} V^\vee \otimes_{\mathbb{C}} \mathrm{Sym}^i Q^\vee \to \mathrm{Sym}^d V^\vee$. The associated graded sheaves of this filtration $G^{i,d} = F^{i,d}/F^{i+1,d}$ are canonically isomorphic to the sheaves $\mathcal{O}_{\mathbb{P}V}(d-i) \otimes_{\mathcal{O}} \mathrm{Sym}^i Q^\vee$.

This filtration is not split on $\mathbb{P}V$. But we can find a covering of $\mathbb{P}V$ by open affine subschemes $G \subset \mathbb{P}V$ over which we do have a splitting. Here by *splitting* we mean an isomorphism of bundles over G

$$\alpha : \mathrm{Sym}^d V^\vee \otimes_{\mathbb{C}} \mathcal{O} \longrightarrow \oplus_{j=0}^d \mathcal{O}_{\mathbb{P}V}(d-j) \otimes_{\mathcal{O}} \mathrm{Sym}^j Q^\vee$$

which maps $F^{i,d}$ to the subbundle $\oplus_{j=i}^d \mathcal{O}_{\mathbb{P}V}(d-j) \otimes_{\mathcal{O}} \mathrm{Sym}^j Q^\vee$ and such that the induced isomorphism $G^{i,d} \to \mathcal{O}_{\mathbb{P}V}(d-i) \otimes_{\mathcal{O}} \mathrm{Sym}^i Q^\vee$ is the isomorphism from above. Given an open affine $G \subset \mathbb{P}V$ we can form the projective bundle $\mathbb{P}_G(F^{1,d}|G)$ over G. Given a splitting α on G, denote by

$$\Delta_j(\alpha) \subset \mathbb{P}_G(F^{1,d}|_G)$$

the closed subscheme which parametrizes pairs $(x, [\Phi])$, $x \in G, \Phi \in F^{1,d}|x$ such that the jth component of $\alpha(\Phi)$ is zero. Thus $\Delta_0(\alpha)$ is all of $\mathbb{P}_G(F^{1,d}|_G)$. And, considering $\mathbb{P}_G(F^{1,d}|_G)$ as an open subscheme of $\mathcal{X}$, $\Delta_1(\alpha)$ is the intersection of $\mathbb{P}_G(F^{1,d}|_G)$ with the singular locus of the projection morphism $\mathcal{X} \to \mathbb{P}\mathrm{Sym}^d V^\vee$ (although $\Delta_0(\alpha)$ and $\Delta_1(\alpha)$ are independent of α, the same is not true of the higher $\Delta_i(\alpha)$'s). The next result follows immediately from the definition of the $\Delta_j(\alpha)$'s.

LEMMA 4. *For $j > 0$ the codimension of $\Delta_j(\alpha)$ in $\mathbb{P}_G(F^{1,d}|_G)$ equals*

$$\text{rank}\ \left(\mathcal{O}_{\mathbb{P}V}(d-j) \otimes_{\mathcal{O}} Sym^j Q^\vee\right) = \left(\begin{array}{c} n-1+j \\ n-1 \end{array} \right).$$

In particular, for $j > 0$ the codimension of $\Delta_j(\alpha)$ in $\mathbb{P}_G(F^{1,d}|_G)$ is at least $n-1+j > n-1$. So to establish that $e > n-1$ it suffices to prove that for each open subscheme $G \subset \mathbb{P}V$, each splitting α, and each point $x \in G$, the locally closed subscheme

$$\mathcal{Y}_{x,\alpha} := \left(\mathcal{Y} \cap pr_1^{-1}(x)\right) - \cup_{j=1}^d \left(\Delta_j(\alpha) \cap pr_1^{-1}(x)\right)$$

of $pr_1^{-1}(x)$ has codimension $> n-1$.

Now on the complement of the closed subset $\Delta(\alpha) := \cup_{j=1}^d \left(\Delta_j(\alpha)\right)$ there is a morphism

$$\mathbb{P}_G\left(F^{1,d}|_G\right) - \Delta(\alpha) \xrightarrow{\ \beta\ } \prod_{j=1}^d \mathbb{P}_G\left(\mathcal{O}_{\mathbb{P}V}(d-j) \otimes_{\mathcal{O}} \mathrm{Sym}^j Q^\vee\right)|_G.$$

Up to a twist, we may identify the space

$$\mathbb{P}_G\left(\mathcal{O}_{d-j} \otimes_{\mathcal{O}} \mathrm{Sym}^j Q^\vee\right)|_G$$

with the space of degree j hypersurfaces in fibers of the projection morphism $\mathbb{P}_G Q|G \to G$. Thus β assigns to each suitable pair $(x, [\Phi])$ a sequence of hypersurfaces in $\mathbb{P}Q|_x$. We denote this sequence by $(X_1, \ldots, X_j, \ldots, X_d)$.

LEMMA 5. *If we denote by X the hypersurface in $\mathbb{P}V$ corresponding to Φ, then $X_1 \cap \cdots \cap X_d$ is the fiber of $F_{0,1}(X)$ over $x \in X$.*

PROOF. This is most easily seen by passing to local coordinates. Let $(x_0, \ldots, x_n)$ be a system of homogeneous coordinates on $\mathbb{P}V$ (i.e. a basis for $V^\vee$) and let x be the point with homogeneous coordinates $[0, \ldots, 0, 1]$. We define a splitting α as follows: for each degree d homogeneous polynomial Φ in $(x_0, \ldots, x_n)$ we have a unique decomposition

$$\Phi = \Phi_d + \Phi_{d-1}x_n + \cdots + \Phi_{d-i}x_n^i + \cdots + \Phi_0 x_n^d$$

where each Φ_i is a homogeneous polynomial of degree i in $(x_0, \ldots, x_{n-1})$. Then the fiber of $F^{1,d}$ at x consists of those polynomials such that $\Phi_0 = 0$ and $\beta(\Phi) = (\Phi_d, \ldots, \Phi_1)$. On any line L passing through x there is a unique point of the form $y = (a_0, \ldots, a_{n-1}, 0)$. Let $\mathbb{P}^1 \to \mathbb{P}V$ be the morphism given by

$$(t_0, t_1) \mapsto (t_1 a_0, t_1 a_1, \ldots, t_1 a_{n-1}, t_0 + t_1 a_n).$$

The image of this morphism is just L. Substituting into Φ yields the polynomial on $\mathbb{P}^1$ given by

$$t_0^d\Phi_d(a_0,\ldots,a_{n-1})+\cdots+t_0^{d-i}t_1^i\Phi_{d-i}(a_0,\ldots,a_{n-1})+\cdots+t_0t_1^{d-1}\Phi_1(a_0,\ldots,a_{n-1}).$$

The line L is contained in X iff this polynomial is identically zero iff each of the terms $\Phi_i(a_0,\ldots,a_n)$ is zero. It is easy to convince oneself that the homogeneous ideal generated by the terms Φ_i is independent of our particular splitting. $\square$

In particular, we conclude that every fiber of β which intersects $\mathcal{Y}$ is contained in $\mathcal{Y}$. Therefore the codimension of $\mathcal{Y}_{x,\alpha}$ in $pr_1^{-1}(x)$ equals the codimension of the subvariety

$$\beta(\mathcal{Y})\subset\prod_{j=1}^{d}\mathbb{P}\left(\mathcal{O}_{\mathbb{P}V}(d-j)\otimes_{\mathcal{O}}\mathrm{Sym}^jQ^\vee\right)|_x.$$

By construction, $\beta(\mathcal{Y})$ is the locus parametrizing sequences of hypersurfaces in $\mathbb{P}Q|_x$, $(X_1,\ldots,X_d)$, of degrees $1,\ldots,d$ respectively such that the intersection

$$X_{(1,\ldots,d)}:=X_1\cap\cdots\cap X_d$$

has dimension greater than $n-d-1$. So we have reduced theorem 3 to the following theorem:

THEOREM 6. *Let Q be a vector space over $\mathbb{C}$ of dimension n and let d be an integer such that $1\leq d\leq n-3$. Let $\mathbb{P}_d$ denote the scheme $\prod_{j=1}^{d}\mathbb{P}Sym^jQ^\vee$. Denote by D_d the closed subscheme of $\mathbb{P}_d$ which parametrizes sequences $(X_1,\ldots,X_d)$ such that*

$$dim\left(X_{(1,\ldots,d)}\right)>n-d-1.$$

The codimension of D_d in $\mathbb{P}_d$ is greater than $n-1$.

PROOF. We will prove this by induction on d. Since $D_1=\emptyset$ and the dimension of $\mathbb{P}_d=\mathbb{P}Q^\vee$ is $n-1$, the result is true for $d=1$.

Let U_d denote the open subscheme of $\mathbb{P}_d$ which is the complement of D_d. Then for $1\leq d\leq n-4$, U_{d+1} is contained in $U_d\times\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$. To see this, note that if $X_{(1,\ldots,d)}$ has dimension larger than $n-d-1$, then $X_{1,\ldots,d+1}$ is nonempty and has dimension greater than $n-d-2$: it is nonempty since X_{d+1} is ample, it has dimension larger than $n-d-1$ by the Hauptidealsatz. So we see that the codimension of D_{d+1} in $\mathbb{P}_{d+1}$ is the minimum of the codimension of D_d in $\mathbb{P}_d$ and the codimension of $D_{d+1}\cap(U_d\times\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee)$ in $U_d\times\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$. So by induction we are reduced to showing that the codimension of $D_{d+1}\cap(U_d\times\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee)$ in $U_d\times\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$ is larger than $n-1$.

Now suppose that $(X_1, \ldots, X_d, X_{d+1})$ is a point in

$$D_{d+1} \cap (U_d \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee).$$

By assumption every irreducible component of $X_{(1,\ldots,d)}$ has dimension $n-d-1$. Since also $X_{(1,\ldots,d+1)}$ has dimension $n-d-1$, we conclude that there is an irreducible component $C \subset X_{(1,\ldots,d)}$ such that $C \subset X_{d+1}$. If $X_{(1,\ldots,d)} = C_1 \cup \cdots \cup C_r$ is the irreducible decomposition, then the fiber of $D_{d+1} \cap (U_d \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee)$ over $(X_1, \ldots, X_d)$ (which we consider as a subscheme of $\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$) is just the union of $i = 1, \ldots, r$ of the set $B_i \subset \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$ parametrizing hypersurfaces X_{d+1} such that $C_i \subset X_{d+1}$. We are reduced to showing that the codimension of each B_i in $\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$ is greater than $n-1$. We prove this in a lemma:

LEMMA 7. *Let $Y \subset \mathbb{P}Q$ be an irreducible subscheme such that $dimY = n-d-1$. Let $B(Y) \subset \mathbb{P}Sym^{d+1}Q^\vee$ be the locus of hypersurfaces X_{d+1} such that $Y \subset X_{d+1}$. The codimension of $B(Y)$ is greater than $n-1$.*

PROOF. Let $\Lambda \subset \mathbb{P}Q$ be a $(d-1)$-plane disjoint from Y. Choose coordinates on $\mathbb{P}Q$, $(x_0, \ldots, x_{n-1})$ with respect to which $\Lambda = Z(x_d, \ldots, x_{n-1})$. Let $\mathbb{G}_m$ denote the "multiplicative group" Spec $\mathbb{C}[t, t^{-1}]$ whose closed points correspond to the torus $\mathbb{C}^*$. Let $s : \mathbb{G}_m \times \mathbb{P}Q \to \mathbb{P}Q$ be the torus action given by

$$t \cdot (x_0, \ldots, x_{d-1}, x_d, \ldots, x_{n-1}) = (t^{-1}x_0, \ldots, t^{-1}x_{d-1}, tx_d, \ldots, tx_{n-1}).$$

Since the Hilbert scheme of $\mathbb{P}Q$ is proper, the valuative criterion implies that the closed subscheme

$$s^{-1}(Y) \subset \mathbb{G}_m \times \mathbb{P}Q$$

which is flat over $\mathbb{G}_m$, extends over 0 to yield a closed subscheme

$$\mathcal{Y} \subset \mathbb{A}^1 \times \mathbb{P}Q$$

which is flat over $\mathbb{A}^1$. It is easy to see that the fiber of $\mathcal{Y}$ over 0 is a scheme whose reduced scheme is just

$$Z(x_0, \ldots, x_{d-1}) \subset \mathbb{P}Q.$$

Now we can form the family

$$\mathcal{B} \subset \mathbb{A}^1 \times \mathbb{P}\mathrm{Sym}^{d+1}Q^\vee, \mathcal{B}_t = B(\mathcal{Y}_t).$$

Over $\mathbb{G}_m$ the fibers of $\mathcal{B}$ are isomorphic. It follows by upper semicontinuity that for $t \neq 0$ we have $\dim(\mathcal{B}_t) \leq \dim(\mathcal{B}_0)$. And of course we have

$$\mathcal{B}_0 = B(\mathcal{Y}_0) \subset B\left(Z(x_0, \ldots, x_{d-1})\right).$$

So we are reduced to proving the lemma for the special case $Y = Z(x_0, \ldots, x_{d-1})$. The set B of hypersurfaces $X_{d+1} \subset \mathbb{P}Q$ which contain $Z(x_0, \ldots, x_{d-1})$ is just the projectivization of the kernel of the surjective linear map

$$H^0(\mathbb{P}Q, \mathcal{O}_{\mathbb{P}Q}(d+1)) \to H^0(Y, \mathcal{O}_Y(d+1)).$$

So the codimension of B in $\mathbb{P}Q$ equals

$$\dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(d+1)) = \binom{n}{d+1}.$$

For $d+1 \leq n-1$ (which is one of our hypotheses) we see that $\binom{n}{d+1} \geq n > n-1$. We conclude that the codimension of $B(Y)$ in $\mathbb{P}\mathrm{Sym}^{d+1}Q^\vee$ is greater than $n-1$. This proves the lemma. □

This completes the proof of theorem 6. □

This completes the proof of theorem 3. While we are discussing lines, let us mention two other results about lines on hypersurfaces.

LEMMA 8 ([Kollár96]. , *Exercise V.4.4.2] For general X and a general line $L \subset X$, the normal bundle $N_{L/X}$ is of the form $\mathcal{O}_L^{\oplus d-1} \oplus \mathcal{O}_L(1)^{\oplus n-1-d}$.*

THEOREM 9 ([Kollár96]. , *Theorem V.4.3.2] For general X, the Fano scheme $F_1(X)$ is smooth. Therefore $F_{0,1}(X)$ is smooth. By generic smoothness, the general fiber of $F_{0,1}(X) \to X$ is smooth.*

1.5. Stable A-graphs and Stable Maps

We follow the notation from [Behrend-Manin95] regarding stable A-graphs. However, we shall only need to use genus 0 trees.

Remark For the purposes of this paper, we shall define stable A-graphs to be stable A-graphs whose underlying modular graph is a genus 0 tree.

DEFINITION 10. *A* graph *τ is a 4-tuple $(F_\tau, W_\tau, j_\tau, \partial_\tau)$ defined as follows:*

(1) *F_τ is a finite set called the set of* flags
(2) *W_τ is a finite set called the set of* vertices
(3) *$j_\tau : F_\tau \to F_\tau$ is an involution*
(4) *$\partial_\tau : F_\tau \to W_\tau$ is a map called the* evaluation *map.*

In addition we have the auxiliary definitions

(1) *the set of* tails *$S_\tau \subset F_\tau$ is the set of fixed points of j_τ*
(2) *the set of* edges *E_τ is the quotient of $F_\tau \setminus S_\tau$ by j_τ*

(3) *for a vertex* $v \in W_\tau$, *the* valence *of* v *is defined to be* $val(v) = \#(\partial^{-1}(v))$.

We shall often write $Flag(\tau)$ *in place of* F_τ, $Vertex(\tau)$ *in place of* W_τ, $Tail(\tau)$ *in place of* S_τ, $Edge(\tau)$ *in place of* E_τ, *and* $\overline{f}$ *in place of* $j_\tau(f)$.

We can associate to a graph its *geometric realization* $|\tau|$ which is a simplicial complex defined as follows. The set of 0-simplices of $|\tau|$ is

$$|\tau|^0 = \text{Vertex}(\tau) \sqcup \text{Tail}(\tau).$$

The set of 1-simplices of $|\tau|$ is

$$|\tau|^1 = \text{Edge}(\tau) \sqcup \text{Tail}(\tau).$$

If $[0,1]$ is a 1-simplex associated to an edge $\{f, \overline{f}\}$, the point 0 is glued to the 0-simplex ∂f, and the point 1 is glued to the 0-simplex $\partial \overline{f}$. If $[0,1]$ is the 1-simplex associated to a tail f, the point 0 is glued to the 0-simplex ∂f, and the point 1 is glued to the 0-simplex f.

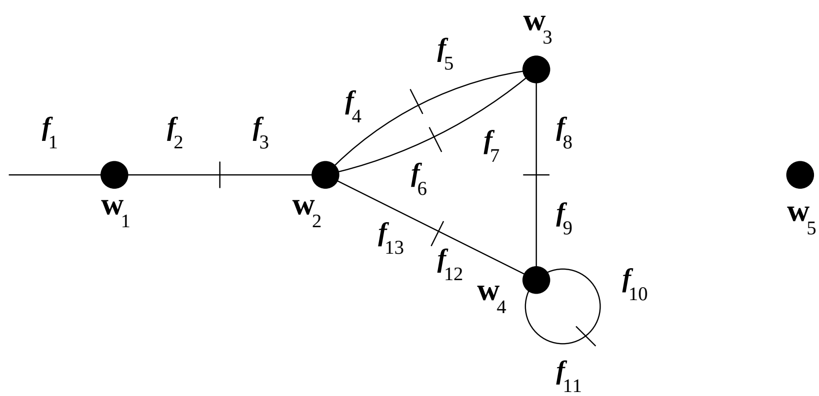

Diagram 1

Example:

In the example the vertex set is $W = \{w_1, \ldots, w_5\}$ and the flag set is $F = \{f_1, \ldots, f_{13}\}$. The only tail is f_1. For all other flags f the opposite flag $j(f)$ is obtained by reflecting through the midpoint of the edge. Notice that there are two edges joining w_2 and w_3, there is an edge joining w_4 to itself, and there are no flags attached to w_5.

Definition 11. *A* tree *is a graph such that* $H_1(|\tau|, \mathbb{Z}) = 0$

We shall use several graphs repeatedly in the proof of theorem 3, so we introduce them now. Of course the most important graph is the *empty graph* $\lambda_\emptyset$, i.e. the unique graph such that $\text{Vertex}(\lambda_\emptyset) = \emptyset$. The second graph will be called λ_0 and is defined to be the graph with one vertex, u_0, and no flags. The third graph will be called λ_1 and is defined to be the graph with one vertex, u_0, one tail, e_0, and no edges. The fourth graph will be called λ_2 and is defined to be the graph with two vertices, u_0 and u_1, no tails, and one edge $\{e_0, e_1\}$ such that $\partial e_i = u_i$. The final graph will be called λ_3 and is defined to be the graph with one vertex, u_0, two tails, e_0 and e_1, and no edges. Notice that all of these graphs are trees.

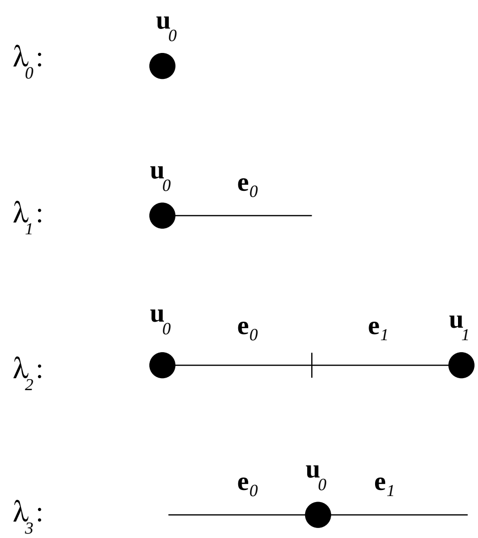

Diagram 2

As mentioned above, we shall only consider graphs which are trees. Trees arise in the study of prestable curves with arithmetic genus 0.

Definition 12. *A* prestable curve *with n marked points* $(C, x_1, \ldots, x_n)$ *is defined to be a datum where C is a complete, reduced, at worst nodal curve and $x_i \in C, i = 1, \ldots, n$ are distinct, nonsingular points of C.*

Suppose that $(C, x_1, \ldots, x_n)$ is a connected, prestable curve whose arithmetic genus is 0. One associates to $(C, x_1, \ldots, x_n)$ a *dual graph*, Δ. The dual graph of $(C, x_1, \ldots, x_n)$ is a tree whose vertices $\{v_1, v_2, \ldots\}$ correspond to the irreducible components $\{C_1, C_2, \ldots\}$ of C, whose edges $\{\{f_1, \overline{f_1}\}, \{f_2, \overline{f_2}\}, \ldots\}$ correspond to the nodes $\{q_1, q_2, \ldots\}$ of C, and whose tails $\{g_1, \ldots, g_n\}$ correspond to the marked points $\{p_1, \ldots, p_n\}$ of C. Given an edge $\{f_i, \overline{f_i}\}$ corresponding to a node q_i, the two vertices $\partial f_1, \partial \overline{f_1}$ are simply the vertices v_j, v_k corresponding to the two (possibly equal) components C_j, C_k making up the branches of the node. A tail g_i corresponding to a marked point p_i is attached to the vertex v_j associated to the (unique) irreducible component C_j containing p_i.

Now suppose X is a projective variety with a fixed ample line bundle L.

DEFINITION 13. *An* A-graph *is a pair* (τ, β_τ) *where* τ *is a tree and*

$$\beta : \mathit{Vertex}(\tau) \to \mathbb{Z}_{\geq 0}$$

is a map called the A-structure. *We shall often abbreviate* (τ, β_τ) *by just writing* τ. *We say that an* A*-graph* τ *is* stable *if for each vertex* $v \in \mathit{Vertex}(\tau)$ *such that* $\beta_\tau(v) = 0$*, there are at least 3 distinct flags* $f \in \mathit{Flag}(\tau)$ *such that* $\partial f = v$ *(i.e. the valence of* v *is at least 3).*

Convention We define $\tau_\emptyset$ to be the unique stable A-graph whose underlying graph is $\lambda_\emptyset$.

One can form a category whose objects are the stable A-graphs. Every morphism in this category is a composition of two basic types of morphisms: *contractions* and *combinatorial morphisms.* The reader is referred to [Behrend-Manin95] for the precise definitions. Essentially a contraction of A-graphs $\phi : \tau \to \sigma$ is a map from the set of vertices of τ onto the set of vertices of σ which maps adjacent vertices to adjacent vertices (here two vertices are adjacent if they are equal or if they are connected by an edge). And a combinatorial morphism $\tau \hookleftarrow \sigma$ is the inclusion of a subgraph σ into a graph τ. The functor which associates to a stable A-graph the corresponding Behrend-Manin stack is covariant for contractions. But it is contravariant for combinatorial morphisms. Therefore we think of a combinatorial morphism $\tau \hookleftarrow \sigma$ as a morphism from τ to σ (which explains our terminology $\tau \hookleftarrow \sigma$ for combinatorial morphisms).

Of special importance for us will be morphisms of graphs which correspond to removing tails. For each stable A-graph τ we define $r_1(\tau)$ to be the stable A-graph obtained by removing every tail $f \in \mathrm{Tail}(\tau)$ such that $\alpha(f) > 0$. We define $\tau \hookleftarrow r_1(\tau)$ to be the canonical combinatorial morphism. For each stable A-graph τ we define $r_2(\tau)$ to be the stabilization of the A-graph obtained by removing all tails $f \in \mathrm{Tail}(\tau)$ such that $\alpha(f) = 0$. Technically the canonical morphism of graphs from τ to $r_2(\tau)$ consists of both a combinatorial morphism and a contraction. But we shall denote it by $\tau \hookleftarrow r_2(\tau)$ just as if it were a combinatorial morphism. Finally, we define $r(\tau) := r_1(r_2(\tau)) = r_2(r_1(\tau))$.

Just as one associates to a connected prestable curve

$$(C, x_1, \ldots, x_n)$$

of arithmetic genus 0 a tree $\Delta(C,x)$, one can associate to a prestable map

$$((C,x_1,\ldots,x_n),C\xrightarrow{h}X)$$

an A-graph $\Delta(C,x,h)$.

DEFINITION 14. *A* prestable map *is a pair*

$$((C,x_1,\ldots,x_n),C\xrightarrow{h}X)$$

where $(C,x_1,\ldots,x_n)$ *is a prestable curve, and where* $C\xrightarrow{h}X$ *is a morphism of* $\mathbb{C}$*-schemes.*

Let us suppose that

$$(C,x_1,\ldots,x_n),(C\xrightarrow{h}X)$$

is a prestable map such that C is connected and such that the arithmetic genus of C is 0. The underlying tree of $\Delta(C,x,h)$ is simply $\Delta(C,x)$. And, given a component C_i of C with corresponding vertex $v_i\in\mathrm{Vertex}(\Delta(C,x))$, one defines

$$\beta(v_i)=\int_{C_i}h_i^*(c_1(L)).$$

The A-graph $\Delta(C,x,h)$ is a *stable* A-graph iff (C,x,h) is a *stable* map.

1.6. Behrend-Manin stacks and the Behrend-Manin decomposition

We refer the reader to [Behrend-Manin96] for the definition of the stacks $\overline{\mathcal{M}}(X,\tau)$. These are proper Deligne-Mumford stacks which parametrize stable maps along with some extra data. We shall occasionally deal with these stacks, but more often we shall deal with associated stacks of *strict maps* $\mathcal{M}(X,\tau)$ which we now define.

DEFINITION 15. *Let X be a variety, L a line bundle on X, and let τ be a stable A-graph. A* strict τ-map *is a datum*

$$((C_v),(h_v:C_v\to X),(q_f))$$

defined as follows:

1. (C_v) *is a set parametrized by* $v\in$ *Vertex*(τ) *of rational curves, i.e. each* $C_v\cong\mathbb{P}^1$
2. $(h_v:C_v\to X)$ *is a set parametrized by* $v\in$ *Vertex*(τ) *of morphisms of* $\mathbb{C}$*-schemes,*
3. (q_f) *is a set parametrized by* $f\in$ *Flag*(τ) *of closed points* $q_f\in C_{\partial f}$

and satisfying the following conditions

1. *for* $v\in$ *Vertex*(τ)*, the degree of* $h_v^*(L)$ *as a line bundle on* C_v *is* $\beta_\tau(v)$*,*

(2) *for* $f_1, f_2 \in Flag(\tau)$ *distinct flags with* $\partial f_1 = \partial f_2$, $q_{f_1} \neq q_{f_2}$,
(3) *for* $f \in Flag(\tau)$, *we have* $h_{\partial f}(q_f) = h_{\partial \overline{f}}(q_{\overline{f}})$.

Convention If $\tau = \tau_\emptyset$, we define a strict τ-map to simply be a point in X. Thus the set of strict τ-maps is simply X.

Definition 16. *If T is a $\mathbb{C}$-scheme, then a* family of strict τ-maps *over T is a datum*

$$((\pi_v : \mathcal{C}_v \to T), (h_v : \mathcal{C}_v \to X), (q_f : T \to \mathcal{C}_{\partial f}))$$

defined as follows:

(1) $(\pi_v : \mathcal{C}_v \to T)$ *is a set parametrized by* $v \in Vertex(\tau)$ *of smooth, proper morphisms whose geometric fibers are rational curves*
(2) $(h_v : \mathcal{C}_v \to X)$ *is a set parametrized by* $v \in Vertex(\tau)$ *of morphisms of $\mathbb{C}$-schemes*
(3) $(q_f : T \to \mathcal{C}_{\partial f})$ *is a set parametrized by* $f \in Flag(\tau)$ *of morphisms of schemes such that* $\pi_{\partial f} \circ q_f = id_T$

and satisfying the following conditions

(1) *for* $v \in Vertex(\tau)$, *the degree of* $h_v^*(L)$ *on each geometric fiber of* $\mathcal{C}_v \to S$ *is* $\beta_\tau(v)$
(2) *for* $f_1, f_2 \in Flag(\tau)$ *distinct flags with* $\partial f_1 = \partial f_2$, q_{f_1} *and* q_{f_2} *are disjoint sections*
(3) *for* $f \in Flag(\tau)$, *we have* $h_{\partial f} \circ q_f = h_{\partial \overline{f}} \circ q_{\overline{f}}$.

Convention If $\tau = \tau_\emptyset$ we define a family of strict τ-maps over T to be a morphism $h : T \to X$.

Suppose given two families of strict τ-maps over S, say

$$\eta = ((\pi_v : \mathcal{C}_v \to T), (h_v : \mathcal{C}_v \to X), (q_f : T \to \mathcal{C}_{\partial f})),$$
$$\zeta = ((\pi'_v : \mathcal{C}'_v \to T), (h'_v : \mathcal{C}'_v \to X), (q'_f : T \to \mathcal{C}'_{\partial f})).$$

Definition 17. *A* morphism *of families of strict τ-maps over S, $\phi : \eta \to \zeta$, is a collection of isomorphisms of S-schemes:*

$$\phi = (\phi_v : \mathcal{C}_v \to \mathcal{C}'_v)$$

indexed by $v \in Vertex(\tau)$ *and satisfying*

(1) *for* $v \in Vertex(\tau)$, $h'_v \circ \phi_v = h_v$
(2) *for* $f \in Flag(\tau)$, $\phi_{\partial f} \circ q_f = q'_f$.

One defines composition of morphisms in the obvious way. Notice that every morphism is an isomorphism. Thus the category of families of strict τ-maps over S is a groupoid. Given a morphism $S' \xrightarrow{u} S$ and a family η of strict τ-maps over S, one has the usual pullback $u^*(\eta)$ which is a family of strict τ-maps over S'. In this way we have the notion of a functor from the category of $\mathbb{C}$-schemes to the category of groupoids which associates to each S the groupoid of families of strict τ-maps over S. We denote this functor by $\mathcal{M}(X,\tau)$.

In every case it is easy to see that $\mathcal{M}(X,\tau)$ is a stack in groupoids over $\mathbb{C}$. In many cases this is even a Deligne-Mumford stack:

THEOREM 18. *If X is projective and L is ample, the functor $\mathcal{M}(X,\tau)$ is a Deligne-Mumford stack which is separated and finite type over $\mathbb{C}$.*

PROOF. There is a 1-morphism $\mathcal{M}(X,\tau) \to \overline{\mathcal{M}}(X,\tau)$ where $\overline{\mathcal{M}}(X,\tau)$ is the refined functor defined in [Behrend-Manin96]. In [Behrend-Manin96] it is proved that $\overline{\mathcal{M}}(X,\tau)$ is a proper Deligne-Mumford stack over $\mathbb{C}$. And it is clear that $\mathcal{M}(X,\tau) \to \overline{\mathcal{M}}(X,\tau)$ is a representable morphism which is an open immersion. Thus $\mathcal{M}(X,\tau)$ is a Deligne-Mumford stack which is separated and finite type over $\mathbb{C}$. $\square$

We shall also need to use a relative version of this construction. Suppose that S is a scheme over $\mathbb{C}$ and $f : X \to S$ is a morphism of schemes. Let L be a relatively ample bundle on X. Define $\overline{\mathcal{M}}(X/S,\tau)$ (resp. $\mathcal{M}(X/S,\tau)$) to be the full substack of $\overline{\mathcal{M}}(X,\tau) \times S$ (resp. $\mathcal{M}(X,\tau) \times S$) such that for each scheme T the objects of $\overline{\mathcal{M}}(X/S,\tau)(T)$ (resp. $\mathcal{M}(X/S,\tau)(T)$) consist of pairs $(\eta, g : T \to S)$ where

$$\eta = \left(\left(\pi_v : \mathcal{C}_v \to T\right), \left(h_v : \mathcal{C}_v \to X\right), \left(q_f : T \to C_{\partial f}\right)\right)$$

is a family of τ-maps (resp. strict τ-maps) and such that for each vertex $v \in \mathrm{Vertex}(\tau)$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C}_v & \xrightarrow{h_v} & \mathrm{X} \\ {\scriptstyle \pi_v}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

It is easy to see that $\overline{\mathcal{M}}(X/S,\tau)$ (resp. $\mathcal{M}(X/S,\tau)$) is indeed a substack of $\overline{\mathcal{M}}(X,\tau) \times S$ (resp. $\mathcal{M}(X,\tau) \times S$).

PROPOSITION 19. *If $f : X \to S$ is projective and L is relatively ample, then $\overline{\mathcal{M}}(X/S,\tau)$ is a Deligne-Mumford stack and the 1-morphism $pr_2 : \overline{\mathcal{M}}(X/S,\tau) \to S$ is proper. Therefore $\mathcal{M}(X/S,\tau)$ is a Deligne-Mumford stack and the 1-morphism $pr_2 : \mathcal{M}(X/S,\tau) \to S$ is separated and finite type.*

PROOF. Consider first the case that $f : X \to S$ is simply the projection $\mathbb{P}^n_C \times S \to S$ and L is simply $\mathrm{pr}_1^*\mathcal{O}(1)$. Then $\overline{\mathcal{M}}(X/S,\tau)$ is simply $\overline{\mathcal{M}}(\mathbb{P}^n,\tau) \times S$ which is obviously a Deligne-Mumford stack. Moreover

$\overline{\mathcal{M}}(X/S,\tau) \to S$ is just the projection morphism which is proper by base-change.

Now the statements that $\overline{\mathcal{M}}(X/S,\tau)$ is a Deligne-Mumford stack and that $\overline{\mathcal{M}}(X/S,\tau) \to S$ is proper are both local on S. In particular, we may suppose that S is affine, that there is an integer n such that $L^{\otimes n}$ is very ample, and that $i : X \to \mathbb{P}^N \times S$ is a closed immersion with $i^*\mathcal{O}(1) = L^{\otimes n}$. Now the ideal sheaf $\mathcal{I}$ of $i(X)$ is coherent. So there is a positive integer m such that $\mathcal{I}\otimes\mathcal{O}(m)$ is generated by global sections. Let $\mathcal{I}\otimes\mathcal{O}(m) \to \mathcal{O}(m)$ be the tensor product of $\mathcal{O}(m)$ with the canonical inclusion.

Define τ' to be the stable A-graph whose underlying modular graph is the same as τ but such that $\alpha'(v) = n\alpha(v)$. As we have seen, $\overline{\mathcal{M}}(\mathbb{P}^N \times S/S,\tau')$ is a Deligne-Mumford stack which is proper over S. Let $\pi : \mathcal{C} \to \overline{\mathcal{M}}(\mathbb{P}^N \times S/S,\tau')$ be the universal curve. Recall that this is a representable 1-morphism of stacks which is proper and flat of relative dimension 1. Let $h : \mathcal{C} \to \mathbb{P}^N \times S$ denote the canonical 1-morphism. The pullback under h of the morphism of coherent sheaves $\mathcal{I}\otimes\mathcal{O}(m) \to \mathcal{O}(m)$ is a morphism of coherent sheaves $h^*(\mathcal{I}\otimes\mathcal{O}(m)) \to h^*\mathcal{O}(m)$. Moreover, since $\mathrm{pr}_2 \circ h = \mathrm{pr}_2 \circ \pi$, and since $\mathcal{I}\otimes\mathcal{O}(m)$ is generated by global sections, we conclude that $\pi^*\pi_*h^*(\mathcal{I}\otimes\mathcal{O}(m)) \to h^*(\mathcal{I}\otimes\mathcal{O}(m))$ is surjective. Since π is representable and proper, the pushforward $\pi_*h^*(\mathcal{I}\otimes\mathcal{O}(m)) \to \pi_*h^*\mathcal{O}_(m)$ is again a morphism of coherent sheaves. The claim is that the canonical 1-morphism $\overline{\mathcal{M}}(X/S,\tau) \to \overline{\mathcal{M}}(\mathbb{P}^N \times S/S,\tau')$ is the zero locus of this morphism of coherent sheaves. This is easy to see: the zero locus of $\pi_*h^*(\mathcal{I}\otimes\mathcal{O}(m)) \to \pi_*h^*\mathcal{O}(m)$ is precisely the maximal closed substack of $\overline{\mathcal{M}}(\mathbb{P}^N \times S/S,\tau')$ over which the morphism $h^*(\mathcal{I}\otimes\mathcal{O}(m)) \to h^*\mathcal{O}(m)$ is identically zero. Using the universal property of closed immersions, we see that this is the maximal closed substack over which $h : \mathcal{C} \to \mathbb{P}^N \times S$ factors through $i(X)$. By definition this is $\overline{\mathcal{M}}(X/S,\tau)$. So we conclude that $\overline{\mathcal{M}}(X/S,\tau)$ is the zero locus of $\pi_*h^*(\mathcal{I}\otimes\mathcal{O}(m)) \to \pi_*h^*\mathcal{O}(m)$. In particular $\overline{\mathcal{M}}(X/S,\tau)$ is a closed substack of the Deligne-Mumford $\overline{\mathcal{M}}(\mathbb{P}^N \times S/S,\tau')$. Thus $\overline{\mathcal{M}}(X/S,\tau)$ is a Deligne-Mumford stack and $\overline{\mathcal{M}}(X/S,\tau) \to S$ is proper.

Given that $\overline{\mathcal{M}}(X/S,\tau)$ is a Deligne-Mumford stack and $\overline{\mathcal{M}}(X/S,\tau) \to S$ is proper, one proves that $\mathcal{M}(X/S,\tau)$ is a Deligne-Mumford stack and $\mathcal{M}(X/S,\tau) \to S$ is seperated and finite type just as in theorem 18. $\square$

The most important A-graphs are the A-graphs $\tau_n(\alpha)$ defined as follows: the underlying graph of $\tau_n(\alpha)$ has a single vertex $\text{Vertex}(\tau_n(\alpha)) = \{u_1\}$ and n flags $\text{Flag}(\tau_n(\alpha)) = \{f_1, \ldots, f_n\}$ all of which are tails, and $\alpha(u_0) = \alpha$. The stack of Behrend-Manin associated to $\tau_n(\alpha)$ is exactly the moduli space of Kontsevich stable maps $\overline{\mathcal{M}}_{0,n}(X, \alpha)$.

One important case to understand is when $\beta(\tau) = 0$. We have already defined $\mathcal{M}(X, \tau) = \overline{\mathcal{M}}(X, \tau) = X$ when τ is the empty graph. For any stable A-graph τ such that $\beta(\tau) = 0$ and such that $\#\text{Tail}(\tau) = r$, we have $\mathcal{M}(X, \tau)$ =X$\times\mathcal{M}(*, \tau)$ where $\mathcal{M}(*, \tau) \subset \overline{\mathcal{M}}_{0,r}$ is the obvious substack.

Suppose that $\phi = (\phi_W, \phi^F) : \tau \to \tau'$ is a contraction of stable A-graphs. There is a corresponding 1-morphism of proper Deligne-Mumford stacks

$$\overline{\mathcal{M}}(X, \phi) : \overline{\mathcal{M}}(X, \tau) \to \overline{\mathcal{M}}(X, \tau').$$

We will denote by $\mathcal{M}(X, \phi)$ the restriction of this 1-morphism to the open substack $\mathcal{M}(X, \tau)$ of $\overline{\mathcal{M}}(X, \tau)$.

In the case that $\phi : \tau \to \tau'$ is a contraction of stable A-graphs such that $\beta(\tau) = \beta(\tau') = 0$, then $\mathcal{M}(X, \tau) \to \overline{\mathcal{M}}(X, \tau')$ is simply the product of $\text{id}_X : X \to X$ with the 1-morphism $\mathcal{M}(*, \phi) : \mathcal{M}(*, \tau) \to \overline{\mathcal{M}}(*, \tau')$. In particular, using the notation of [Behrend-Manin96], consider the case that ϕ is an *isogeny*, i.e. ϕ is the morphism which removes some subset of the set of tails from τ and then stabilizes the resulting (possibly unstable) graph.

LEMMA 20. *Let τ, τ' be stable A-graphs such that $\beta(\tau) = \beta(\tau') = 0$ and let $\phi : \tau \to \tau'$ be an isogeny. Then $\mathcal{M}(X, \phi) : \mathcal{M}(X, \tau) \to \overline{\mathcal{M}}(X, \tau')$ is smooth of relative dimension $dim(X, \tau) - dim(X, \tau')$ with geometrically connected fibers.*

PROOF. Of course it is equivalent to prove that

$$\mathcal{M}(*, \phi) : \mathcal{M}(*, \tau) \to \overline{\mathcal{M}}(*, \tau')$$

is smooth of relative dimension $\dim(X, \tau) - \dim(X, \tau')$ with geometrically connected fibers. Now it follows by proposition 7.4 of [Behrend-Manin96] that $\mathcal{M}(*, \tau)$ and $\overline{\mathcal{M}}(*, \tau')$ have the expected dimension. Thus all we really need to show is that $\mathcal{M}(*, \tau) \to \overline{\mathcal{M}}(*, \tau')$ is smooth with geometrically irreducible fibers. Moreover, since every isogeny is a composition of morphisms obtained by stably removing one tail, we may suppose that $\phi : \tau \to \tau'$ corresponds to stably removing one tail $f \in \text{Tail}(\tau)$.

There are two cases. Suppose first of all that when we remove f from τ, the resulting graph is unstable. But then $\partial f = v$ is a vertex with valence 3. Since a rational curve with 3 marked points has no moduli, we conclude that $\mathcal{M}(*, \tau) \to \overline{\mathcal{M}}(*, \tau')$ is an open immersion.

The second case is that when we remove f from τ, the resulting graph is stable, i.e. the resulting graph is just τ'. But then if $v = \phi(\partial f)$, we conclude that $\phi : \mathcal{M}(*, \tau) \to \overline{\mathcal{M}}(*, \tau')$ is simply an open subset of the universal curve over $\overline{\mathcal{M}}(*, \tau')$ corresponding to the vertex v. In both cases we conclude that $\overline{\mathcal{M}}(*, \tau) \to \overline{\mathcal{M}}(*, \tau')$ is smooth with geometrically connected fibers. □

For each stable A-graph τ define $e = \beta(\tau)$ and define $r = \#\mathrm{Tail}(\tau)$. Then there is a contraction $\phi : \tau \to \tau_r(e)$ which is unique up to a labeling of the tails of τ. We will refer to this contraction as the *canonical contraction* of τ. Corresponding to the canonical contraction we have the 1-morphism

$$\mathcal{M}(X, \phi) : \mathcal{M}(X, \tau) \to \overline{\mathcal{M}}_{0,r}(X, e),$$

which is unique up to an automorphism of $\mathcal{M}(X, \tau)$ corresponding to relabeling the tails. In particular, the image of $\mathcal{M}(X, \phi)$ as a subset of the set $|\overline{\mathcal{M}}_{0,r}(X, e)|$ is well-defined.

PROPOSITION 21. *Let $\phi : \tau \to \tau'$ be a contraction of stable A-graphs. The image of the 1-morphism $\mathcal{M}(X, \phi)$ is a locally closed subset of the topological space $|\overline{\mathcal{M}}(X, \tau)|$.*

PROOF. For notation's sake let's denote the continuous map of topological spaces

$$|\overline{\mathcal{M}}(X, \phi)| : |\overline{\mathcal{M}}(X, \tau)| \to |\overline{\mathcal{M}}(X, \tau')|$$

by

$$f : M \to M'$$

and let's denote the open substack $\mathcal{M}(X, \tau)$ of $\overline{\mathcal{M}}(X, \tau)$ by M^o. Then $f : M \to M'$ is a closed map. And it is easy to see that $f^{-1}(f(M^o)) = M^0$. Therefore $f(M^o) = f(M) - f(M - M^o)$ is a difference of closed sets and so is locally closed. □

We now fix n and α and consider the set S of all images

$$\{\mathrm{Image}(\mathcal{M}(X, \phi))\}$$

as ϕ ranges over all contractions of stable A-graphs to $\tau_n(\alpha)$. The set of isomorphism classes of such contractions is clearly finite. The previous lemma shows that S forms a *locally closed decomposition* of the topological space $|\overline{\mathcal{M}}_{0,n}(X, \alpha)|$, i.e. a partition of $|\mathcal{M}_{0,n}(X, \alpha)$ into locally closed subsets. This partition is what we call the *Behrend-Manin decomposition.*

Definition 22. *One defines certain numerical invariants of an A-graph* τ *as follows:*

(1) *The* class *of* τ *is*

$$\beta(\tau) = \sum_{v \in \mathit{Vertex}(\tau)} \beta(v).$$

(2) *The* virtual dimension *of* τ *is*

$$\mathit{dim}(X, \tau) = -K_X.\beta(\tau) + (\mathit{dim}(X) - 3) + \#(\mathit{Tail}(\tau)) - \#(\mathit{Edge}(\tau)).$$

1.7. Naive Maps

The Behrend-Manin decomposition of the last section is not sufficient to prove theorem 1. We shall define a new decomposition – the *CD decomposition* – which is better suited to our purposes. To define this decomposition we shall need the following modification of the notion of stable map.

Definition 23. *Let* τ *be a stable A-graph. A* naive τ-map *is a datum*

$$((C_v), (h_v : C_v \to X), (q_f))$$

such that

(1) (C_v) *is a set parametrized by* $v \in \mathit{Vertex}(\tau)$ *of rational curves* C_v
(2) $(h_v : C_v \to X)$ *is a set parametrized by* $v \in \mathit{Vertex}(\tau)$ *of morphisms of* $\mathbb{C}$*-schemes,*
(3) (q_f) *is a set parametrized by* $f \in \mathit{Flag}(\tau)$ *of closed points* $q_f \in C_{\partial f}$

and satisfying the following conditions

(1) *for* $v \in \mathit{Vertex}(\tau)$*, the degree of* $h_v^*(L)$ *as a line bundle on* C_v *is* $\beta_\tau(v)$*,*
(2) *if* $f_1, f_2 \in \mathit{Flag}(\tau)$ *are distinct flags with* $\partial f_1 = \partial f_2 = v$ *and if* $\beta(v) = 0$*, then* $q_{f_1} \neq q_{f_2}$*,*
(3) *for* $f \in \mathit{Flag}(\tau)$*, we have* $h_{\partial f}(q_f) = h_{\partial \overline{f}}(q_{\overline{f}})$*,*
(4) *for* $v \in \mathit{Vertex}(\tau)$ *with* $\beta(v) \neq 0$*, the map* $h_v : C_v \to h_v(C_v)$ *is birational,*
(5) *and for distinct* $v, w \in \mathit{Vertex}(\tau)$ *such that* $\beta(v), \beta(w) \neq 0$ *and such that v,w are* adjacent *(i.e. there exists* $\{f, \overline{f}\} \in \mathit{Edge}(\tau)$ *with* $\partial(f) = v, \partial(\overline{f}) = w$*) we have* $h_v(C_v) \neq h_w(C_w)$*.*

Convention If $\tau = \tau_\emptyset$ we define a naive τ-map to simply be a point of X.

Given a k-scheme T, we define a *family of naive* τ*-maps* over T and a *morphism of families of naive* τ*-maps* in the same manner as we defined a family of strict τ-maps. We define $\mathcal{N}(X,\tau)$ to be the groupoid of families of naive τ-maps.

THEOREM 24. *Let τ be a stable A-graph. The groupoid $\mathcal{N}(X,\tau)$ is a Deligne-Mumford stack which is separated and of finite type over $\mathbb{C}$.*

PROOF. It is an easy exercise to see that $\mathcal{N}(X,\tau)$ is a stack in groupoids. First we reduce to the case that $\tau = r_2(\tau)$. If $\tau = \tau_\emptyset$, then clearly $\mathcal{N}(X,\tau) = X$ which is a separated, finite type scheme over $\mathbb{C}$. More generally, if τ is a stable A-graph such that every $v \in \mathrm{Vertex}(\tau)$ has $\beta(\tau) = 0$, then $\mathcal{N}(X,\tau) = \mathcal{M}(X,\tau)$. Now suppose that $\tau \neq r_2(\tau)$. Let V denote the maximal subgraph of τ such that $\mathrm{Vertex}(V)$ is the set of vertices $v \in \mathrm{Vertex}(\tau)$ which are contracted in $r_2(\tau)$ or such that the valence of v in $r_2(\tau)$ is less than the valence of v in τ. We will prove by induction on the size of V that $\mathcal{N}(X,\tau)$ is a separated, finite type Deligne-Mumford stack whenever $\mathcal{N}(X,r_2(\tau))$ is a separated, finite type Deligne-Mumford stack. Let $V_0 \subset V$ be a connected subgraph of V. Let $W_0 \subset r_2(\tau)$ be the image of V_0. Define τ' to be the graph intermediate between τ and $r_2(\tau)$ obtained by contracting V_0 to W_0. Now there are combinatorial morphisms $\tau \hookleftarrow V_0$ and $\tau' \hookleftarrow W_0$. It is clear from the definition of naive maps that the following diagram is Cartesian:

$$\begin{array}{ccc} \mathcal{N}(X,\tau) & \longrightarrow & \mathcal{N}(X,\tau') \\ \downarrow & & \downarrow \\ \mathcal{M}(X,V_0) & \longrightarrow & \mathcal{M}(X,W_0). \end{array}$$

Now a fiber product of separated, finite type Deligne-Mumford stacks is again a separated, finite type Deligne-Mumford stack. We conclude by induction that $\mathcal{N}(X,\tau)$ is a separated, finite type Deligne-Mumford stack whenever $\mathcal{N}(X,r_2(\tau))$ is. Thus we may assume that $\tau = r_2(\tau)$.

Suppose next that τ has a single vertex v, i.e. $\tau = \tau_n(e)$ for some n and e. For each flag $f \in \mathrm{Flag}(\tau)$ we have the combinatorial morphism $\tau \hookleftarrow \tau_1(e)$ which is simply the subgraph whose only flag is f. Consider the product indexed by all $f \in \mathrm{Flag}(\tau)$ of the corresponding morphism $\mathcal{N}(X,\tau) \to \mathcal{N}(X,\tau_1(e))$; for all f the composites with the forgetful map $\mathcal{N}(X,\tau_1(e)) \to \mathcal{N}(X,\tau_0(e))$ are equal. The product map

$$\mathcal{N}(X,\tau) \to \mathcal{N}(X,\tau_1(e)) \times_{\mathcal{N}(X,\tau_0(e))} \cdots \times_{\mathcal{N}(X,\tau_0(e))} \mathcal{N}(X,\tau_1(e))$$

identifies $\mathcal{N}(X,\tau)$ with an open substack of the n-fold fiber product. Thus to prove that $\mathcal{N}(X,\tau)$ is a finite type, separated Deligne-Mumford stack, it suffices to consider the cases $\tau = \tau_0$ and $\tau = \tau_1$. But clearly $\mathcal{N}(X,\tau_0(e))$ is simply the open substack of $\mathcal{M}(X, \tau_0(e))$ which parametrizes stable maps such that $h : C \to h(C)$ is birational (this is an open property of stable maps). And $\mathcal{N}(X, \tau_1(e))$ is simply the fiber product

$$\mathcal{N}(X, \tau_0(e)) \times_{\mathcal{M}(X,\tau_0(e))} \mathcal{M}(X, \tau_1(e)).$$

Thus $\mathcal{N}(X, \tau_0(e))$ and $\mathcal{N}(X, \tau_1(e))$ are both finite type, separated Deligne-Mumford stacks.

Suppose now that τ has more than one vertex. We perform leaf induction: let $v_1 \in \text{Vertex}(\tau)$ be a leaf. Since $\tau = r_2(\tau)$, we know $\beta(v_1) > 0$. Define $\tau \hookleftarrow \tau'$ and $\tau \hookleftarrow \tau''$ to be the maximal subgraphs such that $\text{Vertex}(\tau') = \{v_1\}$ and $\text{Vertex}(\tau'') = \text{Vertex}(\tau) - \{v_1\}$. Further, let us denote by $\{f_1, f_2\}$ the unique edge of τ such that $\partial f_1 = v_1$ and let us denote $v_2 = \partial f_2$.

By the induction assumption we know that $\mathcal{N}(X, \tau')$ and $\mathcal{N}(X, \tau'')$ are separated, finite type Deligne-Mumford stacks. Corresponding to the flags f_1 and f_2 we have evaluation morphisms $e' : \mathcal{N}(X, \tau') \to X$ and $e'' : \mathcal{N}(X, \tau'') \to X$. The combinatorial morphisms $\tau \hookleftarrow \tau'$ and $\tau \hookleftarrow \tau''$ induce 1-morphisms $\mathcal{N}(X, \tau) \to \mathcal{N}(X, \tau')$ and $\mathcal{N}(X, \tau) \to \mathcal{N}(X, \tau'')$. So we have an induced 1-morphism $\mathcal{N}(X, \tau) \to \mathcal{N}(X, \tau') \times_{e',X,e''} \mathcal{N}(X, \tau'')$. It is clear from the definition of naive maps that this 1-morphism is an open immersion of stacks. Since a fiber product of separated, finite type Deligne-Mumford stacks is again a separated, finite type Deligne-Mumford stack, we conclude that $\mathcal{N}(X, \tau)$ is a separated, finite type Deligne-Mumford stack. This completes the proof. $\square$

LEMMA 25. *For each $n \geq 0$ and each stable A-graph τ, the stack $\mathcal{N}(\mathbb{P}^n, \tau)$ is a Deligne-Mumford stack over $\mathbb{C}$ which is smooth of relative dimension $dim(\tau)$. Moreover, for each flag $f \in Flag(\tau)$ the evaluation morphism $e_f : \mathcal{N}(\mathbb{P}^n, \tau) \to X$ is smooth.*

PROOF. First of all, if we prove that $\mathcal{N}(\mathbb{P}^n, \tau)$ is smooth, then it follows by generic smoothness that the evalution morphism $e_f : \mathcal{N}(\mathbb{P}^n, \tau) \to X$ is smooth over a nonempty open subset. But then by homogeneity it follows that $e_f : \mathcal{N}(\mathbb{P}^n, \tau) \to X$ is smooth everywhere.

The remainder of the proof is just as in the last theorem, therefore we will only indicate what needs to be added to that proof. In that proof the morphism $\mathcal{M}(X, V_0) \to \mathcal{M}(X, W_0)$ is smooth – in fact it is simply an open subset of the product $X \times \overline{\mathcal{M}}(*, V_0) \to X \times \overline{\mathcal{M}}(*, W_0)$. Smoothness of $\overline{\mathcal{M}}(*, V_0) \to \overline{\mathcal{M}}(*, W_0)$ is easy (it follows from the fact that each of the spaces $\overline{\mathcal{M}}_{0,d}$ is smooth). By proposition 7.4 of [Behrend-Manin96],

$\mathcal{N}(X, \tau_0(e))$ is smooth. And $\mathcal{N}(X, \tau_1(e)) \to \mathcal{N}(X, \tau_0(e))$ is simply the universal curve so it is smooth. Thus, by induction, all of the spaces $\mathcal{N}(X, \tau_d(e))$ are smooth. Finally, in the leaf induction argument notice that the evaluation morphisms e', e'' are smooth. Thus the fiber product $\mathcal{N}(X, \tau') \times_{e',X,e''} \mathcal{N}(X, \tau'')$ is smooth. So $\mathcal{N}(X, \tau)$ is an open subset of a smooth stack, thus it is smooth. □

Let's just formalize the first part of this proof. We define $n' = n'(\tau)$ to be the number of flags attached to vertices v with $\alpha(v) = 0$, i.e. $n' = \#\{f \in \mathrm{Flag}(\tau) : \alpha(\partial f) = 0\}$. Also we define $c = c(\tau)$ to be the number of vertices $v \in \mathrm{Vertex}(\tau)$ with $\alpha(v) = 0$.

Suppose that $\eta = ((C_v), (h_v), (q_f))$ is a family of naive τ-maps over a base T. By the same procedure as in [Behrend-Manin96], one can associate to η a family $r_2(\eta)$ of naive $r_2(\tau)$-maps over T: The process of removing tails from contracted components and then contracting unstable components results in a family $r_2(\eta)$ which automatically satisfies condition 3. Using the techniques in [Behrend-Manin96] it is trivial to verify the following:

Proposition 26. *The 1-morphism $r_2 : \mathcal{N}(X, \tau) \to \mathcal{N}(X, r_2(\tau))$ which associates to each family η the corresponding family $r_2(\eta)$ is a smooth morphism of relative dimension $n' - 3c$ whose fibers are geometrically connected.*

1.8. Combinatorial Data and the CD decomposition

In this section we will define a refinement of the Behrend-Manin decomposition.

Definition 27. *A* combinatorial datum $\tau_1 \xrightarrow{\phi} \tau_2$ *is the following:*

1. *a stable genus 0 tree τ_1*
2. *a stable genus 0 tree τ_2*
3. *a* degree *map $d : Vert(\tau_1) \to \mathbb{Z}_{>0}$*
4. *a map $\phi_W : Vert(\tau_1) \to Vert(\tau_2)$*
5. *a map $\phi_F : Flag(\tau_1) \to Flag(\tau_2)$*

and satisfying the following conditions

1. *for each $f \in Flag(\tau_1)$, $\partial\phi_F(f) = \phi_W(\partial f)$*
2. *for each $f \in Flag(\tau_1)$, $\phi_F(\overline{f}) = \overline{\phi_F(f)}$*
3. *the induced map $\phi_F : Edge(\tau_1) \cup Tail(\tau_1) \to Edge(\tau_2) \cup Tail(\tau_2)$ is a bijection*
4. *for each $v \in Vertex(\tau_2)$ the preimage $\phi_W^{-1}(v)$ is the set of vertices of a connected subtree of τ_1*
5. *for each $v \in Vertex(\tau_2)$ such that $\alpha_2(v) = 0$, the preimage $\phi_W^{-1}(v)$ consists of exactly one vertex*

(6) *for each* $v \in Vertex(\tau_1)$ *we have* $\alpha_1(v) = d(v)\alpha_2(\phi_W(v))$
(7) *for each* $v \in Vertex(\tau_1)$ *such that* $\alpha_1(v) = 0$, *we have* $d(v) = 1$.

Note that one can recover the combinatorial datum $\tau_1 \xrightarrow{\phi} \tau_2$ (or more precisely its isomorphism class) from τ_1, the *degree* map d, and the decomposition of the graph $\alpha_1^{-1}(\mathbb{Z}_{>0})$ into subtrees corresponding to the decomposition $\{\phi_W^{-1}(v) | v \in \alpha_2^{-1}(\mathbb{Z}_{>0})\}$, subject only to the condition that for each $v \in \alpha_1^{-1}(\mathbb{Z}_{>0})$ $d(v)$ divides $\alpha_1(v)$.

DEFINITION 28. *One defines certain numerical invariants of a combinatorial datum* ϕ *as follows:*

(1) *The* class of ϕ *is* $\beta(\phi) = \beta(\tau_1)$.
(2) *The* defect of ϕ *is*

$$\delta\phi = \sum_{v \in Vertex(\tau_1)} (2d(v) - 2)\,.$$

(3) *The* virtual dimension of ϕ *is*

$$dim(X, \phi) = dim(\tau_2) + \delta(\phi).$$

DEFINITION 29. *If* $\tau_1 \xrightarrow{\phi} \tau_2$ *is a combinatorial datum, then a* strict ϕ-map *is the following:*

(1) *A strict* τ_1*-map* $((C_v), (h_v : C_v \to X), (q_f))$,
(2) *a naive* τ_2*-map* $((C'_v), (h'_v : C'_v \to X), (q'_f))$,
(3) *and for each* $v \in Vertex(\tau_1)$ *a finite, flat morphism* $k_v : C_v \to C'_{\phi_W(v)}$ *of degree* $d(v)$

which satisfies the conditions

(1) *for each* $v \in Vertex(\tau_1), h_v = h'_{\phi_W(v)} \circ k_v$,
(2) *for each* $f \in Flag(\tau_1), k_{\partial f}(q_f) = q'_{\phi_F(f)}$,

DEFINITION 30. *If* T *is a* $\mathbb{C}$*-scheme, then a family of* strict ϕ-maps *over* T *is the following:*

(1) *A family of strict* τ_1*-maps*

$$((\pi_v : \mathcal{C}_v \to T), (h_v : \mathcal{C}_v \to X), (q_f : T \to \mathcal{C}_{\partial f})),$$

(2) *a family of naive* τ_2*-maps*

$$((\pi'_v : \mathcal{C}'_v \to T), (h'_v : \mathcal{C}'_v \to X), (q'_f : T \to \mathcal{C}'_{\partial f})),$$

(3) *and for each* $v \in Vertex(\tau_1)$ *a finite, flat morphism* $k_v : \mathcal{C}_v \to \mathcal{C}'_{\phi_W(v)}$ *of degree* $d(v)$

which satisfies the conditions

(1) *for each* $v \in Vertex(\tau_1), h_v = h'_{\phi_W(v)} \circ k_v$,
(2) *for each* $f \in Flag(\tau_1), h_{\partial f} \circ q_f = q'_{\phi_F(f)}$,

DEFINITION 31. *Given strict ϕ-maps,*

$$\eta = \left(\left(\left(C_v\right), \left(h_v\right), \left(q_f\right)\right), \left(\left(C'_v\right), \left(h'_v\right), \left(q'_f\right)\right), \left(k_v\right)\right),$$
$$\xi = \left(\left(\left(D_v\right), \left(i_v\right), \left(r_f\right)\right), \left(\left(D'_v\right), \left(i'_E v\right), \left(r'_f\right)\right), \left(j_v\right)\right),$$

a morphism of strict ϕ-maps $\alpha : \eta \to \xi$ *is a pair* (u, u')

(1) $u : \left(\left(C_v\right), \left(h_v\right), \left(q_f\right)\right) \to \left(\left(D_v\right), \left(i_v\right), \left(r_f\right)\right)$ *is a morphism of strict τ_1-maps,*

(2) $u' : \left(\left(C'_v\right), \left(h'_v\right), \left(q'_f\right)\right) \to \left(\left(D'_v\right), \left(i'_v\right), \left(r'_f\right)\right)$ *is a morphism of naive τ_2-maps*

and such that for each $v \in$ Vertex(τ_1), $j_v \circ u_v = u'_{\phi_W(v)} \circ k_v$.

DEFINITION 32. *Given T and families of strict ϕ-maps,*

$$\begin{aligned}
\eta &= \left(\left(\left(\pi_v : \mathcal{C}_v \to T\right), \left(h_v : \mathcal{C}_v \to X\right), \left(q_f : T \to \mathcal{C}_v\right)\right)\right. \\
&\quad \left(\left(\pi'_v : \mathcal{C}'_v \to T\right), \left(h'_v : \mathcal{C}'_v \to X\right), \left(q'_f : T \to \mathcal{C}'_v\right)\right), \left.\left(k_v : \mathcal{C}_v \to \mathcal{C}'_{\phi_W(v)}\right)\right), \\
\xi &= \left(\left(\left(\rho_v : \mathcal{D}_v \to T\right), \left(i_v : \mathcal{D}_v \to X\right), \left(r_f : T \to \mathcal{D}_v\right)\right),\right. \\
&\quad \left(\left(\rho'_v : \mathcal{D}'_v \to T\right), \left(i'_v : \mathcal{D}'_v \to X\right), \left(r'_f : T \to \mathcal{D}'_v\right)\right), \left.\left(j_v : \mathcal{D}_v \to \mathcal{D}'_{\phi_W(v)}\right)\right),
\end{aligned}$$

a morphism of families of strict ϕ-maps $\alpha : \eta \to \xi$ *is a pair (u, u') where u is a morphism from the family of strict τ_1-maps of η to the family of strict τ_1-maps of ξ, where u' is a morphism from the family of naive τ_2-maps of η to the family of naive τ_2-maps of ξ and such that for each $v \in$ Vertex(τ_1), $j_v \circ u_v = u'_{\phi_W(v)} \circ k_v$.*

One composes morphisms of strict ϕ-maps in the obvious way. With the obvious definition of morphism, the association to each scheme T of the category of families of strict ϕ-maps over T is easily seen to be a stack in groupoids. We denote by $\mathcal{M}(X, \phi)$ the stack of families of strict ϕ-maps. We denote by

$$F_1 : \mathcal{M}(X, \phi) \to \mathcal{M}(X, \tau_1)$$
$$F_2 : \mathcal{M}(X, \phi) \to \mathcal{N}(X, \tau_2)$$

the canonical 1-morphisms. We will prove that $\mathcal{M}(X, \phi)$ is a Deligne-Mumford stack with the help of the following lemma.

THEOREM 33. *$\mathcal{M}(X, \phi)$ is a separated, finite type Deligne-Mumford stack over $\mathbb{C}$. The 1-morphism F_2 is flat of relative dimension $\delta(\phi)$ (the* defect *of ϕ).*

PROOF. Consider the 1-morphism F_2. We know by lemma 24 that $\mathcal{N}(X, \tau_2)$ is a separated, finite-type Deligne-Mumford stack over $\mathbb{C}$. By lemma 2, to show that $\mathcal{M}(X, \phi)$ is a Deligne-Mumford stack over $\mathbb{C}$, it suffices to show that for every 1-morphism of an affine scheme to $\mathcal{N}(X, \tau_2)$, $g : T \to \mathcal{N}(X, \tau_2)$, the fiber product $\mathcal{X} := T \times_{g, \mathcal{N}(X, \tau_2), F_2} \mathcal{M}(X, \phi)$ is a

Deligne-Mumford stack. To show that $\mathcal{M}(X,\phi)$ is separated and finite type over $\mathbb{C}$, it suffices to show that the projection morphism $\mathrm{pr}_1 : \mathcal{X} \to T$ is separated and finite type. Moreover to show that F_2 is flat of relative dimension $\delta(\phi)$ it suffices to show that pr_1 is flat of relative dimension $\delta\phi$. So we are reduced to studying the 1-morphism of stacks $\mathrm{pr}_1 : \mathcal{X} \to T$.

The 1-morphism $g : T \to \mathcal{N}(X,\tau_2)$ is equivalent to a family of naive τ_2-maps over T,

$$\eta \quad = \quad \left((\pi'_v : \mathcal{C}'_v \to T), (h'_v : \mathcal{C}'_v \to X), \left(q'_f : T \to \mathcal{C}'_{\partial f}\right) \right).$$

The fiber product $\mathcal{X} = T \times_{g,\mathcal{N}(X,\tau_2),F_2} \mathcal{M}(X,\phi)$ is the stack which parameterizes the ways in which η can be completed to a strict family of ϕ-maps.

For each vertex $v \in \mathrm{Vertex}(\tau_1)$, define τ_v to be the subgraph of τ_1 whose only vertex is v and such that $\mathrm{Flag}(\tau_v) = \{f \in \mathrm{Flag}(\tau_1) | \partial f = v\}$. We make this into a stable A-graph by defining $\beta(v) = d(v)$. By proposition 19, we know that $\mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v)$ is a Deligne-Mumford stack and that the 1-morphism $\mathrm{pr}_2 : \mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v) \to T$ is separated and finite-type. For each flag $f \in \mathrm{Vertex}(\tau_v)$ we have a 1-morphism $e_f : \mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v) \to \mathcal{C}'_{\phi(v)}$ obtained by "evaluating" the marked point corresponding to f. We have a second 1-morphism $\mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v) \to \mathcal{C}'_{\phi(v)}$ which is the composition:

$$\mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v) \xrightarrow{\mathrm{pr}_2} T \xrightarrow{q'_{\phi(f)}} \mathcal{C}'_{\phi(v)}.$$

Together these two morphisms define a 1-morphism

$$\mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v) \to \mathcal{C}'_{\phi(v)} \times_T \mathcal{C}'_{\phi(v)}.$$

We define $\mathcal{Y}_{v,f} \to \mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v)$ to be the base-change by this 1-morphism of the diagonal morphism $\Delta : \mathcal{C}'_{\phi(v)} \to \mathcal{C}'_{\phi(v)} \times_T \mathcal{C}'_{\phi(v)}$. Since the diagonal morphism is a closed immersion, we conclude that $\mathcal{Y}_{v,f} \to \mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v)$ is a closed immersion. We define $\mathcal{Y}_v$ to be the intersection over all $f \in \mathrm{Flag}(\tau_v)$ of $\mathcal{Y}_v$. Since $\mathcal{Y}_v \to \mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v)$ is a closed substack and $\mathcal{M}(\mathcal{C}'_{\phi(v)}/T,\tau_v) \to T$ is a separated, finite type morphism of Deligne-Mumford stacks, we conclude that $\mathcal{Y}_v \to T$ is a separated, finite type morphism of Deligne-Mumford stacks.

Define $\mathcal{Y}$ to be the fiber product over T

$$\mathcal{Y} := \prod_{v \in \mathrm{Vertex}(\tau_1)} \mathcal{Y}_v.$$

Then $\mathcal{Y}$ is a Deligne-Mumford stack which is separated and finite-type over T. We will prove that the T-stack $\mathcal{X}$ is equivalent to the T-stack $\mathcal{Y}$.

We begin by constructing a 1-morphism $\mathcal{Y} \to \mathcal{X}$. Such a 1-morphism is equivalent to a family of strict ϕ-maps over $\mathcal{Y}$, i.e. a triple

$$\begin{aligned}(\zeta, c^*\eta, (k_v)) &= (((\mathcal{C}_v), (h_v : \mathcal{C}_v \to X), (q_f : \mathcal{Y} \to \mathcal{C}_{\partial f})), \\ &\quad \left((\mathcal{C}'_v), (h'_v : \mathcal{C}'_v \to X), (q'_f : \mathcal{Y} \to \mathcal{C}'_v)\right), \left(k_v : \mathcal{C}_v \to \mathcal{C}'_{\phi(v)}\right)).\end{aligned}$$

for which the $\pi_v : \mathcal{C}_v \to \mathcal{Y}$, $\pi'_v : \mathcal{C}'_v \to \mathcal{Y}$ 1-morphisms are schematic, for which the $q_f : \mathcal{Y} \to \mathcal{C}_v$, $q'_f : \mathcal{Y} \to \mathcal{C}'_v$ 1-morphisms are closed immersions, and which satisfy all the axioms analogous to those for a family of strict ϕ-maps over a scheme. Of course the strict family of naive τ_2-maps $c^*\eta$ is defined to be the pullback of the strict family of naive τ_2-maps η over T by the canonical 1-morphism $c : \mathcal{Y} \to T$ (or more precisely one of the canonical 1-morphisms $\mathcal{Y} \to T$).

Now we define the family ζ and the morphisms k_v. For each vertex $v \in \mathrm{Vertex}(\tau_1)$, define $\pi_v : \mathcal{C}_v \to \mathcal{Y}$ to be the pullback by the canonical 1-morphism to $\mathcal{M}(\mathcal{C}'_{\phi(v)}/T, \tau_v)$ of the universal curve. And for each flag $f \in \mathrm{Flag}(\tau_1)$, define $q_f : \mathcal{Y} \to \mathcal{C}_{\partial f}$ to be the pullback of the universal section. We define $k_v : \mathcal{C}_v \to \mathcal{C}'_{\phi(v)}$ to be the pullback of the universal evaluation map from the universal curve over $\mathcal{M}(\mathcal{C}'_{\phi(v)}/T, \tau_v)$ to $\mathcal{C}'_{\phi(v)}$. By construction we have that $k_v \circ q_f = q'_{\phi(f)}$. Finally, we define $h_v = h'_{\phi(v)} \circ k_v$.

The claim is that ζ, $c^*\eta$, and (k_v) define a family of strict ϕ-maps. This is easy to see. By construction $c^*\eta$ is a family of naive τ_2-maps. Also by construction, the maps k_v satisfy the axioms for a family of strict ϕ-maps. The only thing left to check is that the maps h_v make ζ a family of strict τ_1-maps. Again the *strictness* follows from the *strictness* of the stacks $\mathcal{M}(\mathcal{C}'_{\phi(v)}/T, \tau_v)$. The only thing that really needs to be checked is the stability condition. But this follows from our demand that $d(v) = 1$ for each vertex $v \in \beta_{\tau_1}^{-1}(0)$: for such a v we have $k_v : \mathcal{C}_v \to \mathcal{C}'_{\phi(v)}$ is an isomorphism. So stability of the $\mathcal{C}_v$ component follows from stability of the $\mathcal{C}'_{\phi(v)}$ component. So we see that $(\zeta, c^*\eta, (k_v))$ do form a family of strict ϕ-maps, i.e they define a 1-morphism $\mathcal{Y} \to \mathcal{X}$.

One defines the inverse 1-morphism $\mathcal{X} \to \mathcal{Y}$ in an analogous way. It is easy to check that these 1-morphisms yield an isomorphism of $\mathcal{X}$ and $\mathcal{Y}$. From this it follows that $\mathcal{X}$ is a separated, finite-type Deligne-Mumford stack over T.

It remains to prove that the morphism $\mathrm{pr}_2 : \mathcal{X} \to T$ is flat of relative dimension $\delta(\phi)$. Since pr_2 is the fiber product of the morphisms

$c_v : \mathcal{Y}_v \to T$, it suffices to prove that each c_v is flat of relative dimension $2d(v) - 2$. If $d(v) = 1$, this is trivial: the morphism $c_v : \mathcal{Y}_v \to T$ is an open immersion. Therefore c_v is flat of relative dimension $0 = 2d(v) - 2$.

So, without loss of generality, we may suppose that $d(v) > 1$. As usual, let $r_1(\tau_v)$ be the *reduced graph* associated to τ_v, i.e. $\tau_v \hookleftarrow r_1(\tau_v)$ is the unique combinatorial morphism such that $r_1(\tau_v)$ has no flags. Consider the Deligne-Mumford stack $\mathcal{M}(\mathcal{C}'_{\phi(v)}/T, r_1(\tau_v))$. Since the fibers of $\pi'_{\phi(v)} : \mathcal{C}'_{\phi(v)} \to T$ are *convex varieties*, it follows from the proof of [Kontsevich95] theorem 1.3.2 that $\mathcal{M}(\mathcal{C}'_{\phi(v)}/T, r_1(\tau_v))$ is smooth of relative dimension $2d(v) - 2$ over T. Let $\mathcal{B}_v \to \mathcal{M}(\mathcal{C}'_{\phi(v)}/T, r_1(\tau_v))$ be the universal curve. For each $f \in \mathrm{Flag}(\tau_v)$, let $\mathcal{D}_f \subset \mathcal{B}_v$ be the base-change of the section $q'_{\phi(f)} : T \to \mathcal{C}'_{\phi(v)}$ by the evaluation morphism $\mathcal{B}_v \to \mathcal{C}'_{\phi(v)}$. It isn't hard to see that $\mathcal{D}_f \to \mathcal{M}(\mathcal{C}'_{\phi(v)}/T, r_1(\tau_v))$ is flat of relative dimension 0. And the canonical morphism $\mathcal{Y}_v \to \mathcal{M}(\mathcal{C}'_{\phi(v)}/T, r_1(\tau_v))$ is an open subscheme of the fiber product over all f of the morphism $D_f \to \mathcal{M}(\mathcal{C}'_{\phi(v)}/T, (\tau_v)_r)$. Thus it is flat of relative dimension 0. So we conclude that $c_v : \mathcal{Y}_v \to T$ is flat of relative dimension $2d(v) - 2$.

□

The following smoothness result will simplify some arguments later:

LEMMA 34. *The 1-morphism $F_2 : \mathcal{M}(X, \phi) \to \mathcal{N}(X, \tau_2)$ is a smooth 1-morphism of relative dimension $\delta(\phi)$. In particular, for each combinatorial datum ϕ, the stack $\mathcal{M}(\mathbb{P}^n, \phi)$ is a smooth Deligne-Mumford stack of dimension $dim(\mathbb{P}^n, \phi)$.*

PROOF. This is straightforward so we will only sketch the proof. By the usual fiber product argument one reduces to the problem: suppose that $\phi : \tau_{r+s}(de) \to \tau_{r+s}(e)$ is a combinatorial datum with $d(v) = d$ and let $\tau_{r+s}(e) \hookleftarrow \tau_r(d)$ be the combinatorial morphism which forgets the last s tails. Define the 1-morphism $G : \mathcal{M}(X, \phi) \to \mathcal{N}(X, \tau_r(e))$ to be the composite of $F_2 : \mathcal{M}(X, \phi) \to \mathcal{N}(X, \tau_{r+s}(e))$ with the 1-morphism $\mathcal{N}(X, \tau_{r+s}(e)) \to \mathcal{N}(X, \tau_r(e))$ induced by the combinatorial morphism. We need to prove that G is a smooth morphism.

Of course we have an analogous combinatorial morphism $\tau_{r+s}(de) \hookleftarrow \tau_r(de)$ and a combinatorial datum $\psi : \tau_r(de) \to \tau_r(e)$ such that G factors as the composite of $\mathcal{M}(X, \phi) \to \mathcal{M}(X, \psi)$ with $F_2 : \mathcal{M}(X, \psi) \to \mathcal{N}(X, \tau_r(e))$. Clearly the morphism $\mathcal{M}(X, \phi) \to \mathcal{M}(X, \psi)$ is just an open substack of the r-fold fiber product of the universal curve over $\mathcal{M}(X,\psi)$. Therefore $\mathcal{M}(X,\phi) \to \mathcal{M}(X,\psi)$ is smooth. So we are reduced to showing that $F_2 : \mathcal{M}(X,\phi) \to \mathcal{N}(X,\tau_r(e))$ is smooth of relative dimension $2d-2$.

Smoothness of F_2 is a local property on $\mathcal{M}(X,\phi)$. Let us denote the universal family over $\mathcal{N}(X,\tau_r(e))$ as $(\mathcal{C}', h', (q_i'))$. Now (étale) locally on $\mathcal{N}(X,\tau_r(e))$ we can find a trivialization $\mathcal{C}' \cong \mathbb{P}^1 \times \mathcal{N}(X,\tau_r(e))$, The sections (q_i') induce a 1-morphism $q' : \mathcal{N}(X,\tau_r(e)) \to (\mathbb{P}^1)^r)$. Now consider the open substack $\mathcal{M}_{0,r}(\mathbb{P}^1,d) \subset \overline{\mathcal{M}}_{0,r}(\mathbb{P}^1,d)$ which parametrizes stable maps with irreducible domain. There is an evaluation morphism $e_r : \mathcal{M}_{0,r}(\mathbb{P}^1,d) \to (\mathbb{P}^1)^r$. And clearly the trivialization above induces a local isomorphism of $\mathcal{M}(X,\phi)$ with an open substack of

$$\mathcal{N}(X,\tau_r(e)) \times_{q',(\mathbb{P}^1)^r,e_r} \mathcal{M}_{0,r}(\mathbb{P}^1,d)$$

such that F_2 corresponds to projection on the first factor. Thus we are reduced to proving that the 1-morphism $e_r : \mathcal{M}_{0,r}(\mathbb{P}^1,d) \to (\mathbb{P}^1)^r$ is smooth of relative dimension $2d-2$.

Suppose that $\kappa : (C,(q_1,\ldots,q_r)) \to \mathbb{P}^1$ is an element of $\mathcal{M}_{0,r}(\mathbb{P}^1,d)$. We partition the set of marked points into those points that are ramification points and those that are not: say $(q_1,\ldots,q_k)$ are ramification points and $(q_{k+1},\ldots,q_r)$ are not ramification points. We have a commutative diagram of 1-morphisms:

$$\begin{array}{ccc} \mathcal{M}_{0,r}(\mathbb{P}^1,d) & \xrightarrow{e_r} & (\mathbb{P}^1)^r \\ \downarrow & & \downarrow \\ \mathcal{M}_{0,k}(\mathbb{P}^1,d) & \xrightarrow{e_k} & (\mathbb{P}^1)^k \end{array}$$

where $\mathcal{M}_{0,r}(\mathbb{P}^1,d) \to \mathcal{M}_{0,k}(\mathbb{P}^1,k)$ is the 1-morphism which forgets the sections $q_{k+1},\ldots,q_r$ and where $(\mathbb{P}^1)^r \to (\mathbb{P}^1)^k$ is projection on the first k factors. And locally near our stable map, the induced morphism $\mathcal{M}_{0,r}(\mathbb{P}^1,d) \to \mathcal{M}_{0,k}(\mathbb{P}^1,d) \times_{(\mathbb{P}^1)^k} (\mathbb{P}^1)^r$ is unramified. So we are reduced to showing that $e_k : \mathcal{M}_{0,k}(\mathbb{P}^1,d) \to (\mathbb{P}^1)^k$ is smooth of relative dimension $2d-2$ near the point $\kappa' : (C,(q_1,\ldots,q_k)) \to \mathbb{P}^1$.

Let $q_1,\ldots,q_k,p_1,\ldots,p_l$ be the ramification points of κ' and let the corresponding ramification indices be $\nu_1,\ldots,\nu_k,\mu_1,\ldots,\mu_l$. The standard analytic description of the Hurwitz scheme $\mathcal{H}_d$ gives a local analytic isomorphism of a neighborhood of the point $k : C \to \mathbb{P}^1$ in $\mathcal{M}_{0,0}(\mathbb{P}^1,d)$ with the product

$$\Delta_1 \times \cdots \times \Delta_k \times \Delta_1' \times \cdots \times \Delta_l'.$$

Here $\Delta_i = \{(c_{i,1}, c_{i,2}, \ldots, c_{i,\nu_i})\}$ is a polydisk of dimension ν_i parametrizing divisors in the disk $\Delta \subset \mathbb{C}$ via

$$(c_{i,1}, \ldots, c_{i,\nu_i}) \mapsto \{t \in \Delta : g_i(c_i; t) = t^{\nu_i} + c_{i,1}t^{\nu_i - 1} + \cdots + c_{i,\nu_i} = 0\}$$

and similarly for the Δ'_j. Then we have a local analytic isomorphism of a neighborhood of $f : (C, (q_1, \ldots, q_k)) \to \mathbb{P}^1$ in $\mathcal{M}_{0,k}(X, d)$ with the product

$$(\Delta_1 \times \Delta) \times (\Delta_2 \times \Delta) \times \cdots \times (\Delta_k \times \Delta) \times \Delta'_1 \times \cdots \times \Delta'_l.$$

Here the factor $\Delta_i \times \Delta$ parametrizes pairs $((c_{i,1}, \ldots, c_{i,\nu_i}), t_i)$. And the evaluation map $e_k : \mathcal{M}_{0,k}(\mathbb{P}^1, d) \to (\mathbb{P}^1)^k$ is locally determined (up to translation of each of the factors $\mathbb{P}^1$) by sending a point in our product to the point

$$\begin{gathered}(g_1(c_1; t_1) = \left(t_1^{\nu_1} + c_{1,1}t_1^{\nu_1 - 1} + \cdots + c_{1,\nu_1}\right), \ldots, \\ g_k(c_k; t_k) = \left(t_k^{\nu_k} + c_{k,1}t_k^{\nu_k - 1} + \cdots + c_{k,\nu_k}\right).\end{gathered}$$

Since each of the partial derivatives $\frac{\partial f_i}{\partial c_{i,\nu_i}}$ is nonzero, we conclude that the fiber over $(0, 0, \ldots, 0)$ is smooth of dimension $\nu_1 + \cdots + \nu_k + \mu_1 + \cdots + \mu_l = 2d - 2$. This proves the lemma.

□

LEMMA 35. *Let $G_2 : \mathcal{M}(X, \phi) \to \mathcal{N}(X, r_2(\tau_2))$ denote the composition of $F_2 : \mathcal{M}(X, \phi) \to \mathcal{N}(X, \tau_2)$ with the canonical 1-morphism $r_2 : \mathcal{N}(X, \tau_2) \to \mathcal{N}(X, r_2(\tau_2))$. Then G_2 is a smooth morphism of relative dimension $n' - 3c$.*

PROOF. This follows from lemma 34 and lemma 20. □

The main result about strict ϕ-maps is the following:

THEOREM 36. *Let τ_1 be a stable genus 0 tree and let $\eta = ((C_v), (h_v : C_v \to X), (q_f))$ be a strict τ_1-map. Then there is a combinatorial datum $\tau_1 \xrightarrow{\phi} \tau_2$ and a strict ϕ-map ξ whose associated strict τ_1-map is η. Moreover ϕ and the strict ϕ-map are unique up to unique isomorphism.*

PROOF. In all cases uniqueness is obvious (it is included as part of the statement in order to simplify the induction argument). We only check existence.

By the same type of argument as in theorem 24, we may reduce to the case where $r_2(\tau_1) = \tau_1$. Now we perform leaf induction. The base case is when there is a single vertex $v \in \mathrm{Vertex}(\tau_1)$. Since $r_2(\tau_1) = \tau_1$, we know $\beta_1(v) > 0$. Let C'_v denote the normalization of $h_v(C_v)$ in X. There is a natural map $k_v : C_v \to C'_v$. This is necessarily finite, define

$d(v)$ to be the degree of this morphism. Define τ_2 to be the stable A-graph whose underlying tree is the tree of τ_1 and such that $\beta_2(v) = \frac{\beta_1(v)}{d(v)}$. Define $\phi : \tau_1 \to \tau_2$ to be the combinatorial morphism which is the identity map on underlying trees and with $d(v)$ defined as above. The natural morphism $h'_v : C'_v \to X$ and the sections $q'_f := k_v \circ q_f$ yield a naive τ_2-map $((C'_v), (h'_v), (q'_f))$. So the data of $((C_v), (h_v), (q_f))$, $((C'_v), (h'_v), (q'_f))$ and $k_v : C_v \to C'_v$ is a strict ϕ-map.

Now suppose that τ_1 has more than 1 vertex. First we fix some notation. Let $v_1 \in \mathrm{Vertex}(\tau_1)$ be a leaf. Since $r_2(\tau) = \tau$, we know $\beta_1(v_1) > 0$. Now define $\{f_1, f_2\}, v_2, \tau'$ and τ'' as in the proof of theorem 24. Denote by η' and η'' the strict τ'-map and strict τ''-map induced by the combinatorial morphism $\tau \hookrightarrow \tau'$ and $\tau \hookrightarrow \tau''$. There is a bifurcation depending on whether or not $h_{v_1}(C_{v_1}) = h_{v_2}(C_{v_2})$.

Suppose first that either $\beta_1(v_2) = 0$ or that $h_{v_1}(C_{v_1}) \neq h_{v_2}(C_{v_2})$. By induction there is a unique combinatorial datum $\phi'' : \tau'' \to \tau''_2$ and a strict ϕ''-map ξ'' whose associated strict τ''-map is η''. By the argument two paragraphs above there is a unique τ'_2 whose underlying tree is the tree of τ', a combinatorial datum $\phi' : \tau' \to \tau'_2$ which is the identity on trees, and a strict ϕ'-map ξ' whose associated strict τ'-map is η'. We define τ_2 to be the stable A-graph obtained by attaching τ'_2 and τ''_2 via $\phi''_W(f_2) = \partial\phi'_W(f_1)$. The two maps ϕ' and ϕ'' extend uniquely to a combinatorial datum $\phi : \tau_1 \to \tau_2$. And by our assumption that either $\beta_1(v_2) = 0$ or $h_{v_1}(C_{v_1}) \neq h_{v_2}(C_{v_2})$, we see that we can concatenate ξ' and ξ'' to form a strict ϕ-map whose associated strict τ_1-map is η.

Finally, suppose that $h_{v_1}(C_{v_1}) = h_{v_2}(C_{v_2})$. As in the last paragraph, there is a unique combinatorial datum $\phi'' : \tau'' \to \tau''_2$ and strict ϕ''-map ξ'' whose associated strict τ''-map is η''. Let $w_2 = \phi''_W(v_2)$ and let C'_{w_2} be the corresponding curve. Since $h'_{w_2} : C'_{w_2} \to h_{v_2}(C_{v_2})$ is birational, we conclude that C'_{w_2} is the normalization of $h_{v_2}(C_{v_2})$. Since C_{v_1} is normal, there is a natural morphism $k_{v_1} : C_{v_1} \to C'_{w_2}$ factoring h_{v_1}. This morphism is necessarily finite. We define the stable A-graph τ_2 to be the result of attaching to τ''_2 a flag for each $f \in \mathrm{Flag}(\tau'), f \neq f_1$ and such that $\partial f = w_2$ in τ_2. We extend ϕ'' to be a combinatorial datum $\phi : \tau_1 \to \tau_2$ by mapping v_1 to w_2, by mapping f_1 to $\phi''_W(f_2)$, by mapping each $f \in \mathrm{Flag}(\tau'), f \neq f_1$ to the same flag in τ_2 and by defining $d(v_1)$ to be the degree of the finite morphism k_{v_1}. We define a strict ϕ-map, ξ, by adding to ξ'' each of the sections $q'_f = k_{v_1} \circ q_f$, $f \in \mathrm{Flag}(\tau'), f \neq f_1$ and by defining $k_{v_1} : C_{v_1} \to C'_{w_2}$ as above. This clearly is a strict ϕ-map, and the theorem is proved.

□

Corollary 37. *The 1-morphism*

$$F_1 : \mathcal{M}(X, \phi) \to \mathcal{M}(X, \tau_1)$$

is a monomorphism whose image is locally closed in $|\mathcal{M}(X, \tau_1)|$. *The collection of all such images as* ϕ *varies over all combinatorial data* (τ_1, τ_2, ϕ) *forms a partition of* $|\mathcal{M}(X, \tau_1)|$ *into locally closed subsets.*

Proof. The only step that needs to be checked is to show that the image of F_1 is locally closed. Since an injective morphism between irreducible varieties has locally closed image, it suffices to check the following: for any two distinct irreducible components A and B of $\mathcal{M}(X, \phi)$, we have $\overline{F_1(A)} \cap B \subset A \cap B$. This is equivalent to the following: Let R be a discrete valuation ring with function field K and residue field k, let $\eta \in \mathcal{M}(X, \tau_1)(\text{Spec } R)$ be a family of strict τ_1-maps, suppose that $\eta_K \in F_1(A)$ and $\eta_k \in F_1(B)$, then $\eta_k \in F_1(A)$. etc. (finish this later)

□

We define the partition in corollary 37 to be the *CD decomposition* of $|\mathcal{M}(X, \tau_1)|$. As we allow τ_1 to vary among all stable A-graphs with class $\beta(\tau_1) = e$ and r tails, the images under the canonical morphism $\mathcal{M}(X, \tau_1) \to \overline{\mathcal{M}}(X, \tau_{0,r}(e)) = \overline{\mathcal{M}}_{0,r}(X, e)$ of the CD decompositions of the stacks $\mathcal{M}(X, \tau_1)$ form a partition of $|\overline{\mathcal{M}}_{0,r}(X, e)|$ into locally closed subsets which we call the *CD decomposition* of $|\overline{\mathcal{M}}_{0,r}(X, e)|$.

1.9. Flatness and Dimension Results

Remark For the remainder of the paper, $X_d \subset \mathbb{P}^n$ will be a fixed hypersurface of degree d which is general in the sense that theorem 3 and lemma 8 hold.

In this section and the next section we will state the basic properties of the CD decomposition which allow us to prove theorem 1. In this section we state the results having to do with the dimension of the components $\mathcal{M}(X, \phi)$ of the CD decomposition and flatness of the evaluation morphisms $e_f : \mathcal{M}(X, \phi) \to X$. We prove some simple lemmas which allow us to reduce the proofs of these results to a manageable statement.

Theorem 38. *For each combinatorial datum* ϕ, $\mathcal{M}(X, \phi)$ *has pure dimension* $dim(X, \phi)$.

Theorem 39. *For each combinatorial datum* ϕ, *the stack* $\mathcal{M}(X, \phi)$ *is a local complete intersection.*

THEOREM 40. *For each combinatorial datum $\phi : \tau_1 \to \tau_2$ and each flag $f \in Flag(\tau_1)$, the evaluation morphism $e_f : \mathcal{M}(X, \phi) \to X$ is flat of relative dimension $dim(X, \phi) - dim(X)$.*

Now we prove some lemmas which begin the proofs of the previous theorems.

LEMMA 41. *Theorem 38 implies theorem 39.*

PROOF. This is standard. We know by lemma 34 that $\mathcal{M}(\mathbb{P}^n, \phi)$ is smooth of dimension $\dim(\mathbb{P}^n, \phi)$. Let $\pi : \mathcal{C} \to \mathcal{M}(\mathbb{P}^n, \phi)$ denote the universal curve and let $h : \mathcal{C} \to \mathbb{P}^n$ denote the universal strict ϕ-map. The coherent sheaf $E(d) := \pi_* h^*(\mathcal{O}_{\mathbb{P}^n}(d))$ is easily seen to be locally free of rank $d\beta(\phi) + 1$ by cohomology and base change, [Mumford70]. A defining equation for X, $s \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ induces a section $t \in H^0(\mathcal{M}(\mathbb{P}^n, \phi), E(d))$. And $\mathcal{M}(X, \phi) \subset \mathcal{M}(\mathbb{P}^n, \phi)$ is simply the zero locus of t. But we exactly have $\dim(X, \phi) = \dim(\mathbb{P}^n, \phi) - (d\beta(\phi) + 1)$. Thus when $\dim(\mathcal{M}(X, \phi)) = \dim(X, \phi)$, we conclude that t is a regular section of a locally free sheaf on a smooth stack. Therefore $\mathcal{M}(X, \phi)$ is a local complete intersection. $\square$

LEMMA 42. *If theorem 38 is valid, then theorem 40 is equivalent to the following: For each combinatorial datum $\phi : \tau_1 \to \tau_2$ and each flag $f \in Flag(\tau_1)$, the evaluation morphism $e_f : \mathcal{M}(X, \phi) \to X$ has constant fiber dimension $dim(X, \phi) - dim(X)$.*

PROOF. By lemma 41, we conclude that $\mathcal{M}(X, \phi)$ is a local complete intersection. And X is a smooth scheme. Thus by theorem 23.1 [Matsumura86], e_f is flat iff e_f has constant fiber dimension. $\square$

LEMMA 43. *Theorem 38 and theorem 40 are implied by the following statement: For each stable A-graph τ, $\mathcal{N}(X, \tau)$ has pure dimension $dim(X, \tau)$ and for each flag $f \in Flag(\tau)$, the evaluation morphism $e_f : \mathcal{N}(X, \tau) \to X$ is flat of relative dimension $dim(X, \tau) - dim(X)$. For theorem 38 it even suffices to consider only stable A-graphs τ such that $r_1(\tau) = \tau$.*

PROOF. This follows immediately from theorem 33 and lemma 34. $\square$

LEMMA 44. *In the previous lemma, it suffices to consider τ such that $r_2(\tau) = \tau$.*

PROOF. This follows immediately from lemma 35. $\square$

LEMMA 45. *The statement in lemma 43 is equivalent to the following statement: For all nonnegative integers r and e, $\mathcal{N}(X, \tau_r(e))$ has dimension $dim(X, \tau_r(e))$ and for each flag $f \in Flag(\tau_r(e))$, the evaluation morphism $e_f : \mathcal{N}(X, \tau_r(e)) \to X$ has constant fiber dimension $dim(X, \tau_r(e)) - dim(X)$.*

PROOF. One direction in the equivalence is obvious. We prove that the statement above implies the statement in lemma 43.

By lemma 44, we may assume that $\tau = r_2(\tau)$. We perform leaf induction. If τ has a single vertex then it is already of the form $\tau_r(e)$ for some r and e, so there is nothing to prove. Suppose that τ has more than 1 vertex. Let $v_1 \in \text{Vertex}(\tau)$ be a leaf. Let $\{f_1, f_2\}$, v_2, τ', and τ'' be as in the proof of theorem 24 Let $e' : \mathcal{N}(X, \tau') \to X$, $e'' : \mathcal{N}(X, \tau'') \to X$ denote the evalution morphisms corresponding to f_1 and f_2. We have seen that $\mathcal{N}(X, \tau)$ is an open substack of the fiber product $\mathcal{N}(X, \tau') \times_{e',X,e''} \mathcal{N}(X, \tau'')$. Now τ' is of the form $\tau_r(e)$ for some r and e. By assumption $\dim(\mathcal{N}(X, \tau')) = \dim(X, \tau')$ and $e' : \mathcal{N}(X, \tau') \to X$ is flat of relative dimension $\dim(X, \tau') - \dim(X)$. Therefore the projection morphism $\mathcal{N}(X, \tau') \times_{e',X,e''} \mathcal{N}(X, \tau'') \to \mathcal{N}(X, \tau'')$ is flat of relative dimension $\dim(X, \tau') - \dim(X)$. By the induction assumption, we also know that $\dim(\mathcal{N}(X, \tau'')) = \dim(X, \tau'')$. By standard dimension theory (exercise II.3.22 of [Hartshorne77]), we conclude that $\mathcal{N}(X, \tau') \times_{e',X,e''} \mathcal{N}(X, \tau'')$ is pure dimensional of dimension $\dim(X, \tau'') + \dim(X, \tau') - \dim(X)$. But this is precisely $\dim(X, \tau)$. Moreover, suppose that $f \in \text{Flag}(\tau)$ is a flag of τ''. Then by the induction assumption $e_f : \mathcal{N}(X, \tau'') \to X$ is flat of relative dimension $\dim(X, \tau'') - \dim(X)$. Then $e_f : \mathcal{N}(X, \tau) \to X$ is the composition of the projection $\mathcal{N}(X, \tau') \times_{e',X,e''} \mathcal{N}(X, \tau'') \to \mathcal{N}(X, \tau'')$ with $e_f : \mathcal{N}(X, \tau'') \to X$. Since a composition of flat morphisms is flat, we conclude that $e_f : \mathcal{N}(X, \tau) \to X$ is flat of relative dimension $(\dim(X, \tau'') - \dim(X)) + (\dim(X, \tau) - \dim(X)) = \dim(X, \tau) - \dim(X)$. But since τ has at least two vertices, and so has at least two leaves, we see that we can always choose the leaf v_1 so that f is a flag in τ''. Thus we conclude that e_f is always flat of relative dimension $\dim(X, \tau) - \dim(X)$. So the lemma is proved by induction. $\square$

LEMMA 46. *In the statement in the last lemma, it suffices to consider only the stable A-graphs $\tau_1(e)$ and $\tau_0(e)$.*

PROOF. This is an obvious reduction. $\square$

1.10. Specializations

The theorems in the last section describe the dimensions and evaluation morphisms for the stacks $\mathcal{M}(X, \phi)$. In this section we will discuss how the components in the CD decomposition of $\overline{\mathcal{M}}_{0,r}(X, e)$ "fit together". We begin with a few definitions.

DEFINITION 47. *A stable A-graph τ is* nonlinear *if $\beta(\tau) \neq 1$ (i.e. $\mathcal{M}(X, \tau)$ does not parametrize lines). A stable A-graph τ is* very stable *if each $v \in Vertex(\tau_1)$ satisfies $\beta_1(f) > 0$. A combinatorial datum $\phi : \tau_1 \to \tau_2$ is* nice *if τ_2 is nonlinear and if τ_1 is very stable.*

DEFINITION 48. *A stable A-graph* τ *is* basic *if each* $v \in$ *Vertex*(τ_1) *satisfies* $\beta(v) = 1$. *A combinatorial datum* $\phi : \tau_1 \to \tau_2$ *is* basic *if* τ_1 *is basic.*

DEFINITION 49. *A combinatorial datum* $\phi : \tau_1 \to \tau_2$ *is* elementary *if* $d(v) = 1$ *for all* $v \in$ *Vertex*(τ_1). *An elementary combinatorial datum* $\phi : \tau_1 \to \tau_2$ *is* simple *if* ϕ *is an isomorphism of the underlying trees.*

LEMMA 50. *Let us assume theorem 38 holds. Let* τ *be a nice stable A-graph (whose canonical contraction is* $\tau \to \tau_r(e)$*) and let* $\phi : \tau \to \tau$ *be the nice, simple combinatorial datum which is the identity map on underlying trees. Then every irreducible component of* $\overline{\mathcal{M}}(X, \tau)$ *intersects* $\mathcal{M}(X, \phi)$. *Moreover, if* $\psi : \sigma_1 \to \sigma_2$ *is a combinatorial datum,* $\sigma_1 \to \tau$ *is a contraction which factors the canonical contraction* $\sigma_1 \to \tau_r(e)$*, and if the image of the induced morphism* $\mathcal{M}(X, \psi) \to \overline{\mathcal{M}}(X, \tau)$ *has codimension 1, then* ψ *is nonlinear and simple.*

PROOF. Recall that we have a CD decomposition of $\overline{\mathcal{M}}(X, \tau)$ which is a partition of $\overline{\mathcal{M}}(X, \tau)$ into a union of the locally closed subsets which are the images of all morphisms $\mathcal{M}(X, \psi) \to \overline{\mathcal{M}}(X, \tau)$. Here $\psi : \sigma_1 \to \sigma_2$ varies among combinatorial data such that the canonical contraction $\sigma_1 \to \tau_r(e)$ factors through a contraction $\sigma_1 \to \tau$ and where $\mathcal{M}(X, \psi) \to \overline{\mathcal{M}}(X, \tau)$ is the morphism associated to this contraction.

Every irreducible component of $\overline{\mathcal{M}}(X, \tau)$ is a disjoint union of its intersections with the locally closed subsets $\mathcal{M}(X, \psi)$ in the CD decomposition. Thus to show every irreducible component intersects $\mathcal{M}(X, \phi)$, it suffices to show that for each ψ other than ϕ, $\dim(X, \psi) < \dim(X, \phi) = \dim(X, \tau)$. Moreover, if $\dim(X, \psi) = \dim(X, \tau) - 1$ we will show that ψ is nice and simple.

First we will show that if $\psi : \sigma_1 \to \sigma_2$ is not simple, then the image of $\mathcal{M}(X, \psi)$ in $\overline{\mathcal{M}}(X, \sigma_1)$ has codimension at least 1. Now $\dim(X, \sigma_1)$ satisfies an obvious additivity with respect to decomposition into subgraphs: namely the rule

$$\dim(X, \tau) = \dim(X, \tau') + \dim(X, \tau'') - \dim(X)$$

which we have used before. Thus it suffices to consider the case where $\sigma_2 = \tau_{r'}(e')$ for some r' and e'. By definition the dimension $\dim(X, \sigma_1)$ is

$$\sum_{v \in \mathrm{Vertex}(\tau)} ((n + 1 - d)\beta_1(v)) + (\dim(X) - 3) + \#\mathrm{Tail}(\tau) - \#\mathrm{Edge}(\tau).$$

By assumption, each $\beta_1(v) = d(v)e$. And by definition the dimension $\dim(X, \psi)$ is

$$(n+1-d)e + (\dim(X) - 3) + \#\mathrm{Tail}(\tau) + \#\mathrm{Edge}(\tau) + \sum_{v \in \mathrm{Vertex}(\tau)} (2d(v) - 2).$$

Thus the difference $\dim(X, \sigma_1) - \dim(X, \psi)$ is simply

$$\sum_{v \in \mathrm{Vertex}(\tau)} (((n+1-d)e - 2)\, d(v) + 2) - (n+1-d)e - 2\#\mathrm{Edge}(\tau).$$

There are two cases to consider. The first case is when there is a single vertex. Then the sum above simplifies to $((n+1-d)e - 2)\,(d(v) - 1)$. Since $(n+1-d)e - 2 \geq n - 1 - d \geq 2$, we conclude the codimension is at least 2 unless $d(v) = 1$, i.e. unless $\psi : \sigma_1 \to \sigma_2$ is simple.

The second case is when there are at least two vertices. Now our inequalities $n \geq 6$, $d \leq \frac{n+1}{2}$ imply that $d \leq n-3$. Thus the term $(n+1-d)e-2$ is always positive. So the difference is minimized when all $d(v) = 1$ and when $e = 1$. In this case the difference reduces to

$$(\#\mathrm{Vertex}(\tau) - 1)\,(n+1-d) - 2\#\mathrm{Edge}(\tau).$$

But of course $\#\mathrm{Edge}(\tau) = \#\mathrm{Vertex}(\tau) - 1$, so the difference is the simple expression:

$$(\#\mathrm{Vertex}(\tau) - 1)\,(n-1-d).$$

If the number of vertices of τ is greater than 1, then we see the codimension is at least $n - 1 - d \geq 2$ since $d \leq n - 3$.

It is equally simple to prove that if $\sigma_1 \to \tau$ is a nontrivial contraction (i.e. is not an isomorphism on trees), then the codimension of $\mathcal{M}(X, \sigma_1)$ in $\mathcal{M}(X, \tau)$ is at least 1. Together with the last paragraph, this proves the lemma.

□

The following is the main result about specializations of the components of the CD decomposition.

THEOREM 51. *Let $\phi : \tau_1 \to \tau_2$ be a nice, simple combinatorial datum which is not basic. Let $\tau_1 \to \tau_r(e)$ be the canonical contraction and let $F_1' : \mathcal{M}(X, \phi) \to \overline{\mathcal{M}}_{0,r}(X, e)$ denote the composition of F_1 with the 1-morphism associated to the canonical contraction. Let $A \subset \mathcal{M}(X, \phi)$ be an irreducible component. There is a nice, simple combinatorial datum $\psi : \sigma_1 \to \sigma_2$ (possibly basic) such that σ_1 contracts to $\tau_r(e)$, and an irreducible component $B \subset \mathcal{M}(X, \psi)$ such that $\overline{F_1'(B)} \subset \overline{F_1'(A)}$ is a codimension 1 closed subset.*

As in the last section, we can reduce theorem 51 to a more manageable statement.

LEMMA 52. *Assuming theorem 38 and theorem 40 to be proved, in the proof of theorem 51 it suffices to restrict to ϕ such that $\tau_1 = \tau_2 = \tau_r(e)$ for some r and e.*

PROOF. Suppose we know theorem 38, theorem 40, and suppose we know theorem 51 whenever $\tau_1 = \tau_2 = \tau_r(e)$. Suppose now that $\phi : \tau \to \tau$ is a general combinatorial datum which is nice, simple and not basic (where $\phi = \mathrm{id}(\tau)$). Let $A \subset \mathcal{M}(X, \phi)$ be an irreducible component. Actually, to simplify things a bit, let's replace A by the maximal open subset of A which is also an open subset of $\mathcal{M}(X, \phi)$. We will prove theorem 51 for ϕ by leaf induction. For any leaf $v_1 \in \mathrm{Vertex}(\tau)$, let $\{f_1, f_2\}$, v_2, τ' and τ'' be as in the proof of theorem 24. Since τ is not basic, we can find a leaf v_1 such that τ'' is also not basic. Let $\tau'' \to \tau_{r''}(e'')$ be the canonical contraction. Let $\phi'' : \tau'' \to \tau''$ be the unique nice, simple combinatorial datum such that $\phi'' = \mathrm{id}_{\tau''}$. Let $A'' \subset \mathcal{M}(X, \phi'') \subset \mathcal{M}(X, \tau'')$ be the image of $A \subset \mathcal{M}(X, \phi) \subset \mathcal{M}(X, \tau)$ under the morphism $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \tau'')$ induced by the combinatorial morphism $\tau \hookleftarrow \tau''$. As we have seen before, the morphism $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \tau'')$ is flat. And $\mathcal{M}(X, \phi) \subset \mathcal{M}(X, \tau)$ is clearly an open substack. Thus A'' is an open subset of $\mathcal{M}(X, \tau'')$. It easily follows that $A'' \subset \mathcal{M}(X, \phi'')$ is a dense open subset of an irreducible component $\overline{A''} \subset \mathcal{M}(X, \phi'')$.

By the induction assumption, we can find a nice, simple combinatorial datum $\psi'' : \sigma'' \to \sigma''$ and an irreducible component $B'' \subset \mathcal{M}(X, \psi'')$ such that $\overline{F_1'}(B'') \subset \overline{F_1'}(A'')$ is a codimension 1 subset. Now let $\mathrm{Spec}\ \kappa \in B''$ be the generic point and let $\zeta_\kappa \in \mathcal{M}(X, \psi'')(\mathrm{Spec}\ \kappa)$ be the corresponding strict ψ''-map. For convenience sake, let's replace κ by its algebraic closure and let's replace ζ_κ by its pullback to $\mathrm{Spec}\ \kappa$. Similarly, let $\mathrm{Spec}\ K \in A''$ be the algebraic closure of the generic point and let ζ_K be the corresponding family. Since $F_1'(B) \subset \overline{F_1'}(A'')$, we can find a (henselian) discrete valuation ring R whose residue field is κ and whose field of fractions is K, and we can find a family $\xi = (C, h, (q_f))$ in $\overline{\mathcal{M}}_{0,r''}(X, e'')(\mathrm{Spec}\ R)$ whose special fiber ξ_κ is the image of ζ_κ in $F_1'(B'')$ and whose general fiber ξ_K is the image of ζ_K in $F_1'(A'')$. Consider the section $q_{f_2} : \mathrm{Spec}\ R \to C$ corresponding to the tail of $\tau_{r''}(e'')$ which on the general fiber ξ_K is the image of the section q_{f_2} of ζ_K. Let us also denote by f_2 the tail of σ'' such that q_{f_2} on ζ_κ maps to q_{f_2} on the special fiber ξ_κ. We define σ to be the stable A-graph obtained by attaching σ'' and τ' by demanding $\overline{f_1} = f_2, \overline{f_2} = f_1$, i.e. that $\{f_1, f_2\}$ is an edge of σ. Let $\psi : \sigma \to \sigma$ be the nice, simple combinatorial datum which is the identity map on trees. The canonical contraction of σ is $\sigma \to \tau_r(e)$, just as for τ. The claim is that there is an irreducible component $B \subset \mathcal{M}(X, \psi)$ such that $\overline{F_1'}(B) \subset \overline{F_1'}(A)$.

Let $p : \text{Spec } R \to X$ be the map obtained by composing $h : C \to X$ with the section $q_{f_2} : \text{Spec } R \to C$. By theorem 40, we know that p factors through the generic point of X, in particular we know p is flat. Consider the fiber product $S := \text{Spec } R \times_{p,X,e'} \mathcal{M}(X, \tau')$. Over S we have a τ-map (not necessarily strict) obtained from the pullback of the universal τ'-map and the pullback of ξ considered as a τ''-map (necessarily not strict). In other words we have a morphism $S \to \overline{\mathcal{M}}(X, \tau)$. Let T be the fiber product $T := S \times_{\overline{\mathcal{M}}(X,\tau)} A$. Then T is an open substack of S. Moreover, again by theorem 40, we know that the fiber of $e' : A \to X$ over the generic point of X is nonempty. Therefore the projection morphism $T \to \text{Spec } R$ is surjective. By construction the special fiber T_κ is in the image of the contraction morphism $\overline{\mathcal{M}}(X, \sigma) \to \overline{\mathcal{M}}(X, \tau)$. Let U be the fiber product $U := T \times_{\overline{\mathcal{M}}(X,\tau)} \overline{\mathcal{M}}(X, \sigma)$. By theorem 38 and theorem 40, it is easy to see that $U \to \overline{\mathcal{M}}(X, \sigma)$ maps dominantly to a union of irreducible components of $\overline{\mathcal{M}}(X, \sigma)$. By lemma 50, we know that each irreducible component of $\overline{\mathcal{M}}(X, \sigma)$ intersects $\mathcal{M}(X, \psi)$. Let $B \subset \mathcal{M}(X, \psi)$ be any irreducible component dominated by U. Then, by construction, $\overline{F_1'}(B) \subset \overline{F_1'}(A)$.

$\square$

1.11. Basic Components

The main result of this section is the following theorem.

THEOREM 53. *Let $\tau_1 \xrightarrow{\phi} \tau_2$ be a very stable, basic datum (i.e. τ_1 is a very stable, basic datum).*

1. *The Deligne-Mumford stack $\mathcal{M}(X, \phi)$ is an integral scheme over $\mathbb{C}$ of dimension $dim(X, \phi)$.*
2. *For each $f \in Flag(\tau_1)$ the evaluation map $e_f : \overline{\mathcal{M}}(X, \phi) \to X$ is faithfully flat and the general fiber is an integral scheme of dimension $dim(X, \phi) - dim(X)$.*
3. *For each contraction of stable genus 0 trees $\alpha : \tau_1 \to \sigma$ the image of $\overline{\mathcal{M}}(X, \phi)$ under the corresponding 1-morphism*
$$\overline{\mathcal{M}}(X, \alpha) : \overline{\mathcal{M}}(X, \tau_1) \to \overline{\mathcal{M}}(X, \sigma)$$
has nonempty intersection with the smooth locus of $\overline{\mathcal{M}}(X, \sigma)$.

PROOF. For a very stable, basic datum ϕ the 1-morphism $G_2 : \mathcal{M}(X, \phi) \to \mathcal{N}(X, r_2(\tau_2))$ is clearly an open immersion. So 1 and 2 above reduce to the following statement:

(1) The Deligne-Mumford stack $\mathcal{N}(X, r_2(\tau_2))$ is an integral scheme over $\mathbb{C}$ of dimension $\dim(X, r_2(\tau_2))$.
(2) For each $f \in \mathrm{Flag}(\tau_2)$ the evaluation map $e_f : \mathcal{N}(X, r_2(\tau_2)) \to X$ is faithfully flat and the (scheme-theoretic) fibers are integral schemes of dimension $\dim(X, r_2(\tau_2)) - \dim(X)$.

For notational convenience, let's replace τ_2 by $r_1(\tau_2)$ (so we don't have to keep writing $r_1(\tau_2)$).

We establish 1 and 2 by induction on $\#(\mathrm{Vertex}(\tau_2))$. If $\#\mathrm{Vertex}(\tau_2) = 0$, clearly there is nothing to prove. If $\#(\mathrm{Vertex}(\tau_2)) = 1$ and $\#(\mathrm{Flag}(\tau_2)) = r$ then we can identify $\mathcal{N}(X, \tau_2)$ with the r-fold fiber product

$$F_{0,1}(X) \times_{F_1(X)} \cdots \times_{F_1(X)} F_{0,1}(X).$$

This is a $(\mathbb{P}^1)^r$-bundle over the integral scheme $F_1(X)$ and so is integral of dimension $r + \dim(F_1(X)) = r + 2n - d - 3 = \dim(\tau_2)$. Also for each $i = 1, \ldots, r$, the evaluation map

$$e_{f_i} : \mathcal{N}(X, \tau_2) \to X$$

factors as

$$F_{0,1}(X) \times_{F_1(X)} \cdots \times_{F_1(X)} F_{0,1}(X) \xrightarrow{\mathrm{pr}_i} F_{0,1}(X) \to X.$$

Now pr_i is faithfully flat with integral fibers since it is a $(\mathbb{P}^1)^{r-1}$-bundle. And by theorem 9 the morphism $F_{0,1}(X) \to X$ is faithfully flat and the general fiber is integral. Thus the general fiber of $e_f : \mathcal{N}(X, \tau_2) \to X$ is integral. This establishes 1 and 2 when $\#(\mathrm{Vertex}(\tau_2)) = 1$.

Now assume that $\#(\mathrm{Vertex}(\tau_2)) = s > 1$. Let $v_1 \in \mathrm{Vertex}(\tau_2)$ be a leaf of τ_2. Since τ_2 is very stable, necessarily we have $\beta_2(v_1) > 0$. Let $\{f_1, f_2\}$, v_2, τ' and τ'' be as in the proof of theorem 24. The combinatorial morphisms $\tau_2 \hookleftarrow \tau'$ and $\tau_2 \hookleftarrow \tau''$ induce 1-morphisms $\mathcal{N}(X, \tau_2) \to \mathcal{N}(X, \tau')$ and $\mathcal{N}(X, \tau_2) \to \mathcal{N}(X, \tau'')$ respectively. The flags f_1, f_2 induce evaluation morphisms $e' : \mathcal{N}(X, \tau') \to X$ and $e'' : \mathcal{N}(X, \tau'') \to X$ respectively. It is clear from the definition that the following diagram is Cartesian:

$$\begin{array}{ccc} \mathcal{N}(X, \tau_2) & \longrightarrow & \mathcal{N}(X, \tau'') \\ \big\downarrow & & \big\downarrow \\ \mathcal{N}(X, \tau') & \longrightarrow & X \end{array}$$

By the induction assumption $\mathcal{N}(X, \tau')$ and $\mathcal{N}(X, \tau'')$ are schemes over $\mathbb{C}$. Thus we conclude that $\mathcal{N}(X, \tau_2)$ is also a scheme over $\mathbb{C}$ By the induction assumption $\mathcal{N}(X, \tau'')$ is integral of dimension $\dim(X, \tau'')$. By the previous paragraph $\mathcal{N}(X, \tau') \to X$ is faithfully flat and the general fiber is integral of relative dimension $\dim(X,\tau') - \dim(X)$. Thus $\mathcal{N}(X,\tau_2) \to \mathcal{N}(X,\tau'')$ is faithfully flat of relative dimension $\dim(X,\tau') - \dim(X)$. Since e' is surjective, we conclude that the general fiber of $\mathcal{N}(X,\tau_2) \to \mathcal{N}(X,\tau'')$ is integral of dimension $\dim(X,\tau'') - \dim(X)$. Thus $\mathcal{N}(X,\tau_2)$ is integral of dimension $\dim(X,\tau') + \dim(X,\tau'') - \dim(X) = \dim(X,\tau_2)$. This proves 1.

Now we consider $f \in \mathrm{Flag}(\tau_2)$. Since $\#(\mathrm{Vertex}(\tau_2)) > 1$, there is a leaf v_1 such that $\partial f \neq v_1$. Thus, without loss of generality we may assume that $f \in \mathrm{Flag}(\tau'')$. So the evaluation morphism corresponding to f factors as

$$\mathcal{N}(X,\tau_2) \to \mathcal{N}(X,\tau'') \xrightarrow{e_f} X.$$

By assumption $\mathcal{N}(X,\tau'') \xrightarrow{e_f} X$ is faithfully flat and the general fiber is integral of dimension $\dim(X,\tau'') - \dim(X)$. And, as established in the last paragraph, $\mathcal{N}(X,\tau_2) \to \mathcal{N}(X,\tau'')$ is faithfully flat and the general fiber is integral of dimension $\dim(X,\tau') - \dim(X)$. Thus the general fiber of the composite is integral of dimension $\dim(X,\tau'') - \dim(X) + \dim(X,\tau') - \dim(X) = \dim(X,\tau_2) - \dim(X)$. This proves 2.

To finish the proof of 1 and 2 in the statement of the theorem, we should prove that the image of $\mathcal{M}(X,\phi)$ in $\mathcal{N}(X,\tau_2)$ is nonempty. This follows easily by an inductive argument similar to the ones above and is left to the reader.

The proof of 3 is standard deformation theory. Given a strict ϕ-map ξ, the obstruction group to deformations of the corresponding stable map in $\overline{\mathcal{M}}(X,\sigma)$ form a subgroup of the obstruction group to deformations of the corresponding stable map in $\overline{\mathcal{M}}_{0,r}(X,d)$. And, by section 1.3.2 of [Kontsevich95], this obstruction group vanishes if $H^1(C,h^*T_X)$ vanishes. We will show that for ξ a general point of $\mathcal{M}(X,\phi)$, $H^1(C,h^*T_X)$ vanishes. The proof is by induction on $\#\mathrm{Vertex}(\tau_1)$. If τ_1 has only one vertex, then we are reduced to showing that the normal bundle of the line $h(C)$ in X is generated by global sections. This follows from lemma 8.

Suppose that $\#\mathrm{Vertex}(\tau_1) > 1$. Let v_1 be a leaf of τ_1 and let $v_2, \{f_1, f_2\}$, τ', and τ'' be as in the proof of theorem 24. Denote by C' (resp. C'') the union of those components of C corresponding to vertices of τ' (resp. τ''). Let $q_1 \in C'$ denote the marked point corresponding to f_1. Then we have a short exact sequence:

$$H^1(C',(h^*T_X)') \longrightarrow H^1(C,h^*T_X) \longrightarrow H^1(C'',h^*T_X).$$

Here $(h^*T_X)'$ is the subsheaf of the restriction of h^*T_X to C' such that the image of $(h^*T_X)'$ in the normal bundle N of $h(C_{\partial f})$ in X is exactly $N(-q_1)$. By induction we know that $H^1(C'', h^*T_X) = \{0\}$. So the only potential nonzero contribution to $H^1(C, h^*T_X)$ is $H^1(C_{\partial f}, N(-q_f))$. But we see by lemma 8, that for general choice of ξ, we have

$$H^1(C_{\partial f_1}, N(-q_f)) = H^1(C_{\partial f_1}, \mathcal{O}(-1)^{\oplus d-1} \oplus \mathcal{O}^{n-d-1}) = \{0\}.$$

$\square$

COROLLARY 54. *For each nice, simple basic datum ϕ there is a unique irreducible component M_ϕ of $\overline{\mathcal{M}}_{0,r}(X,e)$ such that M_ϕ contains the image of $\mathcal{M}(X,\phi)$. Moreover M_ϕ is generically smooth of dimension $(n+1-d)e + (n-4) + r$.*

PROOF. The existence of a unique irreducible component M_ϕ containing $\overline{\mathcal{M}}_{0,r}(X,e)$ follows from theorem 53 parts 1 and 2. That M_ϕ is generically smooth of dimension $(n+1-d)e + (n-4)$ follows from part 3. $\square$

1.12. Almost Basic Components

In this section we will prove that all of the irreducible components M_ϕ of $\overline{\mathcal{M}}_{0,r}(X,e)$ from the previous section are in fact equal. We begin by proving that $\overline{\mathcal{M}}_{0,0}(X,2)$ is irreducible. The proof illustrates the basic idea of the proof of theorem 1, but is much less technical.

DEFINITION 55. *A stable map $(C, h, (q_1, \dots, q_r))$ in $\overline{\mathcal{M}}_{0,r}(X,e)$ is* unobstructed *if $H^1(C, h^*T_X)$ is trivial. We say $\overline{\mathcal{M}}_{0,r}(X,e)$ is* generically unobstructed *if a general stable map is unobstructed.*

PROPOSITION 56. *The stack $\overline{\mathcal{M}}_{0,0}(X,2)$ is irreducible, generically unobstructed and has dimension $3n-2d-2$.*

PROOF. First we consider the *boundary* of $\overline{\mathcal{M}}_{0,0}(X,2)$. Let τ denote the stable genus 0 tree whose modular graph is λ_2 and such that $\alpha(u_0) = \alpha(u_1) = 1$. Let ϕ_1 be the combinatorial datum $(\tau \xrightarrow{\text{id}} \tau)$. Then $\mathcal{M}(X,\phi_1)$ parametrizes stable maps whose image is a pair of lines in X. Now ϕ_1 is a basic datum. Therefore by theorem 53 we see that $\mathcal{M}(X,\phi_1)$ is irreducible of dimension $3n-2d-3$. Moreover the general point of the image of $\mathcal{M}(X,\phi_1)$ in $\overline{\mathcal{M}}_{0,0}(X,2)$ is an unobstructed point.

For the next boundary component, define τ_2 to be the stable genus 0 tree whose modular graph is λ_1 with $\alpha_2(u_0) = 1$. Let τ_1 be the stable genus 0 tree whose modular graph is λ_2 with $\alpha_1(u_0) = \alpha_1(u_1) = 1$. And let $\phi_2 : \tau_1 \to \tau_2$ be the obvious combinatorial datum (in fact there is only one combinatorial datum $\tau_1 \to \tau_2$). Then $\mathcal{M}(X,\phi_2)$ parametrizes stable maps whose domain curve is reducible and whose image is just a line. Again this is a basic datum. So by theorem 53 we see that $\mathcal{M}(X, \phi_2)$ is irreducible. The dimension works out to be $2n - d - 2$.

For the final boundary component, define τ_2 to be the stable genus 0 tree whose modular graph is λ_0 with $\alpha_2(u_0) = 1$. Let τ_1 be the stable genus 0 tree whose modular graph is λ_0 with $\alpha_1(u_0) = 2$. Let $\phi_3 : \tau_1 \to \tau_2$ be the obvious combinatorial datum (again there is a unique combinatorial datum $\tau_1 \to \tau_2$. Now $\mathcal{M}(X, \phi_3)$ parametrizes stable maps whose domain curve is irreducible and which map 2-to-1 to a line in X. By lemma 34 and theorem 53 we see that $\mathcal{M}(X, \phi_3)$ is irreducible of dimension $2n - d - 1$.

Let $M \subset \overline{\mathcal{M}}_{0,0}(X, 2)$ be an irreducible component. Deformation theory (theorem I.2.15 of [Kollár96]) yields an expected dimension of $(n + 1 - d)2 + (n - 4)$, which is always a lower bound for the actual dimension: $\dim(M) \geq 3n - 2d - 2$. Moreover, if a general point of M is unobstructed, then this inequality is an equality. By assumption $d \leq n - 3$ so that $2n - d - 1 \leq 3n - 2d - 4$. Thus the intersection of M with the boundary components $\mathcal{M}(X, \phi_2)$ and $\mathcal{M}(X, \phi_3)$ both have codimension at least 2. We will show that M contains the boundary component $\mathcal{M}(X, \phi_1)$.

Define $Z \subset X \times X$ to be the set of pairs (x_1, x_2) such that there is a stable map $(f : C \to X)$ in M with $x_1, x_2 \in f(C)$. Denote by $r \leq 2n - 2$ the dimension of Z. By dimension theory (exercise II.3.22 of [Hartshorne77]), for a general point $(x_1, x_2) \in Z$ the substack $M_{x_1,x_2} \subset M$ parametrizing $(f : C \to X)$ with $x_1, x_2 \in f(X)$ has dimension

$$\dim(M_{x_1,x_2}) = \dim(Z) + 2 - r \geq (3n - 2d - 2) + 2 - (2n - 2) = n - 2d + 2.$$

By assumption, $d \leq \frac{n+1}{2}$ so that $n - 2d + 2 \geq 1$. So the fiber of Z over this point is a closed substack of dimension at least 1. Now by section 4.3 of [Fulton-Pandharipande96], we know the coarse moduli space of M is a *projective scheme*. And the intersections of M with $\mathcal{M}(X, \phi_2)$, $\mathcal{M}(X, \phi_3)$ each have codimension at least 2. Therefore by choosing the pair (x_1, x_2) to be general, we can find a complete curve in the coarse scheme of M contained in the image of the fiber and which intersects neither $\mathcal{M}(X, \phi_2)$ nor $\mathcal{M}(X, \phi_3)$. Suppose by way of contradiction that the inersection of the fiber and $\mathcal{M}(X, \phi_1)$ has codimension at least 2. Then we can find a complete curve in the coarse scheme of M which doesn't intersect any of the boundary components. We can find a (possibly ramified) finite cover B of this curve such that the morphism of B to the coarse scheme of M

factors through a 1-morphism $\eta : B \to M_{x_1,x_2}$ and which intersects neither $\mathcal{M}(X, \phi_1)$, $\mathcal{M}(X, \phi_2)$ nor $\mathcal{M}(X, \phi_3)$.

Now the 1-morphism η is really just a family of stable maps

$$\eta = (\pi : \mathcal{C} \to B, h : \mathcal{C} \to X) .$$

Since the image of B is disjoint from $\mathcal{M}(X, \phi_2)$ and $\mathcal{M}(X, \phi_3)$, we see that each fiber of π maps isomorphically to its image. Therefore the preimage of the set $\{x_1, x_2\}$ is a divisor in $\mathcal{C}$ which is an étale 2-to-1 cover of B. By performing a base-change of B by a connected component of this 2-to-1 cover we may assume that the preimage of $\{x_1, x_2\}$ is in fact the union of two sections $q_1 : B \to \mathcal{C}$ and $q_2 : B \to \mathcal{C}$. But now by corollary II.5.5.2 of [Kollár96], we conclude that the image of B intersects $\mathcal{M}(X, \phi_1)$, $\mathcal{M}(X, \phi_2)$ or $\mathcal{M}(X, \phi_3)$. This contradiction proves that the intersection of the fiber with $\mathcal{M}(X, \phi_1)$ is codimension 1. Since this holds for a general fiber, we conclude that the intersection of $\mathcal{M}(X, \phi_1)$ with M has codimension 1. Since $\dim(M) \geq 3n-2d-2$, and since $\mathcal{M}(X, \phi_1)$ is irreducible, we conclude that M contains $\mathcal{M}(X, \phi_1)$ and that $\dim(M) = 3n - 2d - 2$.

Since a general point in $p \in \mathcal{M}(X, \phi_1)$ is an unobstructed point of $\overline{\mathcal{M}}_{0,0}(X, 2)$, we conclude that through a general point p there is a unique irreducible component of $\overline{\mathcal{M}}_{0,0}(X, 2)$. By the last paragraph, *every* irreducible component of $\overline{\mathcal{M}}_{0,0}(X, 2)$ contains *every* point of $\mathcal{M}(X, \phi_1)$, in particular it contains a general point p. Therefore there is a unique irreducible component of $\overline{\mathcal{M}}_{0,0}(X, 2)$, it has dimension $3n - 2d - 2$, and a general point is unobstructed. $\square$

COROLLARY 57. *Let τ be the stable genus 0 tree with underlying graph λ_1 such that $\alpha(u_0) = 2$. Let ϕ be the combinatorial datum $\tau \xrightarrow{id} \tau$. The Deligne-Mumford stack $\mathcal{M}(X, \phi)$ is integral. Moreover the evalutation map $\mathcal{M}(X, \phi) \to X$ has dense image.*

PROOF. Since $\mathcal{M}(X, \phi) \to \overline{\mathcal{M}}_{0,0}(X, 2)$ has dense image and smooth, connected fibers, it follows from lemma 56 that $\mathcal{M}(X, \phi)$ is irreducible. The fact that $\mathcal{M}(X, \phi) \to X$ has dense image follows from the proof of lemma 56: a postiori we see the subvariety $Z \subset X \times X$ has dimension $2n - 2$ and so is dense. $\square$

DEFINITION 58. *We call a nice, simple combinatorial datum ϕ an* almost basic datum *if there exists a vertex $w \in Vertex(\tau_1)$ such that $\alpha_1(w) = 2$ and for each vertex $v \in Vertex(\tau_1)$ with $v \neq w$ we have $\alpha_1(v) \leq 1$. We call w the* distinguished vertex *of ϕ.*

PROPOSITION 59. *Let $\tau_1 \xrightarrow{\phi} \tau_2$ be an almost basic datum.*

(1) *The Deligne-Mumford stack $\mathcal{M}(X, \phi)$ is integral.*
(2) *For each flag $f \in Flag(\tau_1)$, the evaluation map $e_f : \mathcal{M}(X, \phi) \to X$ has dense image.*

PROOF. Using lemma 24 and lemma 35, we see that it suffices to prove this for the stacks $\mathcal{N}(X, \tau)$ for τ a stable genus 0 tree which is either basic or almost basic. The basic case is covered by theorem 53. So we may assume that τ is almost basic, i.e. there is a unique vertex $w \in \text{Vertex}(\tau)$ such that $\beta(w) = 2$ and for every other vertex $v \in \text{Vertex}(\tau)$, we have $\beta(v) = 1$.

We prove the result by induction on the number of vertices of τ. If w is the only vertex of τ, the result follows from proposition 56 and corollary 57. Thus we may suppose that τ has at least two vertices. We perform leaf induction. Suppose that v_1 is a leaf of τ such that $\beta(v_1) = 1$. Let $\{f_1, f_2\}$, v_2, τ' and τ'' be as in the proof of theorem 24. The combinatorial morphisms $\tau \hookleftarrow \tau'$ and $\tau \hookleftarrow \tau''$ induce 1-morphisms $\mathcal{M}(X, \tau) \hookleftarrow \mathcal{M}(X, \tau')$ and $\mathcal{M}(X, \tau'')$. Let $e' : \mathcal{M}(X, \tau') \to X$ and $e'' : \mathcal{M}(X, \tau'') \to X$ be the 1-morphisms obtained by evaluation at f_1 and f_2 respectively. Then $\mathcal{M}(X, \tau)$ is identified with an open substack of the fiber product $\mathcal{M}(X, \tau') \times_{e',X,e''} \mathcal{M}(X, \tau'')$. Now by proposition 53 the 1-morphism $e' : \mathcal{M}(X, \tau') \to X$ is flat and the general fiber is irreducible. And by the induction assumption $e'' : \mathcal{M}(X, \tau'') \to X$ has dense image. Therefore the projection morphism $\mathcal{M}(X, \tau') \times_{e',X,e''} \mathcal{M}(X, \tau'') \to \mathcal{M}(X, \tau'')$ is flat and the general fiber is irreducible. But by the induction assumption, $\mathcal{M}(X, \tau'')$ is irreducible. So we conclude that $\mathcal{M}(X, \tau) \times_{e',X,e''} \mathcal{M}(X, \tau'')$ is irreducible. Since $\mathcal{M}(X, \tau)$ is an open substack, we conclude that $\mathcal{M}(X, \tau)$ is also irreducible.

It only remains to prove that for each flag f, the evaluation morphism $\mathcal{M}(X, \tau) \to X$ has dense image. For this we appeal to lemma 56 which shows that the space of conics is irreducible. From this it follows that $\mathcal{M}(X, \tau)$ is dense in the locally closed subvariety of $\overline{\mathcal{M}}(X, \tau)$ parametrizing stable maps for which no component is contracted. So it suffices to show that the image in X of this locally closed subvariety is dense. But it is clear that by replacing w by a pair of vertices w_1, w_2 with $\beta(w_1) = \beta(w_2) = 1$, we can find a basic stable graph σ and a contraction $\sigma \to \tau$ so that the corresponding 1-morphism $\mathcal{M}(X, \sigma) \to \overline{\mathcal{M}}(X, \tau)$ has image in our locally closed subvariety. By theorem 53 we know that the evaluation map $\mathcal{M}(X, \sigma) \to X$ induced by f is flat, and so has dense image. Thus it follows that $\mathcal{M}(X, \tau) \to X$ has dense image.

□

THEOREM 60. *For each r and e there is an irreducible component $M \subset \overline{\mathcal{M}}_{0,r}(X,e)$ such that for every nice, simple, basic datum ϕ whose canonical contraction is $\tau_1 \to \tau_r(e)$, the image of $\mathcal{M}(X,\phi) \to \overline{\mathcal{M}}_{0,r}(X,e)$ is contained in M. We call M the* distinguished component *of $\overline{\mathcal{M}}_{0,r}(X,e)$.*

PROOF. Up to corollary 54, we need to show that for all nice, simple, basic data ϕ contracting to $\tau_r(e)$, the irreducible components M_ϕ are equal. First of all, it is clear that we may reduce to the case that $r = 0$, i.e. each datum ϕ has no tails. So we will assume that $r = 0$.

This is a simple combinatorial argument. Suppose that ϕ_1 and ϕ_2 are two basic data and suppose that ψ is an almost basic datum such that we have contractions $\phi_1 \to \psi$ and $\phi_2 \to \psi$. Then the images of $\mathcal{M}(X,\phi_1)$ and $\mathcal{M}(X,\phi_2)$ in $\overline{\mathcal{M}}_{0,0}(X,e)$ are contained in the closure of the image of $\mathcal{M}(X,\psi)$. By proposition 59, $\mathcal{M}(X,\psi)$ is irreducible. Thus we can find an irreducible component M of $\overline{\mathcal{M}}_{0,0}(X,e)$ which contains the image of $\mathcal{M}(X,\psi)$. Then M also contains the images of $\mathcal{M}(X,\phi_1)$ and $\mathcal{M}(X,\phi_2)$. So we conclude that $M = M_{\phi_1}$ and also $M = M_{\phi_2}$, i.e. $M_{\phi_1} = M_{\phi_2}$. So we are reduced to the combinatorial statement that for any two nice, simple, basic data there is a sequence of nice, simple, basic data $\phi_1, \ldots, \phi_n$ (with ϕ_1 and ϕ_n the original two data), and a sequence of almost basic data $\psi_1, \ldots, \psi_{n-1}$ such that we have contractions $\psi_i \to \phi_i$ and $\psi_i \to \phi_{i+1}$. In case such sequences exist, let's say that ϕ_1 and ϕ_n are *linked*. Also, let's say that ϕ_i and ϕ_{i+1} are *directly linked*.

Let us define a tree τ to be a *chain* if every vertex has valence at most 2 (i.e. each vertex is attached to either 1 or 2 edges). To prove that any two nice, simple, basic data are linked, it suffices to show that all the nice, simple, basic data are linked to the unique basic datum σ whose underlying tree is a chain. Let us define the length of a nice, simple, basic datum τ to be the maximum of the number of vertices of all subgraphs of τ which are chains. We will prove that τ is linked to σ by downward induction on the length of τ.

If the length of τ is equal to the length of σ then clearly $\tau = \sigma$. Thus suppose that τ has length less than σ. We will prove that τ is directly linked to a nice, simple, basic datum σ' whose length is 1 greater. Then the result will follow by induction. Let τ_1 be a chain in τ of maximal length. Since the length of τ is less than the length of σ, we know that there is a vertex $v_1 \in \text{Vertex}(\tau_1)$ whose valence in τ_1 is 2 but whose valence in τ is at least 3. Let $\{f_1, f_2\}$ be an edge in τ such that $\partial f_1 = v_1$ and such that f_1 is not a flag of τ_1. Define $v_2 = \partial f_2$. Let τ' be the maximal (possibly disconnected) subgraph of τ whose set of vertices is $\text{Vertex}(\tau') = \text{Vertex}(\tau) - \{v_1, v_2\}$. We construct an almost basic datum τ'' as follows: The set of vertices is $\text{Vertex}(\tau'') = \text{Vertex}(\tau') \cup \{w\}$, where w is the distinguished vertex. Also τ' is identified with the maximal subgraph of τ'' whose set of vertices is $\text{Vertex}(\tau')$. Finally, for each tail $f \in \text{Tail}(\tau')$, there is an edge $\{f, \overline{f}\}$ such that $\partial \overline{f} = w$. There is an obvious contraction of τ to τ''. We construct the nice, simple, basic datum σ' as follows: The set of vertices is just $\text{Vertex}(\sigma) = \text{Vertex}(\tau)$. Let $\{g_1, h_1\}, \{g_2, h_2\}$ be the two edges in τ_1 attached to v_1 and such that $\partial h_1 = v_1, \partial g_2 = v_1$. The set of flags of σ is defined to be $\text{Flag}(\sigma) = \text{Flag}(\tau)$. And the operation $f \mapsto \overline{f}$ is the same in σ' as in τ. For all flags f other than g_2, we define ∂f to be as in τ. And we define $\partial g_2 = v_2$. Clearly σ' also contracts to τ'' in an obvious manner. And there is a chain in σ' whose set of vertices is $\text{Vertex}(\tau_1) \cup \{v_2\}$. Thus σ' has greater length than τ and τ is directly linked to σ'. This proves the theorem.

$\square$

1.13. Bend-and-break

Together theorem 51 and theorem 60 imply that $M \subset \overline{\mathcal{M}}_{0,r}(X, e)$ is the unique irreducible component of $\overline{\mathcal{M}}_{0,r}(X, e)$, i.e. $\overline{\mathcal{M}}_{0,r}(X, e)$ is irreducible. Moreover theorem 38 and theorem 39 imply that $\overline{\mathcal{M}}_{0,r}(X, e)$ is a local complete intersection stack of dimension $(n + 1 - d)e + (n - 4) + r$.

Using the lemmas from the sections on dimension, flatness, and specialization properties of the CD decomposition, all that remains to finish the proofs of theorem 38, theorem 40 and theorem 51 is to prove that for each $\tau_r(e)$ and corresponding nice, simple combinatorial datum $\phi : \tau_r(e) \to \tau_r(e)$

1. $\mathcal{M}(X, \phi)$ has dimension $\dim(X, \phi) = (n + 1 - d)e + (n - 4) + r$,
2. each evaluation map $e : \mathcal{M}(X, \phi) \to X$ has constant fiber dimension $\dim(X, \phi) - \dim(X)$, and
3. for each irreducible component $A \subset \mathcal{M}(X, \phi)$ there is a nice, simple combinatorial datum $\psi : \sigma_1 \to \sigma_2$ whose canonical contraction is $\sigma_1 \to \tau_r(e)$, and an irreducible component $B \subset \mathcal{M}(X, \psi)$ such that $\overline{F_1'}(B) \subset \overline{F_1'}(A)$ is a codimension 1 subvariety.

Of course proving this for $\tau_r(e)$ is clearly equivalent to proving this for $\tau_1(e)$. And when $e = 2$, this is the content of proposition 56 (along with its proof). So we may suppose that $e \geq 3$. Moreover, having reduced to the case $\tau_1(e)$, we are free to prove the result instead for $\tau_2(e)$ (our reason for doing this will become clear). Let $\phi : \tau_2(e) \to \tau_2(e)$ be the unique nice, simple datum which is the identity map on underlying trees. We will prove 1, 2, and 3 above for each ϕ by induction on e (where $e = 2$ is the base case).

Suppose that $A \subset \mathcal{M}(X, \phi)$ is an irreducible component. Without loss of generality, identify A as an open subset of an irreducible component of $\overline{\mathcal{M}}_{0,2}(X, e)$. Let $E : \overline{\mathcal{M}}_{0,2}(X, e) \to$X$\times X$ denote the evaluation map coming from the two flags f_1, f_2 of $\tau_2(e)$. Let $Y \subset X \times X$ denote the image of $\overline{A}$ under E. Let $(p, q) \in Y$ be any point such that $p \neq q$ (since the stable maps parametrized by A are non-constant, such a pair exists). Now the virtual dimension $\dim(X, \phi)$ is always a lower bound for the dimension of A. Thus $\dim(A) \geq \dim(X, \phi) = (n + 1 - d)e + (n - 2)$. So the dimension of the fiber $E^{-1}(p, q) \subset \overline{A}$ is at least

$$\dim(A) - \dim(Y) \geq (n + 1 - d)e + (n - 2) - (2n - 2) =$$

$$(n + 1 - d)e - n \geq (n + 1 - d)3 - n = 1 + (2n + 2 - 3d).$$

Since $d \leq \frac{n+1}{2}$, certainly $d \leq \frac{2(n+1)}{3}$. We conclude that the dimension of every irreducible component of the fiber of $E^{-1}(p, q)$ is at least 1.

Let $D \subset E^{-1}(p, q)$ be any irreducible component. The claim is that $D \cap (\overline{A} - A) \subset D$ is a codimension 1 subvariety. Now by section 4.3 of [Fulton-Pandharipande96], we know that the coarse moduli scheme of $\overline{\mathcal{M}}_{0,2}(X, e)$ is *projective*, thus the coarse moduli scheme of D (with the reduced induced structure) is projective. Therefore, to establish that $D \cap (\overline{A} - A) \subset D$ is codimension 1, it suffices to establish that for any complete curve $K \subset D$, $K \cap (\overline{A} - A) \neq \emptyset$. Suppose the contrary, i.e. suppose that there is a complete curve $K \subset D \cap A$. Now after replacing K by a ramified cover of K, we may assume that K actually factors through the stack of D and not just the coarse moduli scheme of D. We may also assume that K is smooth. Let $\pi : C \to K$ be the pullback to K of the universal curve, let $s_1, s_2 : K \to C$ be the two markings and let $h : C \to X$ be the map to X. Since the image of K is contained in A, π is smooth. Now after again replacing K by a ramified cover of K, we may suppose that there is a third section $s_3 : K \to C$ such that $h(s_1(K)), h(s_2(K))$ and $h(s_3(K))$ are disjoint: simply take any hyperplane section H of X which contains neither p nor q and replace K by the normalization of the multisection $h^{-1}(H)$ of π. Since any rational curve with three distinct marked points is canonically isomorphic to $(\mathbb{P}^1, 0, 1, \infty)$, we may identify $(\pi : C \to K, s_1, s_2, s_3)$ with $(\pi_1 : K \times \mathbb{P}^1, t_0, t_1, t_\infty)$ where $t_i(x) = (x, i) \in K \times \mathbb{P}^1$ for all $x \in K$. But then the morphism $h : \mathbb{P}^1 \times K \to X$ along with the sections t_0, t_1 satisfies the hypothesis of a version of Mori's bend-and-break lemma:

Theorem 61 (Corollary II.5.5, (Kollár95)). *Let C be an irreducible, proper and smooth curve and X a proper variety. Let $p_1, \ldots, p_k \in C$ be k distinct points $g : \{p_1, \ldots, p_k\} \to X$ a morphism (i.e. a choice of k not necessarily distinct points of X).*

Assume that there is a smooth, irreducible, proper curve B, an open set $B^0 \subset B$ and a morphism

$$[h^0 : C \times B^0 \to X \times B^0] \in Hom(C, X, g)(B^0)$$

such that $h^0(C \times B^0)$ and $p_X \circ h^0(c \times B^0)$ are one dimensional for some $b \in B^0$ and $c \in C$ where $p_X : X \times B \to X$ is the natural projection.

There is a unique normal compactification $S \supset C \times B^0$ such that h^0 extends to a finite morphism $h : S \to X \times B$. Let $p : S \to B$ denote the natural projection.

If $g(C) = 0$, dim $im(p_X \circ h^0) = 2$ and $k \geq 2$, then for some $b \in B - B^0$ the 1-cycle $h_*(p^{-1}(b))$ is either reducible or nonreduced.

This contradicts that $K \subset \mathcal{M}(X, \phi)$. So we conclude that $D \cap (\overline{A} - A) \subset D$ is a codimension 1 subvariety. Since this is true for every irreducible component, we conclude that $\overline{A} - A \subset \overline{A}$ is a codimension 1 subvariety. By the proof of corollary 37, we know that $\overline{A} - A$ is contained in $\overline{\mathcal{M}}_{0,2}(X, e) - \mathcal{M}(X, \phi)$. Let us consider the other components of the CD decomposition of $\overline{\mathcal{M}}_{0,2}(X, e)$. One of the combinatorial data we have to consider is the simple combinatorial datum $\phi' : \tau \to \tau$ where the underlying tree of τ is the tree with two vertices v_1, v_2, one edge $\{f_1, f_2\}, \partial f_1 = v_1, \partial f_2 = v_2$ and two tails $\partial f_3 = v_2, \partial f_4 = v_4$ and such that $\beta(v_1) = e, \beta(v_2) = 0$. But clearly $\mathcal{M}(X, \phi')$ will not intersect any of the fibers $E^{-1}(p, q)$ with $p \neq q$. But, by the induction assumption that theorem 38 has been proved for all $e' < e$, and by the lemmas in the section on dimension and flatness, for every combinatorial datum $\psi : \sigma_1 \to \sigma_2$ with canonical contraction $\sigma_1 \to \tau_2(e)$ and $\psi \neq \phi, \phi'$, we know that $\dim(\mathcal{M}(X, \psi)) = \dim(X, \psi)$. Moreover, by lemma 50 we conclude that the only ψ such that $\dim(X, \psi) \geq \dim(X, \phi) - 1$ is simple and nonlinear. It easy to check that it must also be nice. Since $\overline{A} - A$ is the union of its (locally closed) intersections with each of the components $\mathcal{M}(X, \psi)$, we conclude that there is a nice, simple combinatorial datum ψ and an irreducible component $B \subset \mathcal{M}(X, \psi)$ such that $\overline{B} \subset (\overline{A} - A)$ is an irreducible component, i.e. $\overline{B} \subset \overline{A}$ is a codimension 1 subvariety. But then we conclude $\dim(A) = \dim(B) + 1 = \dim(X, \psi) + 1 \leq (\dim(X, \phi) - 1) + 1 = \dim(X, \phi)$. This proves theorem 38 and theorem 51 for $\tau_2(e)$.

Finally we prove that the evaluation morphism $e_1 : A \to X$ has constant fiber dimension $\dim(X,\phi) - \dim(X)$. Since $e_1 : B \to X$ is flat, we know that $e_1 : A \to X$ is dominant. Let $p \in e_1(A)$ be any point. Then the preimage $e_1^{-1}(p)$ is simply $A_p := E^{-1}(\{p\} \times X)$. Moreover, clearly the preimage of $\{p\} \times (X - \{p\})$ is dense in A_p. So let us replace A_p by this preimage. Consider the evaluation morphism $e_2 : A_p \to X$. We have shown that for each $q \in e_2(A_p)$, and for every irreducible component D of the fiber $e_2^{-1}(q)$, $D \cap (\overline{A} - A)$ has codimension 1 in D. But this implies that $\overline{A_p} \cap B$ has codimension one in $\overline{A_p}$. Since we know that $e_1 : B \to X$ is flat of relative dimension $\dim(X,\psi) - \dim(X)$, we conclude that the fiber A_p has dimension $1 + \dim(X,\psi) - \dim(X) = \dim(X,\phi) - \dim(X)$. So theorem 40 is also proved for $\tau_2(e)$.

This finishes the proof that $\overline{\mathcal{M}}_{0,r}(X,e)$ is an irreducible, local complete intersection stack of dimension $(n+1-d)e + (n-4) + r$. Moreover, we conclude that $\mathcal{M}(X,\phi)$ is a dense open subset of $\overline{\mathcal{M}}_{0,r}(X,e)$.

Finally, we know by corollary 54, we know that the general point of $\overline{\mathcal{M}}_{0,r}(X,e)$ has trivial obstruction group. It is an easy exercise in deformation theory to conclude that for any $h : C \to X$ in $\overline{\mathcal{M}}_{0,r}(X,e)$ for which the obstruction group vanishes, a general first order deformation of $h : C \to X$ is actually an embedding and not just birational. Since the obstruction group vanishes, we conclude that $h : C \to X$ admits deformations which are embeddings. Thus $R_e(X) \subset \overline{\mathcal{M}}_{0,0}(X,e)$ is a nonempty, therefore dense, open set. This proves theorem 1.

CHAPTER 2

Rational Curves on Cubic Threefolds

2.1. Introduction

In the last chapter we proved that for $n \geq 6$ and $d \leq \frac{n+1}{2}$, for a *general* hypersurface $X_d \subset \mathbb{P}^n$ of degree d, and for each e the space $R_e(X)$ is an integral scheme of dimension $(n+1-d)e+(n-4)$ and is a local complete intersection scheme. In this chapter we shall prove the following theorem:

THEOREM 62. *Let $X \subset \mathbb{P}^4$ be any smooth cubic hypersurface. For each integer $e > 1$, the scheme $R_e(X)$ is an integral, local complete intersection scheme of dimension $2e$.*

The basic proof is exactly the same as in the last chapter. We again embed $R_e(X)$ into the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(X,e)$. We again study the CD decomposition of $\overline{\mathcal{M}}_{0,0}(X,e)$. We again study the specializations of nice components of the CD decomposition and prove that in the boundary of every nice component there is a nice basic component (i.e. a *nonlinear* basic component). And we again show that there is a unique irreducible component M of $\overline{\mathcal{M}}_{0,0}(X,e)$ such that for every basic component B of $\overline{\mathcal{M}}_{0,0}(X,e)$, M contains B and M is the unique irreducible component which contains B. As before, the general point of M is smooth and $\dim(M) = 2e$. Finally $R_e(X)$ is a (nonempty) open subset of M, which proves that $R_e(X)$ is irreducible of dimension $2e$.

Having outlined the similarities to the last chapter, let us outline the new complications of this chapter. The first complication is that $\overline{\mathcal{M}}_{0,0}(X,e)$ is *reducible*. Indeed $\overline{\mathcal{M}}_{0,0}(X,e)$ has two irreducible components: the irreducible component M from the last paragraph and a new irreducible component N. The general point of N parametrizes a stable map $f : C-- > X$ which is an $e-to-1$ map to a line in X, i.e. f is linear (this is why we include nonlinearity in the definition of nice). We have to take special care in our specialization arguments to avoid the component N.

The second complication comes from the fact that there are smooth cubic threefolds X such that the projection map $\pi_1 : F_{0,1}(X) \to X$ is not flat. However, we know the flattening stratification of π_1: there is a finite subset $E \subset X$ consisting of the classical Eckardt points of a cubic threefold. Define the *flat locus* X_f to be $X - E$. Then π_1 is flat over X_f. Moreover the fiber dimension of π_1 only increases by 1 over E. By an induction argument as in the last chapter, we conclude that for all (appropriate) combinatorial data ϕ and evaluation morphisms $e_f : \mathcal{M}(X, \phi) \to X$, e_f is flat over X_f and the fiber dimension of e_f increases by at most 1 over E. This allows us to perform a refined version of "leaf induction".

2.2. Classical Results

The standard reference for results about cubic threefolds are [Clemens-Griffiths] and [Tjurin]. The classical results we need all have to do with the Fano scheme $F_1(X)$ parametrizing lines on X. Suppose that $L \subset X$ is a line. The normal bundle of L in $\mathbb{P}^4$ is simply $N_{L/\mathbb{P}^4} = \mathcal{O}_L(1) \oplus \mathcal{O}_L(1)$. We have a normal bundle sequence

$$0 \longrightarrow N_{L/X} \longrightarrow N_{L/\mathbb{P}^4} \longrightarrow N_{X/\mathbb{P}^4}|_L \longrightarrow 0.$$

Since $N_{X/\mathbb{P}^4}|_L = \mathcal{O}_L(3)$, we see that the only possibilities for $N_{L/X}$ are $\mathcal{O}_L \oplus \mathcal{O}_L$ and $\mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)$. Both of these possibilities occur for some points $[L] \in F_1(X)$. A line is said to be of *type I* if $N_{L/X} = \mathcal{O}_L \oplus \mathcal{O}_L$ and is said to be of *type II* if $N_{L/X} = \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1)$. Notice that in both cases, $H^1(L, N_{L/X}) = 0$. Since $H^1(L, N_{L/X})$ is the obstruction space to extending first order deformations of L to higher order, we conclude that the scheme $F_1(X)$ is smooth of dimension $h^0(L, N_{L/X}) = 2$. This is classically called the *Fano surface* of X.

Inside $F_1(X)$ the locus $D \subset F_1(X)$ parametrizing lines L of type II form a divisor, i.e. a general point of $F_1(X)$ is a line of type I. The surface $F_1(X) \subset \mathrm{G}(2,5) \subset \mathbb{P}^9$ is canonically embedded. Moreover, there is an abelian variety $J = J_3(X)$ canonically associated to X which is isomorphic to the Albanese variety of $F_1(X)$. In fact the Albanese map $F_1(X) \to J$ is an embedding.

Finally, for a general X we know $F_{0,1}(X) \to X$ is flat. But there are smooth X such that $F_{0,1}(X) \to X$ is not flat – for example the Fermat cubic $X = \{[x_0, \ldots, x_4] | x_0^3 + \ldots x_4^3 = 0\}$. But $F_{0,1}(X) \to X$ is flat over an open set $X_f \subset X$ whose complement E is a finite set. The points $p \in E$ are called *Eckardt points*. They are characterized as those points $p \in X$ such that the intersection of X with the osculating 3-plane H_p to X at p, $H_p \cap X$ is a cone over a smooth plane cubic curve inside of H_p. For $p \in X_f$ a general point, there are 6 lines passing through p.

2.3. Flatness and Dimension Results

In this section we state the results having to do with the dimensions of the components $\mathcal{M}(X, \phi)$ of the CD decomposition and flatness of the evaluation morphisms $e_f : \mathcal{M}(X, \phi) \to X$. We prove some simple lemmas which allow us to reduce the proofs of these results to a manageable statement. We give one new definition:

DEFINITION 63. *A stable A-graph τ is* moderate *if $r_2(\tau)$ is very stable (here the empty graph is considered very stable). A combinatorial datum $\phi : \tau_1 \to \tau_2$ is* moderate *if τ_2 is moderate.*

Using lemma 35, we see that being moderate is essentially the same as being very stable. We begin with the statement of the main results about moderate data:

THEOREM 64. *For each moderate combinatorial datum ϕ, $\mathcal{M}(X, \phi)$ has pure dimension $dim(\mathcal{M}(X, \phi)) = dim(X, \phi)$.*

THEOREM 65. *For each moderate combinatorial datum ϕ, $\mathcal{M}(X, \phi)$ is a local complete intersection.*

THEOREM 66. *For each moderate combinatorial datum $\phi : \tau_1 \to \tau_2$ and each tail $f \in Tail(\tau_1)$ such that $\beta(\partial f) > 0$, the evaluation morphism $e_f : \mathcal{M}(X, \phi) \to X$ is flat over X_f of relative dimension $dim(X, \phi) - 3$. For each Eckardt point $p \in E$, every irreducible component $A \subset e_f^{-1}(p)$ has dimension $dim(A) \leq dim(X, \phi) - 2$.*

In the proof of theorem 62 we will largely ignore combinatorial data which are not very stable. The following result justifies this:

THEOREM 67. *Suppose that $\phi : \tau_1 \to \tau_2$ is a combinatorial datum which is not moderate. For each irreducible component $A \subset \mathcal{M}(X, \phi)$, we have $dim(A) \leq 2\beta(\tau_1) - 3 + \#Tails(\tau_1)$.*

Now we prove some lemmas which begin the proofs of the previous theorems.

LEMMA 68. *Theorem 64 implies theorem 65.*

PROOF. The proof is exactly the same here as the proof of lemma 41. □

LEMMA 69. *If theorem 64 is valid, then theorem 66 is equivalent to the following: For each moderate combinatorial datum $\phi : \tau_1 \to \tau_2$ and each tail $f \in Flag(\tau_1)$ such that $\beta(\partial f) > 0$, the evaluation morphism $e_f : \mathcal{M}(X, \phi) \to X$ has constant fiber dimension $dim(X, \phi) - 3$ over X_f and has fiber dimension at most $dim(X, \phi) - 2$ over each Eckardt point $p \in E$.*

PROOF. The proof is exactly the same as the proof of lemma 42. □

LEMMA 70. *Theorem 64 and theorem 66 are implied by the following statement: For each very stable A-graph τ, $\mathcal{N}(X,\tau)$ has pure dimension $dim(X,\tau)$ and for each tail $f \in Tail(\tau)$, the evaluation morphism $e_f : \mathcal{N}(X,\tau) \to X$ is flat of relative dimension $dim(X,\tau) - 3$ over X_f and has fiber dimension at most $dim(X,\tau) - 2$ over each Eckardt point $p \in E$. For theorem 38 it even suffices to consider only stable A-graphs τ such that $r_1(\tau) = \tau$.*

PROOF. This follows immediately from theorem 33 and lemma 35. $\square$

LEMMA 71. *The statement in lemma 70 is equivalent to the following statement: For all nonnegative integers r and e, $\mathcal{N}(X,\tau_r(e))$ has dimension $dim(X,\tau_r(e))$ and for each tail $f \in Tail(\tau_r(e))$, the evaluation morphism $e_f : \mathcal{N}(X,\tau_r(e)) \to X$ has constant fiber dimension $dim(X,\tau_r(e)) - 3$ over X_f and has fiber dimension at most $dim(X,\tau_r(e)) - 2$ over each Eckardt point $p \in E$.*

PROOF. Suppose that the above statement has been proved for all r and e. We will prove the statement in lemma 70 by our "refined leaf induction".

If τ has a single vertex then it is already of the form $\tau_r(e)$ for some r and e, thus suppose that τ has more than 1 vertex. Let $v_1 \in \text{Vertex}(\tau)$ be a leaf. Let $\{f_1, f_2\}$ be the unique edge attached to v_1 and oriented so that $\partial f_1 = v_1$. Let $v_2 = \partial f_2$. Define τ' to be the subgraph of τ such that $\text{Vertex}(\tau') = \{v_1, v_2\}$, and such that $\text{Flag}(\tau') = \{f \in \text{Flag}(\tau) : \partial f = v_1\} \cup \{f_2\}$. Define τ'' to be the subgraph of τ such that $\text{Vertex}(\tau'') = \text{Vertex}(\tau) - \{v_1\}$ and such that $\text{Flag}(\tau'') = \text{Flag}(\tau) - \text{Flag}(\tau')$. Finally define τ_0 to be the subgraph of τ whose only vertex is v_2 and which has no flags.

The combinatorial morphisms $\tau' \hookleftarrow \tau_0$, $\tau'' \hookleftarrow \tau_0$ induce 1-morphisms $\mathcal{N}(X,\tau') \to \mathcal{N}(X,\tau_0)$ and $\mathcal{N}(X,\tau'') \to \mathcal{N}(X,\tau_0)$. Let $\mathcal{N}(X,\tau') \times_{\mathcal{N}(X,\tau_0)} \mathcal{N}(X,\tau'')$ denote the fiber product of these morphisms. The combinatorial morphisms $\tau \hookleftarrow \tau'$ and $\tau \hookleftarrow \tau''$ induce 1-morphisms $\mathcal{N}(X,\tau) \to \mathcal{N}(X,\tau')$ and $\mathcal{N}(X,\tau) \to \mathcal{N}(X,\tau'')$, i.e. they induce a morphism $\mathcal{N}(X,\tau) \to \mathcal{N}(X,\tau') \times_{\mathcal{N}(X,\tau_0)} \mathcal{N}(X,\tau'')$. Moreover, from the definition of $\mathcal{N}(X,\tau)$, we see this 1-morphism is an open immersion.

The claim is that $\mathcal{N}(X,\tau') \to \mathcal{N}(X,\tau_0)$ has constant fiber dimension $2\beta(v_1) - 1 + \#\text{Tail}(\tau')$. Since the expected dimension $\dim(X,\tau')$ is always a lower bound, every irreducible component of $\mathcal{N}(X,\tau')$ has dimension $\geq \dim(X,\tau') = 2\beta(v_1) + 2\beta(v_2) - 1 + \#\text{Tail}(\tau')$. On the other hand, by hypothesis we know that every irreducible component of $\mathcal{N}(X,\tau_0)$ has dimension $2\beta(v_2)$. Therefore every irreducible component of every fiber has dimension at least $2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau')$; so we need to show every irreducible component of very fiber has dimension at most $2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau')$.

To see this let us introduce the subgraph τ''' of τ' which is the graph whose only vertex is v_1 and such that $\mathrm{Flag}(\tau''') = \mathrm{Flag}(\tau') - \{f_2\}$. Define $e_1 : \mathcal{N}(X, \tau''') \to X$ to be the evaluation morphism associated to the tail $f_1 \in \mathrm{Tail}(\tau''')$. Suppose that $h : C \to X$ is a naive τ_0-map. The fiber of $\mathcal{N}(X, \tau') \to \mathcal{N}(X, \tau_0)$ over the moduli point $[h : C \to X]$ is an open subset of the fiber product $\mathcal{N}(X, \tau''') \times_{e_1, X, h} C$. Now the general point of C maps to a point in X_f. Since e_1 is flat over X_f by theorem 66, we conclude that every irreducible component of $\mathcal{N}(X, \tau''') \times_{e_1, X, h} C$ over X_f has dimension equal to the sum of the fiber dimension of e_1 and the dimension of C, i.e. $2\beta(v_1) - 3 + \#\mathrm{Tail}(\tau''') + 1 = 2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau')$. On the other hand, there are only finitely many points of C which map to Eckardt points $p \in E$, and for each such point the preimage $e_1^{-1}(p)$ has dimension at most $2\beta(v_1) - 2 + \#\mathrm{Tail}(\tau''') = 2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau''')$. Thus each irreducible component of the fiber product $\mathcal{N}(X, \tau''') \times_{e_1, X, h} C$ has dimension at most $2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau''')$. Thus every irreducible component of every fiber of $\mathcal{N}(X, \tau') \to \mathcal{N}(X, \tau_0)$ has dimension at most $2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau''')$.

Since every irreducible component of every fiber of $\mathcal{N}(X, \tau') \to \mathcal{N}(X, \tau_0)$ has dimension $2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau''')$, we conclude by induction that every irreducible component of the fiber product $\mathcal{N}(X, \tau') \times_{\mathcal{N}(X, \tau_0)} \mathcal{N}(X, \tau'')$ has dimension $2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau''') + \dim(X, \tau'') = \dim(X, \tau)$. Thus $\mathcal{N}(X, \tau)$ has pure dimension $\dim(X, \tau)$. Moreover suppose that $f \in \mathrm{Tail}(\tau)$ is a tail. Since τ has at least two tails, without loss of generality we may suppose that f is attached to τ''. By induction $e_f : \mathcal{N}(X, \tau') \to X$ has constant fiber dimension $\dim(X, \tau') - 3$ over X_f and has fiber dimension at most $\dim(X, \tau') - 2$ over each Eckardt point $p \in E$. Since the projection $\mathcal{N}(X, \tau) \to \mathcal{N}(X, \tau'')$ has constant fiber dimension $2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau')$, and since $2\beta(v_1) - 1 + \#\mathrm{Tail}(\tau') + \dim(X, \tau') = \dim(X, \tau)$, we conclude that $e_f : \mathcal{N}(X, \tau) \to X$ has constant fiber dimension $\dim(X, \tau') - 3$ over X_f and has fiber dimension at most $\dim(X, \tau') - 2$ over each Eckardt point $p \in E$. $\square$

LEMMA 72. *In the statement in the last lemma, it suffices to consider only stable A-graphs $\tau_1(e)$ and $\tau_0(e)$.*

PROOF. This is an obvious reduction. $\square$

LEMMA 73. *Theorem 67 is equivalent to the following statement: For each stable A-graph τ which is not moderate, every irreducible component of $\mathcal{N}(X,\tau)$ has dimension at most $2\beta(\tau) - 3 + \#Tails(\tau)$.*

PROOF. Suppose that $\phi : \tau_1 \to \tau_2$ is a combinatorial datum which is not moderate. Then τ_2 is not moderate. Therefore by hypothesis $\dim(\mathcal{N}(X,\tau_2)) \leq 2\beta(\tau_1) - 3$. By theorem 33, $\mathcal{M}(X,\phi) \to \mathcal{N}(X,\tau_2)$ is flat of relative dimension $\delta(\phi)$. Now consider the difference

$$D = (2\beta(\tau_1) - 3 + \#\text{Tails}(\tau_v)) - (2\beta(\tau_2) - 3 + \#\text{Tails}(\tau) + \delta(\phi)).$$

Exactly what we need to prove is that $D \geq 0$. But D satisfies an obvious additivity: D is the sum over vertices $v \in \text{Vertex}(\tau_2)$ of D_v where D_v is the difference one gets above when we replace τ_2 by the subgraph whose only vertex is v and whose flags are the tails of τ_2 attached to v and when we replace τ_1 by the preimage of τ_2 under ϕ. Thus to establish $D \geq 0$, it suffices to consider the case when $\tau_2 = \tau_r(e)$ for some r and e.

In case $\tau_2 = \tau_r(e)$, we see that $r = \#\text{Edge}(\tau_1) + \#\text{Tail}(\tau_1)$. And $\beta(\tau_1) = \sum_{v \in \text{Vertex}(\tau_1)} d(v)e$. Thus D is simply

$$\sum_{v \in \text{Vertex}(\tau_1)} ((2e-2)d(v) + 2) - 2e - \#\text{Edge}(\tau_1).$$

Clearly this is minimized when $e = 1$. And in this case the sum simply reduces to $2(\#\text{Vertex}(\tau_1) - 1) - \#\text{Edge}(\tau_1)$. But $\#\text{Edge}(\tau_1) = \#\text{Vertex}(\tau_1)$. So in this case, D reduces to $\#\text{Edge}(\tau_1)$ which is certainly nonnegative. So we conclude in every case that $D \geq 0$. $\square$

LEMMA 74. *To prove theorem 67 it suffices to prove the statement in lemma 73 for only τ such that $\tau = r_2(\tau)$.*

PROOF. Since τ is not moderate, $r_2(\tau)$ is not very stable. Thus, by assumption $\dim(\mathcal{N}(X, r_2(\tau))) \leq 2\beta(\tau) - 3 + \#\text{Tail}(r_2(\tau))$. It follows from lemma 35 that $\dim(\mathcal{N}(X,\tau)) \leq \dim(\mathcal{N}(X, r_2(\tau))) + \#\text{Tail}(\tau) - \#\text{Tails}(\tau)$. Thus we conclude that $\dim(\mathcal{N}(X,\tau)) \leq 2\beta(\tau) - 3 + \#\text{Tail}(\tau)$. $\square$

LEMMA 75. *Theorem 64 and theorem 66 imply theorem 67.*

PROOF. Using lemma 67, lemma 73, and lemma 74, it suffices to prove the statement in lemma 73 for τ such that $\tau = r_2(\tau)$. Of course it is trivial to reduce to the case that also $\tau = r_1(\tau)$.

Let $v_1 \in \text{Vertex}(\tau)$ be a leaf. Since $\tau = r_2(\tau)$, we know $\beta(v_1) > 0$. Again let $\{f_1, f_2\}$ be the unique edge attached to v_1 oriented so that $\partial f_1 = v_1$. Let $v_2 = \partial f_2$. Let τ'' denote the stable A-graph obtained by stabilizing the subgraph $(\tau'')_{\text{pre}}$ of τ whose vertex set is $V = \text{Vertex}(\tau) - \{v_1\}$ and whose flag set is $F = \{f \in \text{Flag}(\tau) : \partial f \in V, f \neq f_2\}$. There are two cases.

The first case is that τ'' is not very stable. Then by the induction assumption we know that every irreducible component of $\mathcal{N}(X, \tau'')$ has dimension at most $2\beta(\tau'') - 3$. Now it follows from theorem 66 that the every irreducible component of every fiber of the morphism $\mathcal{N}(X, \tau) \to \mathcal{N}(X, \tau'')$ has dimension at most $2\beta(v_1) - 1$. We conclude that every irreducible component of $\mathcal{N}(X, \tau)$ has dimension at most $2\beta(\tau'') - 3 + 2\beta(v_1) - 1 = 2\beta(\tau) - 4 < 2\beta(\tau) - 3$.

The next case is that τ'' is very stable. But then the vertex v_2 is contracted to an edge of τ''. Since τ'' is very stable and has at least one edge, by theorem 64, $\dim(\mathcal{N}(X, \tau'')) = \dim(X, \tau'') \leq 2\beta(\tau'') - 1$. Moreover, now the morphism $\mathcal{N}(X, \tau) \to \mathcal{N}(X, \tau'')$ has fiber dimension at most $2\beta(v_1) - 2$ since now the universal marked point q_f on the universal curve over $\mathcal{N}(X, \tau)$ must map to the universal marked point on the universal curve over $\mathcal{N}(X, \tau'')$ corresponding to the edge to which v_2 is contracted (rather than a whole component of the stable curve as in the last case). Thus every irreducible component of $\mathcal{N}(X, \tau)$ has dimension at most $2\beta(\tau'') - 1 + 2\beta(v_1) - 2 = 2\beta(\tau) - 3$. $\square$

2.4. Specializations

The theorems in the last section describe the dimensions and evaluation morphisms for the stacks $\mathcal{M}(X, \phi)$. In this section we will discuss how the components in the CD decomposition "fit together". We begin with one definition.

DEFINITION 76. *A point p* $\overline{\mathcal{M}}_{0,r}(X, e)$ *is called a* type I, nice, basic point *if there is a nice, basic combinatorial datum* $\phi : \tau_1 \to \tau_2$ *such that* $\tau_1 \to \tau_r(e)$ *is the canonical contraction, and if there is a strict* ϕ*-map* $\left(((C_v), (h_v), (q_f)), ((C'_v), (h'_v), (q'_f)), (k_v)\right)$ *in* $\mathcal{M}(X, \phi)$ *whose image in* $\overline{\mathcal{M}}_{0,r}(X, e)$ *is p and such that each* C'_v *corresponds to a line of type I.*

The main result is the following:

THEOREM 77. *If* $\phi : \tau_1 \to \tau_2$ *is a nonlinear, moderate combinatorial datum, if* $\tau_1 \to \tau_r(e)$ *is the canonical contraction, and if* $A \subset \mathcal{M}(X, \phi)$ *is an irreducible component, then the closure of the image of A,* $\overline{F'_1(A)} \subset \overline{\mathcal{M}}_{0,r}(X, e)$ *contains a type I, nice, basic point.*

In the next two sections we will show that there is a unique irreducible component $M \subset \overline{\mathcal{M}}_{0,r}(X, e)$ which contains every type I, nice, basic point. It then follows from theorem 77 that the image of $\mathcal{M}(X, \tau)$ is contained in M.

The following lemmas allow us to reduce the proof of theorem 77 to a more manageable statement.

LEMMA 78. *In theorem 77 it suffices to consider only nonlinear, moderate, combinatorial data ϕ which are also elementary. More precisely, if $\phi : \tau_1 \to \tau_2$ is a nonlinear, moderate combinatorial datum which is not elementary and if $A \subset \mathcal{M}(X, \phi)$ is an irreducible component, then there exists a nonlinear, moderate, elementary combinatorial datum $\psi : \sigma_1 \to \sigma_2$ such that $\sigma_1 \to \tau_r(e)$ is the canonical contraction, and there exists an irreducible component $B \subset \mathcal{M}(X, \psi)$ such that $\overline{F_1'(B)} \subset \overline{F_1'(A)}$.*

PROOF. Suppose that ϕ is not elementary. We will produce $\psi : \sigma_1 \to \sigma_2$ which is "one step closer" to being elementary, i.e. such that

$$d(\psi) = \sum_{v \in \mathrm{Vertex}(\sigma_1)} (d_\psi(v) - 1) = \sum_{v \in \mathrm{Vertex}(\tau_1)} (d_\phi(v) - 1) = d(\phi).$$

We will also produce an irreducible component $B \subset \mathcal{M}(X, \psi)$ such that $\overline{B} \subset \overline{A}$ is a codimension 1 subvariety. The lemma then follows by induction on $d(\phi)$.

Let $v_1 \in \mathrm{Vertex}(\tau_1)$ be a vertex such that $d(v_1) > 1$. Let $w_1 = \phi_V(v_1)$, let $d = d(v_1)$, $m = \beta_{\tau_2}(w_1)$ (so that $\beta(v_1) = dm$). We define σ_1 to be the stable A-graph obtained as follows. The tree of σ_1 is obtained from τ_1 by adding one new vertex v_2 and one new edge $\{f_1, f_2\}$ such that $\partial f_1 = v_1$, $\partial f_2 = v_2$. We define β_{σ_1} by

$$\beta_{\sigma_1}(v) = \begin{cases} m(d-1) & v = v_1 \\ m & v = v_2 \\ \beta_{\tau_1}(v) & \text{otherwise} \end{cases}$$

We construct the stable A-graph σ_2 by attaching one new tail g to τ_2 such that $\partial g = w_1$. We define $\psi : \sigma_1 \to \sigma_2$ as follows. The maps ψ_V and ψ_F are obtained by extending ϕ_V and ϕ_F by $\psi_V(v_2) = w_1$, $\psi_F(f_1) = \psi_F(f_2) = g$. Finally we define d_ψ by

$$d_\psi(v) = \begin{cases} d-1 & v = v_1 \\ 1 & v = v_2 \\ d_\phi(v) & \text{otherwise} \end{cases}$$

Since ϕ is nonlinear and moderate, it is clear that ψ is also nonlinear and moderate. Moreover, by construction we have $d(\psi) = d(\phi) - 1$.

Let $A \subset \mathcal{M}(X, \phi)$ be an irreducible component. By lemma 34, we know that $G_1 : \mathcal{M}(X, \phi) \to \mathcal{N}(X, r_1(\tau_2))$ is smooth with irreducible fibers. So if we define $A' = G_1(A)$, we see that $A = G_1^{-1}(A')$. Now the obvious combinatorial morphism $\sigma_2 \to \tau_2$ induces a morphism $\mathcal{N}(X,\sigma_2) \to \mathcal{N}(X,\tau_2)$. Clearly this morphism is smooth with irreducible fibers since it is just an open subset of the universal curve $\mathcal{C}_{w_2} \to \mathcal{N}(X,\tau_2)$. Thus the composition $\mathcal{N}(X,\sigma_2) \to \mathcal{N}(X,\tau_2) \xrightarrow{r_1} \mathcal{N}(X,r_1(\tau_2))$ is also smooth with irreducible fibers. Denote this morphism by $H : \mathcal{N}(X,\sigma_2) \to \mathcal{N}(X,r_1(\tau_2))$. Define $B' = H^{-1}(A')$. Then B' is irreducible and $\dim(B') = \dim(A') + 1$. We define $B \subset \mathcal{M}(X,\psi)$ to be the preimage of B' under the 1-morphism $G_1 : \mathcal{M}(X,\psi) \to \mathcal{N}(X,\sigma_2)$. Since $d_\psi(v_1) = d_\phi(v_1) - 1$, we see that the defect $\delta\psi = \delta\phi - 2$. Thus $\dim(X,\psi) = \dim(X,\phi) + 1 - 2 = \dim(X,\phi) - 1$ (and $\dim(B) = \dim(A) - 1$).

The claim is that $\overline{F_1'(B)} \subset \overline{F_1'(A)}$, i.e. $F_1'(B) \subset \overline{F_1'(A)}$. This is easy: suppose given a strict ψ-map

$$\xi = \left(\left((C_v),(h_v),(q_f)\right),\left((C_v'),(h_v'),\left(q_f'\right)\right),(k_v)\right).$$

Suppose moreover that this is general in the sense that for each $f \in \operatorname{Flag}(\sigma_1)$ with $\partial f = v_1$, the corresponding point $q_f \in C_{v_1}$ is not a ramification point of k_{v_1}. Let $C = C_{v_1} \cup C_{v_2}$ be the connected sum of C_{v_1} and C_{v_2} along the points $q_{f_1} \in C_{v_1}$ and $q_{f_2} \in C_{v_2}$. Let $k : C \to C'_{w_2}$ be the unique morphism whose restriction to C_{v_1} is k_{v_1} and whose restriction to C_{v_2} is k_{v_2}. If we consider this as an admissible cover, there is no obstruction to smoothing. In other words, there is a flat proper morphism $\pi : \tilde{C} \to \operatorname{Spec} \mathbb{C}[[t]]$ whose generic fiber is simply $\mathbb{P}^1_{\mathbb{C}[[t]]}$ and whose special fiber is C and there is a finite morphism of $\operatorname{Spec} \mathbb{C}[[t]]$-schemes $\tilde{k} : \tilde{C} \to \operatorname{Spec} \mathbb{C}[[t]] \times C'_{w_2}$ whose generic fiber is simply a finite morphism of degree d and whose special fiber is k defined as above. Now for each $f \in \operatorname{Flag}(\sigma_1)$ such that $\partial f = v_1$ and $f \neq f_1$, the corresponding point $q_f \in C_v \subset C$ is a smooth point of π. Thus we can extend this to a section $\tilde{q}_f : \operatorname{Spec} \mathbb{C}[[t]] \to \tilde{C}$.

We define an element of $\eta = \left(\left(\tilde{C}_v\right),\left(\tilde{h}_v\right),(\tilde{q}_f)\right)$ in $\overline{\mathcal{M}}(X,\tau_1)(\operatorname{Spec} \mathbb{C}[[t]])$ as follows: For each $v \in \operatorname{Vertex}(\tau_1)$ with $v \neq v_1$, we define $\tilde{C}_v = \operatorname{Spec} \mathbb{C}[[t]] \times C_v$ and $\tilde{h}_v$ is just the base-change of h_v. For each flag $f \in \operatorname{Flag}(\tau_1)$ with $\partial f \neq v_1$, we define $\tilde{q}_f$ to simply be the base-change of q_f. We define $\tilde{C}_{v_1} = \tilde{C}$ from the last paragraph. And for each $f \in \operatorname{Flag}(\tau_1)$ with $\partial f = v_1$, we define $\tilde{q}_f$ from the last paragraph. Clearly this is an element of $\overline{\mathcal{M}}(X,\tau_1)(\operatorname{Spec} \mathbb{C}[[t]])$. Moreover it is easy to see that the general fiber of this family is an element of $F_1(A)$. We conclude that $F_1'(B) \subset \overline{F_1'(A)}$. $\square$

LEMMA 79. *In theorem 77 it suffices to consider $\phi : \tau_1 \to \tau_2$ which are nonlinear, moderate and simple.*

PROOF. By lemma 78, we see that we may restrict to $\phi : \tau_1 \to \tau_2$ which are elementary. Suppose we have proved theorem 77 for all moderate, nonlinear combinatorial data which are simple.

Let $\psi : \tau_2 \to \tau_2$ denote the unique simple, moderate, nonlinear combinatorial datum such that ψ is the identity map on trees. By lemma 34, we know that $\mathcal{M}(X, \phi) \to \mathcal{N}(X, r_1(\tau_2))$ is flat. Thus the image of A in $\mathcal{N}(X, r_1(\tau_2))$ is an irreducible component. Let $B \subset \mathcal{M}(X, \psi)$ denote the preimage of this irreducible component.

By assumption we can find a DVR Spec R with function field K and residue field $k = \mathbb{C}$ and a family of stable maps

$$\zeta = \big((\pi' : \mathcal{C}' \to \text{Spec } R), (q'_f : \text{Spec } R \to \mathcal{C}'), (h' : \mathcal{C}' \to X)\big)$$

whose general fiber $\zeta_K = (h'_K : \mathcal{C}'_K \to X)$ lies in $F'_1(B)$ and whose special fiber $\zeta_k = (h'_k : \mathcal{C}'_k \to X)$ is a type I, nice, basic point.

For each vertex $v \in \text{Vertex}(\tau_2)$, define $\mathcal{C}'_v \subset \mathcal{C}$ to be the closure of the irreducible component of $\mathcal{C}'_K$ corresponding to v. For each tail $f \in \text{Tail}(\tau_2)$ with $\partial f = v$, choose a point in the special fiber $q'_f \in (\mathcal{C}'_v)_k$ which is not a special point of $\mathcal{C}'_k$. After replacing Spec R by a finite cover, we may suppose that each q'_f extends to a section $q'_f : \text{Spec } R \to \mathcal{C}$.

Once we add in all the sections q'_f from the last paragraph, the generic fiber is now a point in the image of $\mathcal{N}(X, \tau_2)$. Of course there is a unique point $\eta \in \mathcal{M}(X, \phi)(\text{Spec } K)$ such that $R_2(\eta)$ in $\mathcal{N}(X, \tau_2)$ maps to the general fiber. Moreover, we can construct a family of stable maps over Spec R whose general fiber is η as follows: for each vertex $v \in \text{Vertex}(\tau_1)$, let $\mathcal{C}_v = \mathcal{C}'_{\phi_V(v)}$.

For each flag $f \in \text{Flag}(\tau_1)$ with $\partial f = v$, let $q_f : \text{Spec } R \to \mathcal{C}_v$ be the section corresponding to $q_{\phi_F(f)} : \text{Spec } R \to \mathcal{C}'_{\phi_V v}$. And let $h_v : \mathcal{C}_v \to X$ correspond to $h'_{\phi_V(v)} : \mathcal{C}_{\phi_V(v)} \to X$. Then $((\mathcal{C}_v), (h_v), (q_f))$ is an element of $\overline{\mathcal{M}}(X, \tau_1)(\text{Spec } R)$. The general fiber is simply η. Let us take the image of this family in $\overline{\mathcal{M}}_{0,r}(X, e)$. By construction, the special fiber is obtained by gluing together components of the type I, nice, basic point ζ_k along non-special points. Thus the special fiber is also a type I, nice, basic point. $\square$

The proof of theorem 77 has been reduced to a statement about simple combinatorial data ϕ. The following theorem – whose proof will be proved in the next few sections – then provides the induction step to finish the proof of theorem 77

THEOREM 80. *Let $\phi : \tau \to \tau$ be a simple, moderate, nonlinear combinatorial datum such that $\tau \to \tau_r(e)$ is the canonical contraction. Suppose that τ is not basic. Let $A \subset \mathcal{M}(X, \phi)$ be an irreducible component. There exists a moderate, nonlinear combinatorial datum $\psi : \sigma_1 \to \sigma_2$ (possibly basic and not necessarily simple) such that $\sigma_1 \to \tau_r(e)$ is the canonical contraction, and there exists an irreducible component $B \subset \mathcal{M}(X, \psi)$ such that $\overline{F_1'(B)} \subset \overline{F_1'(A)}$ is a codimension 1 subvariety.*

From this theorem one finishes the proof of theorem 77 by induction on the dimension of A. The following lemmas allow us to reduce the proof of theorem 80 to a manageable statement.

LEMMA 81. *Let us assume theorem 64 and theorem 66. Then in the proof of theorem 80 it suffices to consider only τ of the form $\tau_r(e)$.*

PROOF. First of all we may use lemma 35 to reduce to the case that $\tau = r_2(\tau)$, i.e. τ is very stable. So let us assume that τ is very stable. Then we will prove by induction that ψ and B exist so that ψ is even nice (not just moderate and nonlinear). If $\#\text{Vertex}(\tau) = 1$, then τ is already of the form $\tau_r(e)$. Thus suppose that τ has more than 1 vertex.

We again use our refined leaf induction. Suppose that $v_1 \in \text{Vertex}(\tau)$ is a leaf. Let $\{f_1, f_2\}$ be the unique edge attached to v_1, let $\partial f_1 = v_1$ and let v_2 denote ∂v_2. Let τ', τ'' and τ_0 be as defined in the proof of lemma 71. Since τ is not basic, we can choose the leaf v_1 so that τ'' is also not basic. In particular, τ'' is nonlinear.

Define $\phi'' : \tau'' \to \tau''$ to be the unique simple combinatorial datum which is the identity map on underlying trees. The combinatorial morphism $\tau \hookleftarrow \tau''$ induces a 1-morphism $\mathcal{M}(X, \phi) \to \mathcal{M}(X, \phi'')$. Just as in the proof of lemma 71, we conclude that this 1-morphism is dominant and has constant fiber dimension $2\beta(v_1) - 1 + \#\text{Tail}(\tau_1)$ (I bet that we can prove this 1-morphism is even flat). In particular, the image of the irreducible component $A \subset \mathcal{M}(X, \phi)$ in $\mathcal{M}(X, \phi'')$ is a dense subset of an irreducible component $A'' \subset \mathcal{M}(X, \phi'')$.

By induction we know that there is a nice combinatorial datum $\psi'' : \sigma_1'' \to \sigma_2''$ and an irreducible component $B'' \subset \mathcal{M}(X, \psi'')$ such that

$\overline{F_1'(B'')} \subset \overline{F_1'(A'')}$ is a codimension 1 subvariety. Let $\mathcal{C}_{v_2}$ denote the irreducible component of the universal curve over $F_1'(A'')$ corresponding to v_2. Let $\overline{\mathcal{C}}_{v_2}$ denote the closure of $\mathcal{C}_{v_2}$ inside the universal curve over $\overline{F_1'(A'')}$. The restriction of $\overline{\mathcal{C}}_{v_2}$ over $F_1'(B'')$ is a union of irreducible components of the universal curve. Let $w_2 \in \text{Vertex}(\sigma'')$ be a vertex corresponding to one of these irreducible components. Let τ''' be the subgraph of τ' as defined in the proof of lemma 71. Define σ_2 to be the stable A-graph obtained by gluing σ_2'' and τ''' via an edge $\{f_1, f_2\}$ such that $\partial f_1 = v_1$ and $\partial f_2 = \psi_V''(w_2)$. Similarly define σ_1 to be the stable A-graph obtained by gluing σ_1'' and τ''' via an edge $\{f_1, f_2\}$ such that $\partial f_1 = v_1$ and $\partial f_2 = w_2$. There is a unique combinatorial datum $\psi : \sigma_1 \to \sigma_2$ which restricts to ψ'' on σ_1'' and which restricts to the identity on τ'''.

The combinatorial morphisms $\sigma_1 \hookleftarrow \sigma_1''$ and $\sigma_2 \hookleftarrow \sigma_2''$ induce a 1-morphims $\mathcal{M}(X, \psi) \to \mathcal{M}(X, \psi'')$. Consider the preimage of B'' under this 1-morphism. The claim is that there is an irreducible component $B \subset \mathcal{M}(X, \psi)$ such that $\overline{F_1'(B)} \subset \overline{F_1'(A)}$. This is easy to see: for a general point in B'', we can find a DVR Spec R and a morphism Spec $R \to \overline{\mathcal{M}}_{0,r''}(X, e'')$ whose generic point maps into $F_1'(A'')$ and whose special point maps to the image of our point in $F_1'(B'')$. Since $\mathcal{M}(X, \phi) \to \mathcal{M}(X, \phi'')$ is dominant, after replacing Spec R by a finite cover, we may lift the generic fiber of Spec R, Spec $K \to \overline{\mathcal{M}}_{0,r''}(X, e'')$ to a map Spec $K \to A$. We can compose with the map $F_1' : A \to F_1'(A)$. By the valuative criterion of properness for $\overline{\mathcal{M}}_{0,r}(X, e)$, after replacing Spec R by a finite cover again, we can extend this to a map Spec $R \to \overline{F_1'(A)}$. And we may choose Spec $K \to A$ so that the special fiber of Spec $K \to A$ factors through a point of B which maps to our original point in B''. Thus we conclude that $\overline{F_1'(B)} \subset \overline{F_1'(A)}$ and the codimension is the same as the codimension of $\overline{F_1'(B'')} \subset \overline{F_1'(A'')}$, i.e. it is a codimension 1 subvariety. $\square$

LEMMA 82. *In fact in the proof of theorem 80 it suffices to consider only τ of the form $\tau_0(e)$ and $\tau_1(e)$.*

PROOF. This is an obvious reduction. $\square$

2.5. Basic Components

LEMMA 83. *Let τ_2 denote the stable A-graph whose underlying graph is just λ_2 and with $\beta(u_0) = \beta(u_1) = 1$. Let τ_0 denote the stable A-graph whose underlying graph is λ_0 with $\beta(u_0) = 1$. Then $\mathcal{M}(X, \tau_2)$ is equivalent to a smooth, irreducible $\mathbb{C}$-scheme of dimension 3, $M(X, \tau_2)$. The obvious combinatorial morphism $\tau_2 \hookleftarrow \tau_0$ induces a morphism $M(X, \tau_2) \to M(X, \tau_0)$.*

This morphism is faithfully flat and the fiber over a general point is irreducible.

PROOF. Let τ_1 be the stable A-graph obtained from λ_1 by setting $\beta(u_0) = 1$.We have two combinatorial morphisms $\alpha_0 : \tau_2 \hookleftarrow \tau_1$ and $\alpha_1 : \tau_2 \hookleftarrow \tau_1$ obtained by sending u_0 to u_0 and u_1 respectively. We have induced morphisms of stacks $\mathcal{M}(X, \tau_2) \xrightarrow{\alpha_0} \mathcal{M}(X, \tau_1)$ and $\mathcal{M}(X, \tau_2) \xrightarrow{\alpha_1} \mathcal{M}(X, \tau_1)$. Additionally we have the morphism obtained by evaluation at e_0, $\mathcal{M}(X, \tau_1) \to X$. Together these morphisms exhibit $\mathcal{M}(X, \tau_2)$ as an open substack of the fiber product $\mathcal{X}$

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\alpha_0} & \mathcal{M}(X, \tau_1) \\ {\scriptstyle \alpha_1}\downarrow & & \downarrow \\ \mathcal{M}(X, \tau_1) & \longrightarrow & X \end{array} .$$

Of course $\mathcal{M}(X, \tau_1)$ is equivalent to the universal line over the Fano variety, $F_{0,1}(X) = \mathbb{P}_{F_1(X)}(S)$, and so is a finite type, separated $\mathbb{C}$-scheme. Therefore $\mathcal{X}$ is equivalent to a finite type, separated $\mathbb{C}$-scheme and so is $\mathcal{M}(X, \tau_2)$.

And the Zariski tangent space to $\mathcal{M}(X, \tau_2)$ at some $[(C, x, h)]$ is

$$\left(H^0 \left(C_{w_0}, N_{h_{w_0}} \right) \times_{N_{h_{w_0}}|_p} TX|_p \right) \times_{TX|_p} \left(H^0 \left(C_{w_1}, N_{h_{w_1}} \right) \times_{N_{h_{w_1}}|_p} TX|_p \right).$$

If C_{w_i} is a line of type I for either $i = 0$ or $i = 1$, then this tangent space has dimension 3. In fact the only way that the dimension could be greater than 3 is if both lines are of type II, p is the special point on both lines, and the osculating 2-planes of the two lines coincide. Since X is smooth, it contains no 2-planes. Therefore the intersection of X with the osculating 2-plane considered as a Cartier divisor on the osculating 2-plane equals $2[C_{w_0}] + 2[C_{w_1}]$, and this contradicts Bezout's theorem. So the dimension of the Zariski tangent space is always 3 and therefore $\mathcal{M}(X, \tau_2)$ is smooth of dimension 3, in particular it is reduced.

Moreover, by lemma 10.5 of [Clemens-Griffiths], there is an open subset $U \subset \mathcal{M}(X, \tau_0)$ such that for each closed point $[L] \in U$, the fiber over $[L]$ of $\mathcal{M}(X, \tau_2) \to \mathcal{M}(X, \tau_0)$ is irreducible. Since $\mathcal{M}(X, \tau_0) = F_1(X)$ is irreducible, this implies that $\mathcal{M}(X, \tau_2)$ is irreducible.

Let $\mathcal{M}(X, \tau_1) \to \mathcal{M}(X, \tau_0)$ be the morphism induced by the obvious combinatorial morphism $\tau_1 \hookleftarrow \tau_0$. Composing this with (α_0, α_1) induces an embedding $\mathcal{M}(X, \tau_2) \to \mathcal{M}(X, \tau_0) \times \mathcal{M}(X, \tau_0)$. The closure of $\mathcal{M}(X, \tau_2)$ is an effective Cartier divisor D. And to test whether the projection of a

Cartier divisor $D \xrightarrow{\pi_0} \mathcal{M}(X,\tau_0)$ is flat, it suffices to show that D contains no associated points of fibers of $\mathcal{M}(X,\tau_0) \times \mathcal{M}(X,\tau_0) \xrightarrow{\pi_0} \mathcal{M}(X,\tau_0)$. This follows since there is no line $L \subset X$ such that for every line $L' \subset X$, $L \cap L' \neq \emptyset$. Thus $\mathcal{M}(X,\tau_2) \to \mathcal{M}(X,\tau_0)$ is flat. Moreover, for every line $L \subset X$ there exists a line $L' \subset X$ distinct from L such that $L \cap L' \neq \emptyset$. Thus the map is even faithfully flat. □

Suppose that τ is a genus 0 tree such that $\beta_\tau \equiv 1$. We define a poset $(I, \prec)$ by $I = \mathrm{Flag}(\tau) \sqcup \mathrm{Vertex}(\tau) \sqcup (\mathrm{Flag}(\tau)/(f \sim \overline{f}))$ and $f \prec \partial f$, $f \prec \{f, \overline{f}\}$. For each $[(C,x,h)] \in$ obj $\mathcal{M}(X,\tau)(\mathrm{Spec}\ \mathbb{C})$ one defines a compatible family of $\mathbb{C}$-vector spaces indexed by I as follows: For each $\{f,\overline{f}\} \in \mathrm{Flag}(\tau)/\sim$, i.e. for each tail $\{f\}$ and for each edge $\{f,\overline{f}\}$, we define $T_{\{f,\overline{f}\}} := TX|_{h_w(x_f)}$ where $w = \partial f$. For each $w \in \mathrm{Vertex}(\tau)$ we define $T_w := H^0(C_w, N_{h_w})$ where N_{h_w} is the normal bundle of the map h_w. And for each $f \in \mathrm{Flag}(\tau)$ we define

$$T_f := H^0(C_w, N_{h_w}) \times_{N_{h_w}|_{x_f}} TV|_{h_w(x_f)}$$

where $w = \partial f$. The linear maps corresponding to $f \prec \partial f$ and $f \prec \{f,\overline{f}\}$ are the obvious ones. We define $T_{[(C,x,h)]}$ to be the inverse limit of this compatible family.

PROPOSITION 84. *Let τ and $T_{[(C,x,h)]}$ be as above. Then $\mathcal{M}(X,\tau)$ is represented by a separated, finite type $\mathbb{C}$-scheme $\mathcal{M}(X,\tau)$. Let σ_0 be the A-graph whose underlying graph is λ_0 with $\beta(u_0) = 1$. We have the following:*

(1) *$\mathcal{M}(X,\tau)$ is integral of dimension $\mathrm{dim}(X,\tau)$.*
(2) *For $w \in \mathrm{Vertex}(\tau)$, let $\tau \hookleftarrow \sigma_0$ be the combinatorial morphism $u_0 \mapsto w$. The corresponding morphism $\mathcal{M}(X,\tau) \to \mathcal{M}(X,\sigma_0)$ is faithfully flat.*
(3) *The Zariski tangent space to $\mathcal{M}(X,\tau)$ at $[(C,x,h)]$ is $T_{[(C,x,h)]}$. In particular, $[(C,x,h)]$ is a smooth point if all the lines $h_w(C_w)$ are of type I.*

PROOF. Let $\tau \hookleftarrow r_1(\tau)$ be the combinatorial morphism which strips all tails from τ. Clearly the induced morphism $\mathcal{M}(X,\tau) \to \mathcal{M}(X,r_1(\tau))$ is representable, surjective and smooth of relative dimension $\#\mathrm{Tail}(\tau)$ and has connected geometric fibers. If we denote by $[(C,x,h)^r] \in \mathcal{M}(X,r_1(\tau))(\mathrm{Spec}\ \mathbb{C})$ the image of a point $[(C,x,h)] \in \mathcal{M}(X,\tau)(\mathrm{Spec}\ \mathbb{C})$, then the vertical tangent space to this morphism at a point $[(C,x,h)]$ is identified with

$$\prod_{f \in \mathrm{Tail}(\tau)} TC_{\partial f}|_{x_f}.$$

This is also the kernel of the induced map of inverse limits $T_{[(C,x,h)]} \to T_{[(C,x,h)^r]}$. So, without loss of generality, we may assume that τ has no tails.

We proceed by induction on $\beta(\tau)$. If $\beta(\tau) = 1$, then τ is just a single vertex w with $\beta(w) = 1$. Every stable map of a rational curve to X with image class $[L]$ is just an isomorphism to a line in X. Therefore $\overline{\mathcal{M}}(V, \tau) = \mathcal{M}(X, \tau)$ is identified with the Fano variety of lines $F_1(X)$. The fact that $F_1(X)$ is smooth of dimension 2 is well-known (c.f.[Clemens-Griffiths]). And for any $[L] \in F_1(X)$, the Zariski tangent space at $[L]$ is identified with $H^0(L, N_{L/X})$. So the proposition is true when $\beta(\tau) = 1$.

Now suppose $\beta(\tau) > 1$. Then τ has at least two vertices. Choose a leaf $w_0 \in \text{Vertex}(\tau)$ (i.e. a vertex of valence 1). Let $f_0 \in \text{Flag}(\tau)$ be the unique flag such that $\partial f_0 = w_0$, and let w_1 denote $\partial \overline{f_0}$. We form new A-graphs $\tau' = (\text{Flag}(\tau) - \{f_0, \overline{f_0}\}, \text{Vertex}(\tau) - \{w_0\}, j_\tau, \partial_\tau)$ and $\tau'' = (\{f_0, \overline{f_0}\}, \{w_0, w_1\}, j_\tau, \partial_\tau)$ both with $\beta \equiv 1$.

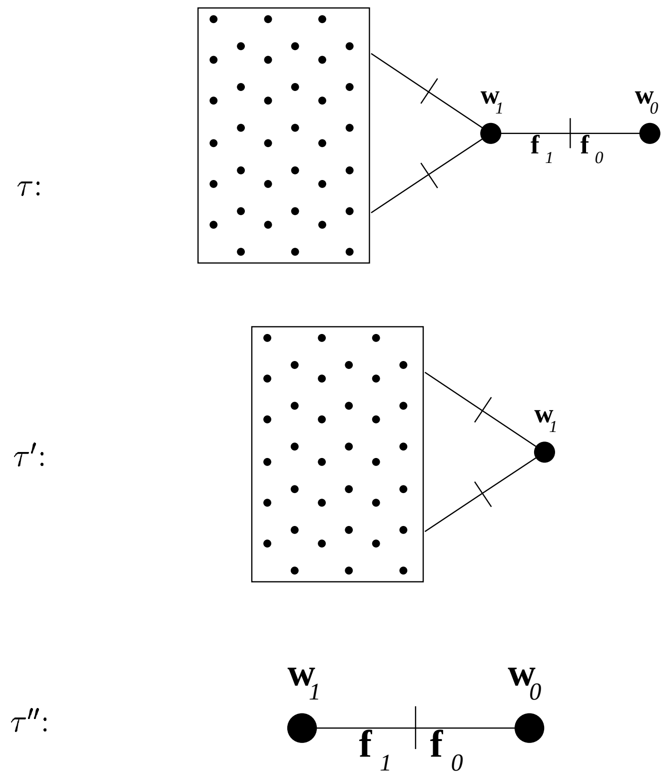

Diagram 5

We have combinatorial morphisms $\tau \hookleftarrow \tau'$ and $\tau \hookleftarrow \tau''$ inducing morphisms of stacks $\mathcal{M}(X,\tau) \to \mathcal{M}(X,\tau')$ and $\mathcal{M}(X,\tau) \to \mathcal{M}(X,\tau'')$. Let τ_0 denote the A-graph whose underlying graph is λ_0 with $g(u_0) = 0, \beta(u_0) = 1$. We have combinatorial morphisms $\tau' \hookleftarrow \tau_0$ and $\tau'' \hookleftarrow \tau_0$ by sending u_0 to w_1. Let $\mathcal{Y}$ denote the fiber product stack:

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{M}(X,\tau'') \\ \downarrow & & \downarrow \\ \mathcal{M}(X,\tau') & \longrightarrow & \mathcal{M}(X,\tau_0) \end{array}.$$

Then we have an open immersion of $\mathcal{M}(X,\tau)$ into $\mathcal{Y}$. We know $\mathcal{M}(X,\tau_0)$ is equivalent to $F_1(X)$. And by the induction assumption $\mathcal{M}(X,\tau')$ is equivalent to a scheme. By lemma 83, $\mathcal{M}(X,\tau'')$ is equivalent to a smooth, irreducible $\mathbb{C}$-scheme of dimension 3. Since the fiber product of separated, finite type $\mathbb{C}$-schemes is still a separated, finite type $\mathbb{C}$-schemes, $\mathcal{Y}$ is equivalent to a separated, finite type $\mathbb{C}$-scheme Y. So $\mathcal{M}(X, \tau)$ is also equivalent to a separated, finite type $\mathbb{C}$-scheme.

Also by induction $\mathcal{M}(X, \tau') \to \mathcal{M}(X, \tau_0)$ and $\mathcal{M}(X, \tau'') \to \mathcal{M}(X, \tau_0)$ are faithfully flat. By base change $X \to \mathcal{M}(X, \tau'')$ and $X \to \mathcal{M}(X, \tau')$ are faithfully flat. And $\mathcal{M}(X, \tau_0)$, $\mathcal{M}(X, \tau')$ and $\mathcal{M}(X, \tau'')$ are integral schemes. So, by a standard argument, Y is reduced (at least in characteristic 0; in positive characteristic we would need to prove that our morphisms are separably generated). Also by lemma 7, the fiber of $\mathcal{M}(X, \tau'') \to \mathcal{M}(X, \tau_0)$ over a general point is irreducible. Since $\mathcal{M}(X, \tau') \to \mathcal{M}(X, \tau_0)$ is faithfully flat, we conclude that the fiber of $Y \to \mathcal{M}(X, \tau')$ over a general point is irreducible. Since $\mathcal{M}(X, \tau')$ is irreducible we conclude that Y is also irreducible. Therefore Y is integral. By elementary dimension theory, theorem 5.6 of [Matsumura86], $\dim(Y) = \dim(\mathcal{M}(X, \tau')) + \dim(\mathcal{M}(X, \tau'')) - \dim(\mathcal{M}(X, \tau_0))$. And this is $\dim(X, \tau') + \dim(X, \tau'') - \dim(X, \tau_0) = \dim(X, \tau)$.

Next we show that $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \tau')$ is surjective. Suppose that $[(C, x, h)] \in \mathcal{M}(X, \tau')$ is a closed point. To extend this to a stable map in $\mathcal{M}(X, \tau)$, it suffices to show there exists a line $C_{w_0} \subset V$ such that $C_{w_0} \cap C_{w_1} \neq \emptyset$ but $C_{w_0} \cap C_w = \emptyset$ for all $w \in W_{\tau'}$ which are connected to w_1 by an edge. By construction, all such C_w are distinct from C_w.

LEMMA 85. *Let $L_1, L_2 \subset V$ be distinct lines. There is a finite closed subset $G \subset L_1$ such that $L \cap G \neq \emptyset$ for all lines $L \subset V$ for which $L \cap L_1 \neq \emptyset$ and $L \cap L_2 \neq \emptyset$.*

PROOF. Suppose first that $L_1 \cap L_2 \neq \emptyset$ (this is the only case of importance to us), say $p \in L_1 \cap L_2$. If we have $L \cap L_1 \neq \emptyset$, $L \cap L_2 \neq \emptyset$ and $p \notin L$, then necessarily $L \subset$ span $(L_1, L_2) \cap X$ which is a cubic plane curve. Since $p \notin L$, we see that L is the residual line to $L_1 \cup L_2$ in this plane curve. There is at most one such L and we define $G = \{p\} \cup (L \cap L_1)$.

Next suppose that $L_1 \cap L_2 = \emptyset$. Then span (L_1, L_2) is a $\mathbb{P}^3$ which intersects X in a cubic surface S. If L is a line intersecting both L_1 and L_2, then it lies in span (L_1, L_2) and so in S. If S contains only finitely many lines, say $\{L_1, \ldots, L_N\}$, then we may take $G = \cup_{i=3}^{N} L_i \cap L$. If S contains infinitely many lines then S must be a cone over a smooth plane cubic. This follows, for example, from the fact that $F_1(X)$ is embedded in the abelian variety $J(V)$ and so contains no rational curves. At any rate, this implies that $L_1 \cap L_2 \neq \emptyset$ which contradicts our hypothesis.

$\square$

We define G to be the union of all the finite sets G_w defined as above for $L_1 = C_{w_1}$ and $L_2 = C_w$ for $w \in W_{\tau'}$ connected to w_1 by an edge. Then any line L intersecting C_{w_1} in a point not lying in G will suffice for C_{w_0}. Through all but at most one point on any line there will be two or more lines (if there are not two lines through a point, then the one line must be a line of type II and the point must be the special point on a line of type II). Thus we may find such L. So $\mathcal{M}(X,\tau) \to \mathcal{M}(X,\tau')$ is surjective and so is faithfully flat. Using induction we conclude that for $w \neq w_0$, the morphism $\mathcal{M}(X,\tau) \to \mathcal{M}(X,\sigma_0)$ is the composite of the faithfully flat morphisms $\mathcal{M}(X,\tau) \to \mathcal{M}(X,\tau') \to \mathcal{M}(X,\sigma_0)$ and thus is faithfully flat. Since τ has at least two leaves, we conclude that $\mathcal{M}(X,\tau) \to \mathcal{M}(X,\sigma_0)$ is faithfully flat for all $w \in \mathrm{Vertex}(\tau)$. In particular $\mathcal{M}(X,\tau) \subset Y$ is a nonempty open set. So $\mathcal{M}(X,\tau)$ is integral of dimension $\dim(Y,\tau)$.

So it only remains to establish (3). This follows easily by induction and our description of $\mathcal{M}(X,\tau)$ as an open subset of the fiber product $\mathcal{M}(X,\tau') \times_{\mathcal{M}(X,\tau_0)} \mathcal{M}(X,\tau'')$. In particular, let C_{w_0} be a line of type I and let $T_{[(C,x,h)']}$ be the Zariski tangent space to $\mathcal{M}(X,\tau')$ at $[(C,x,h)']$. The kernel of the derivative map $T_{[(C,x,h)]} \to T_{[(C,x,h)']}$ is the inverse limit of the system

$$\begin{array}{ccccc} & & H^0(C_{w_0}, N_{h_{w_0}}) \times_{N_{h_{w_0}}|_p} TX|_p & \longrightarrow & H^0(C_{w_0}, N_{h_{w_0}}) \\ & & \downarrow & & \\ TC_{w_1}|_p & \longrightarrow & TX|_p & & \end{array}$$

i.e. $H^0(C_{w_0}, N_{h_{w_0}}) \times_{N_{h_{w_0}}|_p} TC_{w_1}|_p$. Since C_{w_1} and C_{w_0} are distinct, $TC_{w_1}|p$ maps injectively into $N_{h_{w_0}}|_p$. Since C_{w_0} is a line of type I, the normal bundle is generated by global sections. So the inverse limit has codimension 1 in $H^0(C_{w_0}, N_{h_{w_0}})$. Therefore $\dim T_{[(C,x,h)]} \leq \dim\, T_{[(C,x,h)']} + 1$. If we assume that all the lines C_w are lines of type I, then we may make the induction assumption $\dim T_{[(C,x,h)']} = \dim(X,\tau')$. And then we conclude that $\dim\, T_{[(C,x,h)]} = \dim\,(X,\tau)$. So $[(C,x,h)]$ is a smooth point of $\mathcal{M}(X,\tau)$. $\square$

<u>Remarks:</u>

(1) Notice that the inductive proof that $\mathcal{M}(X,\tau)$ is irreducible only used that the morphism $\mathcal{M}(X,\tau) \to \mathcal{M}(X,\sigma_0)$ is dominant even though we proved that this morphism is even faithfully flat. This will be important later.

(2) Notice that $\mathcal{M}(X,\tau'')$ is smooth even at points not satisfying (3). Thus (3) is sufficient but not necessary.

COROLLARY 86. *Let $\phi : \tau_2 \to \tau_1$ be a combinatorial datum such that $\beta_{\tau_1} \equiv 1$. Then $\mathcal{M}(X, \phi)$ is integral and generically smooth of dimension $dim(X, \phi)$ (as a stack).*

PROOF. This follows from lemma 34 and proposition 84. $\square$

PROPOSITION 87. *Let σ be a genus 0 tree, let τ be a basic genus 0 tree, and let $\psi : \tau \to \sigma$ be a contraction of A-graphs. For each $((C_w), (h_w), (q_f)) \in \mathcal{M}(X, \tau)(Spec\ \mathbb{C})$ such that each $h_w(C_w)$ is a line of type I, the image point in $\overline{\mathcal{M}}(X, \sigma)$ is smooth.*

Let σ be a genus 0 tree, let $\phi : \tau_1 \to \tau_2$ be a combinatorial datum such that τ_1 is basic, and let $\psi : \tau_1 \to \sigma$ be a contraction. For each

$$\left(((C_w), (h_w), (q_f)), \left((C'_w), (h'_w), (q'_f)\right), (k_w)\right) \in \mathcal{M}(X, \phi)(Spec\ \mathbb{C})$$

such that each $h_w(C_w)$ is a line of type I, the image point in $\overline{\mathcal{M}}(X, \sigma)$ is smooth.

PROOF. The proof of this result follows the pattern of the proof of theorem 2 in [Fulton-Pandharipande96]. There are only two changes to that proof which need to be made to prove our proposition. The first change is that one needs to allow for deformations which preserve those nodes coming from edges in σ or σ_2. This change is trivial and will be left to the reader. The second change that needs to be made is to provide a proof that for each component $C = C_w$, and for each point $p \in C$, one has $H^1(C, h^*T_X) = H^1(C, h^*T_X(-p)) = 0$. This fact is the only place in the proof of theorem 2 [Fulton-Pandharipande96] where convexity is used.

By hypothesis, the map $h : C \to X$ factors through the map of a line of type I to X, $h = h' \circ k, h' : L \to X, k : C \to L$. And on the line of type I we have $(h')^*T_X = O_L \oplus O_L \oplus O_L(2)$. So, if k has degree β, then $k * T_X = O_C \oplus O_C \oplus O_C(2\beta)$. Since $H^1(C, O_C(r)) = 0$ for all $r \geq -1$, we have the result and the proposition is proved. $\square$

2.6. Almost Basic Components

If $C \subset X$ is a smooth conic, then span$(C) \cap X$ is a plane cubic which contains C as an irreducible component. The residual component is then a line $L \subset X$. Conversely, given a line $L \subset X$ and a 2-plane P containing L, we have that $P \cap X$ is a plane cubic which contains L as an irreducible component. The residual to L will then be conic (possibly degenerate). In this way we see that $\overline{\mathrm{M}}_{0,0}(X, 2)^o$ admits a natural open immersion into the projective bundle $\mathbb{P}_{F_1(X)}(Q)$ where Q is the universal rank 3 quotient bundle of $\mathbb{C}^5$ supported on $F_1(X)$.

LEMMA 88. *Let $I \subset M_{0,0}(X,2) \times M_{0,0}(X,1)$ be the incidence correspondence $\{([C],[L]), C \cap L \neq \emptyset\}$. Then I is irreducible of dimension 5. Also both projections $I \to M_{0,0}(X,2)$ and $I \to M_{0,0}(X,1)$ are dominant and the generic fibers are irreducible.*

PROOF. We may view the product $\mathrm{M}_{0,0}(X,2) \times \mathrm{M}_{0,0}(X,1)$ as an open subscheme of $\mathbb{P}_{F_1(X)}(Q) \times_{\mathbb{C}} F_1(X)$. One sees that the closure of I, $\overline{I}$, is a subvariety of the incidence correspondence

$$D \subset \mathbb{P}_F(Q) \times F, D = \{((([P],[L]),[L'])|L \subset P, L' \cap P \neq \emptyset\}$$

. We notice that D is precisely the pullback of the universal family of hyperplane sections of F by the natural map $\mathbb{P}_F(Q) \to \text{ Grass }(3,\mathbb{C}^5) = \text{ Grass }(2,\mathbb{C}^5)^\vee$. This shows us that D is a Cartier divisor. And $\overline{I}$ is simply the closure of the open subscheme of D, $I^o = \{((([P],[L]),[L']) \in D | L \cap L' = \emptyset\}$. Therefore $\overline{I}$ is an irreducible component of D. So the reduced scheme of $\overline{I}$ is also a Weil divisor which is even Cartier since $\mathbb{P}_F(Q) \times_k F$ is smooth. Since $\overline{I}^{\text{red}}$ is a Cartier divisor, it is a complete intersection and so is certainly Cohen-Macaulay. And F is regular. Therefore, according to the corollary to theorem 23.1 in [Matsumura86], to prove that the projections $\overline{I}_{\text{red}} \to F$, $\overline{I}_{\text{red}} \to \mathbb{P}_F(Q)$ are flat, it suffices to prove that all nonempty fibers of these maps have dimension 3 and 1 respectively.

Suppose that $[L'] \in F$ is a line and $L \subset X$ is a line skew to L'. Let p be a point not contained in span (L, L'). Then $P =$ span (L,p) is a $\mathbb{P}^2$ skew to L'. So $((([P],[L]),[L']) \in \mathbb{P}_F(Q) \times F$ is a point lying over $[L']$ but not in $\overline{I}$. So the fiber of $\overline{I}$ over $[L']$ has dimension 3. Thus $\overline{I}_{\text{red}} \to F$ is flat.

Similarly suppose given $([P],[L]) \in \mathbb{P}_F(Q)$. Since the Fano surface $F_1(X)$ is nondegenerate, i.e. lies in no hyperplane in $\mathbb{P}^{14}$, not every line $L' \subset X$ intersects P. If $L' \cap P = \emptyset$, then $((([P],[L]),[L'])$ is a point lying over $([P],[L])$ but not in $\overline{I}$. So the fiber of $\overline{I}$ over $[L']$ has dimension 1. Thus $\overline{I}^{\text{red}} \to \mathbb{P}_F(Q)$ is flat.

Now we consider the fibers of $\overline{I} \to F$. Fix a line $[L'] \in F$ and consider the open subset U $L' \times F, U = \{(p,[L])|\forall L''$ such that $p \in L'', L \cap L'' = \emptyset\}$. Then U embeds in the fiber $\overline{I}_{[L']}$ via $(p,[L]) \mapsto (($ span $(p,L),[L]),[L'])$. Even if L' contains an Eckardt point, the complement of U in $\overline{I}_{[L']}$ is still only 2 dimensional. Since $\overline{I}_{[L']}$ has pure dimension 3, we see that U is dense. Since U is irreducible, $\overline{I}_{[L']}$ is irreducible.

Since both projection maps are faithfully flat and since the fibers of $\overline{I}_{\mathrm{red}} \to F$ are irreducible, a standard argument shows that $\overline{I}_{\mathrm{red}}$ itself is irreducible. And the morphism $\overline{I}^{\mathrm{red}} \to \mathbb{P}_F(Q)$ admits a section, $([P],[L]) \to (([P],[L]),[L])$. So another standard argument shows that the generic fiber of $\overline{I}_{\mathrm{red}} \to \mathbb{P}_F(Q)$ is irreducible. □

PROPOSITION 89. *Let τ be an almost basic genus 0 tree such that Tail$(\tau) = \emptyset$. Then $\mathcal{M}(X,\tau)$ is equivalent to a finite type, separated $\mathbb{C}$-scheme, $\mathcal{M}(X,\tau)$. And we have the following:*

(1) *$\mathcal{M}(X,\tau)$ is irreducible of dimension dim (X,τ).*
(2) *Let $w \in$ Vertex(τ) be a vertex. Let σ_0 be the A-graph with underlying graph λ_0 such that $\beta(u_0) = \beta_\tau(w)$. Let $\tau \hookleftarrow \sigma_0$ be the combinatorial morphism which sends u_0 to w. The induced morphism $\mathcal{M}(X,\tau) \to \mathcal{M}(X,\sigma_0)$ is dominant.*

PROOF. The proof is similar to the proof of proposition 84. We perform induction on $\beta(\tau)$. If $\beta(\tau) = 2$, the result is trivial. Thus suppose that $\beta(\tau) > 2$. We may find a leaf $w_0 \in \mathrm{Vertex}(\tau)$ such that $w \neq \tilde{w}$. Let f_0 be the unique flag such that $\partial f_0 = w_0$. Let $w_1 = \partial \overline{f}$. There are two cases.

Case I: $w_1 \neq \tilde{w}$ Let τ' and τ'' be as defined in the proof of proposition 87. Then $\mathcal{M}(X,\tau)$ is an open subfunctor of the fiber product stack $\mathcal{Y}$:

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{M}(X,\tau'') \\ \downarrow & & \downarrow \\ \mathcal{M}(X,\tau') & \longrightarrow & \mathcal{M}(X,\tau_0) \end{array}.$$

By induction $\mathcal{M}(X,\tau')$ is equivalent to a finte type, separated $\mathbb{C}$-scheme. By lemma 7, $\mathcal{M}(X,\tau'')$ is equivalent to a finite type, separated $\mathbb{C}$-scheme. Therefore $\mathcal{Y}$ is equivalent to a finite type, separated $\mathbb{C}$-scheme and so is $\mathcal{M}(X,\tau)$. Since $\mathcal{M}(X,\tau'') \to \mathcal{M}(X,\tau_0)$ is faithfully flat, so is $X \to \mathcal{M}(X,\tau')$. Since $\mathcal{M}(X,\tau') \to \mathcal{M}(X,\tau_0)$ is dominant and since $\mathcal{M}(X,\tau'') \to \mathcal{M}(X,\tau_0)$ is flat, we see that $X \to \mathcal{M}(X,\tau'')$ is dominant.

Now suppose that $U_1, U_2 \subset Y$ are nonempty open subsets. Let U_1' and U_2' denote the images of these sets in $\mathcal{M}(X,\tau')$. Since $Y \to \mathcal{M}(X,\tau')$ is fppf, we see that U_1' and U_2' are nonempty open sets. Let $U \subset \mathcal{M}(X,\tau_0)$ be the open set from the proof of proposition 84. Since $\mathcal{M}(X,\tau') \to \mathcal{M}(X,\tau_0)$ is dominant and since $\mathcal{M}(X,\tau_0)$ is irreducible, the preimage of U in $\mathcal{M}(X,\tau')$ is a nonempty open set, U'. By induction $\mathcal{M}(X,\tau')$ is irreducible. Therefore $U_1' \cap U_2' \cap U' \neq \emptyset$. Therefore U_1 and U_2 intersect a fiber of $Y \to \mathcal{M}(X, \tau')$ over a point in U'. Since such a fiber is irreducible, we conclude that $U_1 \cap U_2 \neq \emptyset$. Therefore Y is irreducible. And the dimension of Y is $\dim\mathcal{M}(X, \tau') + \dim\mathcal{M}(X, \tau'') - \dim\mathcal{M}(X, \tau_0) = \dim(X, \tau') + \dim(X, \tau'') - \dim(X, \tau_0) = \dim(X, \tau)$.

Suppose that $[(C, x, h)] \in \mathcal{M}(X, \tau')(\text{Spec } \mathbb{C})$ is some point. If $\tilde{w}$ is not connected to w_1 by an edge, then lemma 85 applies directly to show there is a point in $\mathcal{M}(X, \tau)$ lying above $[(C, x, h)]$. Suppose that $\tilde{w}$ is connected to w_1 by an edge. Let $G \subset C_{w_1}$ be the union of all the sets G_w as in lemma 85 for all w connected to w_1 by an edge such that $\beta_{\tau'}(w) = 1$. In fact the proof of lemma 85 shows that, up to enlarging G by a finite set of closed points, we may assume that $L \not\subset \text{span } (C_{\tilde{w}}, C_{w_1})$. If $L \subset X$ is a line such that $L \cap C_{w_1} \neq \emptyset$, $L \cap G = \emptyset$, and L is not contained in span $(C_{\tilde{w}}, C_{w_1})$, then $C_{w_0} = L$ yields a point of $\mathcal{M}(X, \tau)$ lying above $[(C, x, h)]$. Thus $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \tau')$ is surjective. In particular $\mathcal{M}(X, \tau)$ is a nonempty open subset of Y. Therefore $\mathcal{M}(X, \tau)$ is irreducible of dimension $\dim(X, \tau)$.

For all $w \neq w_0$ we conclude that $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \sigma_0)$ equals the composite of the dominant morphisms $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \tau') \to \mathcal{M}(X, \sigma_0)$, and so is itself dominant. The final case $w = w_0$ factors as $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \tau'') \to \mathcal{M}(X, \sigma_0)$. Since it is the flat basechange of a dominant morphism, $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \tau'')$ is dominant. And by lemma 83, $\mathcal{M}(X, \tau'') \to \mathcal{M}(X, \sigma_0)$ is faithfully flat. Therefore $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \sigma_0)$ is dominant. So $\mathcal{M}(X, \tau) \to \mathcal{M}(X, \sigma_0)$ is dominant for all $w \in \text{Vertex}(\tau)$. So the proposition is proved in this case.

Case 2: $w_1 = \tilde{w}$ We define τ' and τ'' as before. Yet again we have that $\mathcal{M}(X, \tau)$ is an open subfunctor of the fiber product $\mathcal{Y}$:

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{M}(X, \tau'') \\ \downarrow & & \downarrow \\ \mathcal{M}(X, \tau') & \longrightarrow & \mathcal{M}(X, \tau_0) \end{array} .$$

This time $\mathcal{M}(X, \tau'')$ is different than the previous cases. The proof, however, is exactly analogous to the proof of the last case where instead of using lemma 83 we now use lemma 88. Additionally, to get irreducibility of Y we first prove irreducibility of the fiber product Y' defined as above where $\mathcal{M}(X, \tau'')$ is replaced by the reduced scheme of $\mathcal{M}(X, \tau'')$. By the proof of lemma 88, the map from $Y' \to \mathcal{M}(X, \tau')$ is faithfully flat. Since the reduced scheme of Y equals the reduced scheme of Y', we conclude that Y is irreducible. It will be left to the reader to make the necessary changes. $\square$

COROLLARY 90. *Let τ be a genus 0 tree such that there exists $\tilde{w} \in$ Vertex(τ) with $\beta_\tau(\tilde{w}) = 2$ and $\beta_\tau(w) = 1$ for all $w \neq \tilde{w}$. Then $\mathcal{M}(X, \tau)$ is equivalent to a finite type, separated $\mathbb{C}$-scheme $M(X, \tau)$ which is integral of dimension dim(X, τ).*

PROOF. We are simply putting the tails back into τ. If $\tau \hookleftarrow r_1(\tau)$ is the combinatorial morphism which strips all tails, then $\mathcal{M}(X,\tau) \to \mathcal{M}(X, r_1(\tau))$ is representable, flat, separated, and finite type with geometric fibers which are irreducible of dimension $\#\text{Tail}(\tau)$. By the previous result $\mathcal{M}(X, r_1(\tau))$ is equivalent to a finite type, separated $\mathbb{C}$-scheme $M(X, r_1(\tau))$ which is irreducible of dimension $\dim(X, r_1(\tau))$. Thus $\mathcal{M}(X,\tau)$ is equivalent to a finite type, separated $\mathbb{C}$-scheme $M(X,\tau)$ which is integral of dimension $\dim(X, r_1(\tau)) + \#\text{Tail}(\tau) = \dim(X,\tau)$. $\square$

PROPOSITION 91. *Let $\phi : \tau_1 \to \tau_2$ be an almost basic combinatorial datum such that $\tau_1 = r_1(\tau_1)$. Then $\mathcal{M}(X, \phi)$ is equivalent to a finite type, separated $\mathbb{C}$-scheme, $M(X, \phi)$. And we have the following:*

1. *$M(X, \phi)$ is integral of dimension dim(X, ϕ).*
2. *Let $w \in$ Vertex$_{\tau_1}$ be a vertex. Then $M(X, \phi) \to M(X, \{w\})$ is flat and dominant.*

PROOF. Doubtless the reader is growing weary of seeing the same argument repeated. Therefore we shall just indicate what changes must be made to the proof of proposition 89 in order to prove the current proposition.

One proceeds by induction on $\beta(\tau_2)$. One chooses a leaf $w_0 \in \text{Vertex}(\tau_1)$. If the preimage of $\phi_V(w_0)$ is just w_0, then one proceeds just as in the proof of proposition 89.

Thus suppose that $\phi_V(w_0)$ consists of more than one vertex. Necessarily we have $\beta_{\tau_2}(\phi_V(w_0)) = 1$. Let τ_2'' be the A-graph whose underlying graph is λ_1 with $\beta(u_0) = 1$. Let τ_1'' be the A-graph whose underlying graph is λ_2 with $\beta(u_0) = \beta(u_1) = 1$. Let $\phi'' : \tau_1'' \to \tau_2''$ be the elementary combinatorial datum such that $\phi_V''(u_0) = \phi_V''(u_1) = u_0$, and $\phi_F''(e_0) = \phi_F''(e_1) = e_0$. One has combinatorial morphisms $\tau_1 \hookleftarrow \tau_1''$ and $\tau_2 \hookleftarrow \tau_2''$ as follows: Send u_0 in σ_2 to $\phi_V(w_0)$ and e_0 in σ_2 to $\phi_F(f_0)$. And send u_0, u_1 in σ_1 to w_0, w_1; e_0, e_1 to $f_0, \overline{f_0}$. These morphisms induce a 1-morphism $\mathcal{M}(X,\phi) \to \mathcal{M}(X, \phi'')$.

Define $\phi' : \tau_1' \to \tau_2'$ to be the combinatorial datum obtained by pruning f_0 and w_0 from τ_1. There is an obvious 1-morphism $\mathcal{M}(X, \phi) \to \mathcal{M}(X, \phi')$.

Finally define τ_0 to be the same old A-graph: the underlying graph is λ_0 with $g(u_0) = 0, \beta(u_0) = 1$. Let $\upsilon : \tau_0 \to \tau_0$ be the unique simple combinatorial datum which is the identity on underlying trees. We have combinatorial morphisms $\tau_1' \hookleftarrow \tau_0$ and $\tau_2' \hookleftarrow \tau_0$ by sending u_0 to w_1 in $\text{Vertex}(\tau_1')$ and to $\phi_V'(w_1)$ in $\text{Vertex}(\tau_2')$. We have combinatorial morphisms $\tau_1'' \hookleftarrow \tau_0$ and $\tau_2'' \hookleftarrow \tau_0$ by sending u_0 to u_1 in $\text{Vertex}(\tau_1'')$ and to u_0 in $\text{Vertex}(\tau_2'')$. These combinatorial morphisms induce obvious 1-morphisms: $\mathcal{M}(X, \phi') \to \mathcal{M}(X, \upsilon), \mathcal{M}(X, \phi'') \to \mathcal{M}(X, \upsilon)$. These morphisms exhibit $\mathcal{M}(X, \phi)$ as an open substack of the fiber product stack $\mathcal{Y}$:

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{M}(X, \phi'') \\ \downarrow & & \downarrow \\ \mathcal{M}(X, \phi') & \longrightarrow & \mathcal{M}(X, \upsilon) \end{array}.$$

The only nontrivial new result one needs is the following: $\mathcal{M}(X, \phi'')$ is equivalent to a finite type, separated $\mathbb{C}$-scheme, $\mathcal{M}(X, \phi'')$. Moreover this is a smooth, connected scheme of dimension 3. And the morphism $\mathcal{M}(X, \phi'') \to \mathcal{M}(X, \upsilon)$ is faithfully flat with irreducible geometric fibers.

Indeed, an element

$$\left(((C_v), (h_v), (q_f))\,, \left((C_v'), (h_v'), (q_f')\right)\,, (k_v)\right) \in \mathcal{M}(X, \phi'')$$

is uniquely determined by $\left((C_v'), (h_v'), (q_f')\right)$. Thus $\mathcal{M}(X, \phi'')$ is a nonempty open subscheme of the universal line over $F_1(X)$, $F_{0,1}(X)$. And the morphism $\mathcal{M}(X, \phi'') \to \mathcal{M}(X, \upsilon)$ just corresponds to the projection $F_{0,1}(X) \to F_1(X)$.

One uses this fact in exactly the analogous manner to lemma 83 and lemma 88. We leave the necessary changes to the reader. $\square$

COROLLARY 92. *Let $\phi : \tau_1 \to \tau_2$ be any almost basic combinatorial datum. Then $\mathcal{M}(X, \phi)$ is equivalent to a finite type, separated $\mathbb{C}$-scheme which is integral of dimension dim (X, ϕ).*

PROOF. This is the same as the proof of corollary 90. $\square$

Finally we come to the construction of the distinguished component.

PROPOSITION 93. *There is an irreducible component $M_\beta \subset \overline{\mathcal{M}}_{0,0}(X, \beta)$ with the following property. Let τ be any nice, basic genus 0 tree such that $\beta(\tau) = \beta$ and Tail(τ) $= \emptyset$. Then M_β is the unique irreducible component through which the morphism $\mathcal{M}(X, \tau) \to \overline{\mathcal{M}}_{0,0}(X, \beta)$ factors. Let $\phi : \tau_1 \to \tau_2$ be any nice, basic combinatorial datum such that $\beta(\tau_1) = \beta$ and Tail(τ_1) $= \emptyset$. Then M_β is the unique irreducible component through* *which the morphism* $\mathcal{M}(X,\phi) \to \overline{\mathcal{M}}_{0,0}(X,\beta)$ *factors. We call* M_β *the* distinguished component.

PROOF. First of all note that the stacks $\mathcal{M}(X,\tau)$ are equivalent to integral schemes and $\mathcal{M}(X,\phi)$ are integral stacks. And by prop 87, the image of the morphism intersects the smooth locus of $\overline{\mathcal{M}}_{0,0}(X,\beta)$. Thus we conclude that for each morphism there is a unique component (which a priori depends on the morphism) such that the morphism factors through that component. Let us refer to this component as the distinguished component of τ or the distinguished component of ϕ respectively. We wish to prove that these distinguished components all coincide.

Reduction to Basic A-graphs: First we reduce the case of a basic combinatorial datum to the case of a basic A-graph. Let $l = \#\mathrm{Vertex}(\tau_1) - \#\mathrm{Vertex}(\tau_2$. If $l > 1$, we will prove that there is a basic stable pair $\phi' : \sigma_1 \to \sigma_2$ whose distinguished component is the same as the distinguished component of ϕ and such that $\#\mathrm{Vertex}(\sigma_1) - \#\mathrm{Vertex}(\sigma_2) < l$. If $l = 1$, we will prove that there is a basic A-graph σ whose distinguished component is the same as the distinguished component of ϕ. We do this by smoothing an "exterior" node (i.e. a node corresponding to an edge in τ_2) and then allowing the smooth component to become reducible in such a way as to reduce the size of l.

Let $\{f, \overline{f}\}$ be an edge in τ_2 such that the preimage of ∂f contains at least 2 vertices. Since $\phi : \tau_1 \to \tau_2$ is a combinatorial datum, there is a unique edge $\{f', \overline{f'}\}$ which maps under ϕ_F to $\{f, \overline{f}\}$. We define $w_0 = \partial f', w_1 = \partial \overline{f'}$. We form a new graph τ' along with a factorization of $\phi : \tau_1 \to \tau_2$ by $\phi_1 : \tau_1 \to \tau', \phi_2 : \tau' \to \tau_2$. We define $\tau' \to \tau_2$ to be the smallest "covering" of τ_1 with the following property: for every vertex w whose distance from $\phi_{1,V}(w_0)$ or $\phi_{1,V}(w_1)$ is at most 1 edge, the preimage $\phi_{1,V}^{-1}(w)$ consists of a single vertex.

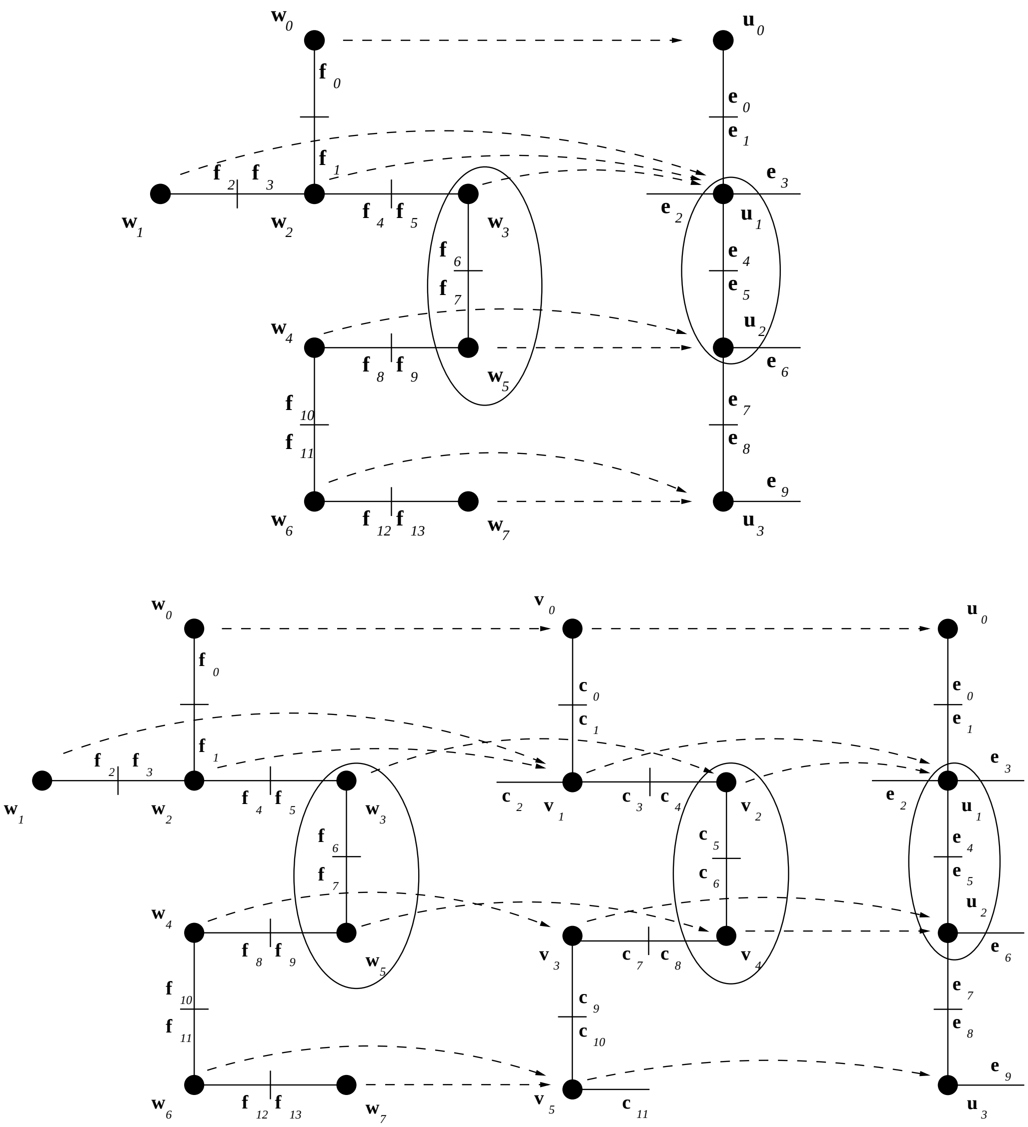

Diagram 6

For illustrative purposes we have changed notation in diagram 6. Here the role of f_0 is played by e_4. The first picture shows a basic combinatorial datum. The second picture shows the factorization of ϕ into ϕ_1 and ϕ_2.

There are two possibilities for ϕ_1.

Case I: ϕ_1 nontrivial. If ϕ_1 is a nontrivial covering map, then we consider $\mathcal{M}(X,\phi)$ as embedded in $\overline{\mathcal{M}}(X,\phi_1)$. Now let $\psi_1 : \tau_1 \to \sigma_1'$ and $\psi_2 : \tau_2 \to \sigma_2'$ be the smallest contractions which contract the edge $\{f', \overline{f'}\}$

and such that we have a combinatorial morphism $\phi'_1 : \sigma'_1 \to \sigma'_2$ such that $\phi'_1 \circ \psi_1 = \psi_2 \circ \phi_1$. Let $w' \in \text{Vertex}(\sigma'_2)$ be the vertex corresponding to this contracted edge. Then $\beta(w') = 2$ and one sees that $\phi'_1 : \sigma'_1 \to \sigma'_2$ is an almost basic combinatorial datum. One uses the contractions ψ_1 and ψ_2 to obtain an obvious 1-morphism $\mathcal{M}(X, \phi_1) \to \mathcal{M}(X, \phi'_1)$. This induces a map of the (set) closures closure($\mathcal{M}(X, \phi_1)$) $\to$ closure($\mathcal{M}(X, \phi'_1)$. By proposition 87, for a general point

$$\zeta = \big(((C_v), (h_v), (q_f)) \,, \big((C'_v), (h'_v), (q'_f)\big) \,, (k_v)\big) \in \mathcal{M}(X, \phi),$$

the image point in closure($\mathcal{M}(X, \phi'_1)$) will be a smooth point. We may construct the universal deformation of this point. Clearly the generic point of this deformation will smooth the node corresponding to $\{f', \overline{f'}\}$. Therefore the image of the generic point of the universal deformation is contained in $\mathcal{M}(X, \phi'_1)$. Since ζ is a smooth point of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ and since $\mathcal{M}(X, \phi'_1)$ is irreducible, we see that the distinguished component of ϕ contains $\mathcal{M}(X, \phi'_1)$.

But now we also have the contraction $(\phi_1 : \tau_1 \to \tau') \xrightarrow{\psi} (\phi'_1 : \sigma'_1 \to \sigma'_2)$ and $\phi_1 : \tau_1 \xrightarrow{\phi_1} \tau'$ is a basic datum. The same argument as above shows that $\mathcal{M}(X, \phi'_1)$ is contained in the distinguished component of ϕ_1. Since the distinguished component is close, it also contains $\mathcal{M}(X, \phi)$. Therefore the distinguished component of ϕ equals the distinguished component of ϕ. And $\#\text{Vertex}(\tau_1) - \#\text{Vertex}(\tau')$ is smaller than $\#\text{Vertex}(\tau_1) - \#\text{Vertex}(\tau_2)$ by at least 1 since now there is a single vertex in τ' lying above $\phi_{1,V}(w_0)$.

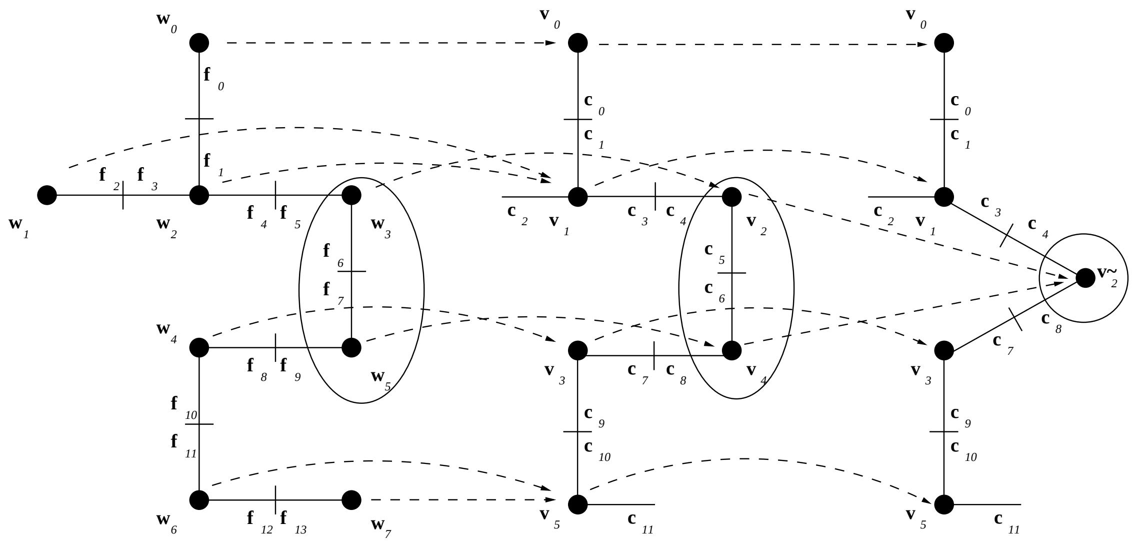

Diagram 7

This example is a continuation of the example in diagram 6. The first map illustrated is the covering map ϕ_1. The second map is the contraction $\psi_1 : \tau' \to \sigma_2$, (the contraction $\psi_2 : \tau_1 \to \sigma_1$ is easily deduced from diagram 6 and so is not illustrated) in which $\tilde{v}_2$ is the unique vertex with $\beta(\tilde{v}_2) = 2$.

Case II: ϕ_1 trivial. If $\phi_1 : \tau_1 \to \tau'$ is trivial, then we can repeat the argument in case I and now we deduce that the distinguished component of $\phi : \tau_1 \to \tau_2$ equals the distinguished component of the basic A-graph τ'.

Therefore we are reduced to considering the case of a basic A-graph τ.

Linking Basic A-graphs. Let l be the length of a maximal chain inside of τ. If $l < \beta$ we will prove that there is a basic A-graph σ which contains a chain of length $> l$ and such that the distinguished component of τ equals the distinguished component of σ. To prove this, let $\{f, \overline{f}\}$ be an edge such that ∂f is contained in a maximal chain but $\partial \overline{f}$ is not contained in this maximal chain. Let $\tau \xrightarrow{\psi} \sigma'$ be the contraction which only contracts $\{f, \overline{f}\}$ to a vertex w' such that $\beta(w') = 2$. We obtain an induced morphism $\mathcal{M}(X, \tau) \to \overline{\mathcal{M}}(V, \sigma')$. If

$$\zeta = ((C_v), (h_v), (q_f)) \in \mathcal{M}(X, \tau)$$

is a general $\mathbb{C}$-valued point, then the image of this point in $\overline{\mathcal{M}}(X, \sigma')$ is smooth by proposition 87. We may form the universal deformation space of this point. Clearly the generic point of this universal deformation space will smooth the node corresponding to $\{f, \overline{f}\}$. Thus the generic point of the universal deformation space will lie in $\mathcal{M}(X, \sigma')$. Since ζ is a smooth point of $\overline{\mathcal{M}}_{0,0}(X, \beta)$ and since $\mathcal{M}(X, \sigma')$ is irreducible, we conclude that distinguished component of τ contains $\mathcal{M}(X, \sigma')$.

Now we let $\sigma \xrightarrow{\psi'} \sigma'$ be a contraction obtained by expanding the vertex w' into an edge in such a way that we increase the length of the maximal chain. By the same argument as above, we conclude that the distinguished component of σ contains $\mathcal{M}(X, \sigma')$. Therefore it contains $\mathcal{M}(X, \tau)$. Therefore the distinguished component of τ equals the distinguished component of σ.

Example:

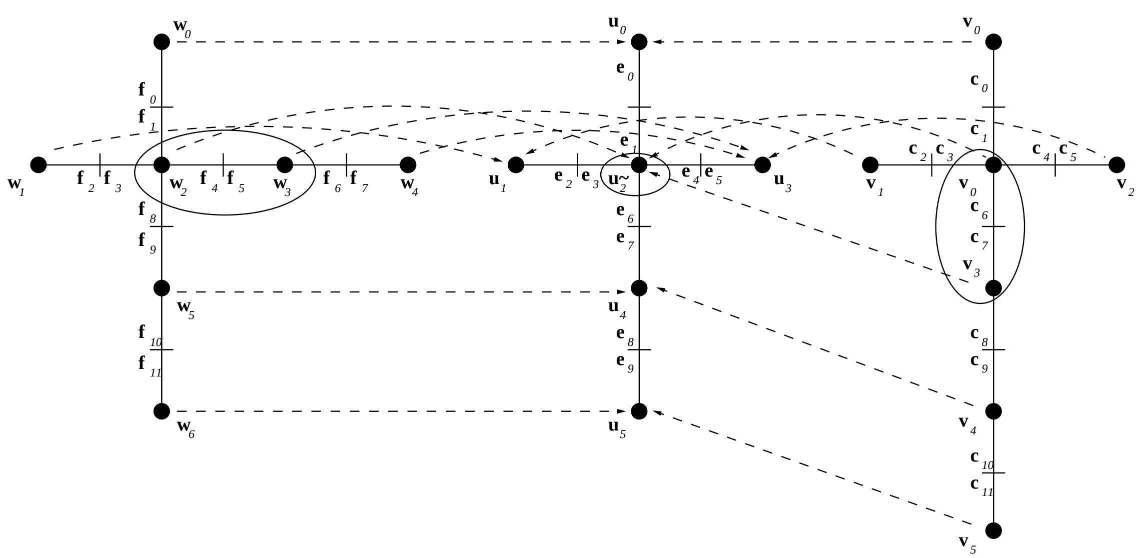

Diagram 8

The diagram illustrates how the basic A-graph on the left and the basic A-graph on the right both contract to the same almost basic A-graph in the middle. Here the contracted edges are circled and $\tilde{u}_2$ is the unique vertex with $\beta(\tilde{u}_2) = 2$.

Since there is a unique basic A-graph which contains a chain of length β, we see that all of the distinguished components are actually equal.

□

Corollary 94. *For each r and e, there is an irreducible component $M \subset \overline{\mathcal{M}}_{0,r}(X,e)$ such that for every type I, nice, basic point $p \in \overline{\mathcal{M}}_{0,r}(X,e)$, M is the unique irreducible component containing p. Moreover M is generically smooth of dimension $2e+r$.*

Proof. By lemma 34, it suffices to consider the case when $r=0$. By the last theorem, there is a distinguished component $M_e \subset \overline{\mathcal{M}}_{0,0}(X,e)$ which contains every nice, basic component, in particular it contains every type I, nice, basic point. But a type I, nice, basic point is a smooth point of $\overline{\mathcal{M}}_{0,0}(X,e)$ and therefore lies on a unique irreducible component. Finally since $\overline{\mathcal{M}}_{0,0}(X,e)$ is smooth of dimension $2e$ at each type I, nice, basic point, M is also smooth of dimension $2e$ at each type I, nice, basic point. □

2.7. Bend-and-Break

Together theorem 77 and cor 94 imply that $M \subset \overline{\mathcal{M}}_{0,r}(X,e)$ is the unique irreducible component which contains every nonlinear, moderate component $\mathcal{M}(X,\phi)$. In this section we will finish the proofs of theorem 64, theorem 65, theorem 66, and theorem 80. Using the lemmas from section 2.3 and section 2.4, all that remains to finish the proofs is to prove that for each $\tau_r(e)$ and for each corresponding simple combinatorial datum $\phi : \tau_r(e) \to \tau_r(e)$

1. $\mathcal{M}(X,\phi)$ has pure dimension $2e+r$,
2. each evaluation 1-morphism $e : \mathcal{M}(X,\phi) \to X$ has constant fiber dimension $2e+r-3$ over X_f and has fiber dimension at most $2e+r-2$ over each Eckardt point $p \in E$, and
3. for each irreducible component $A \subset \mathcal{M}(X,\phi)$, there is a moderate, nonlinear combinatorial datum $\psi : \sigma_1 \to \sigma_2$ with canonical contraction $\sigma_1 \to \tau_r(e)$, and an irreducible component $B \subset \mathcal{M}(X,\psi)$ such that $\overline{F_1'}(B) \subset \overline{F_1'}(A)$ is a codimension 1 subvariety.

Of course to prove this for all $\tau_r(e)$, it is clearly equivalent to prove this for $\tau_1(e)$. For $e=2$ this is proved in section 2.6. Thus we may suppose that $e \geq 3$. Moreover, having reduced to the case $\tau_1(e)$, we are free to prove the result for $\tau_2(e)$ instead (our reason for doing this will become clear). Let $\phi : \tau_2(e) \to \tau_2(e)$ be the unique simple datum which is the identity map on underlying trees. We will prove 1, 2, and 3 above for each ϕ by induction on e (where $e=2$ is the base case).

Suppose that $A \subset \mathcal{M}(X,\phi)$ is an irreducible component. Without loss of generality, identify A as an open subset of an irreducible component of $\overline{\mathcal{M}}_{0,2}(X,e)$. Let $E : \overline{\mathcal{M}}_{0,2}(X,e) \to X \times X$ denote the evaluation map coming from the two flags f_1, f_2 of $\tau_2(e)$. Let $Y \subset X \times X$ denote the image of $\overline{A}$ under E. Let $(p, q) \in Y$ be any point such that $\{p, q\}$ lies on no line $L \subset X$ (since the stable maps parametrized by A are birational to their image, we can find such a pair on the image of each such stable map). Now the virtual dimension $2e+2$ of $\mathcal{M}(X, \phi)$ is always a lower bound, i.e. $\dim(A) \geq 2e+2$. So the dimension of the fiber $E^{-1}(p,q) \subset \overline{A}$ is at least $\dim(A) - \dim(Y) \geq 2e+2-6 = 2e-4$. Since $e \geq 3$, this is at least 2 (in particular it is positive).

Let $D \subset E^{-1}(p,q)$ be any irreducible component. The claim is that $D \cap (\overline{A} - A) \subset D$ is a codimension 1 subvariety. By [Cornalba95], we know that the coarse moduli scheme of $\overline{\mathcal{M}}_{0,2}(X, e)$ is projective, thus the coarse moduli scheme of D (with the reduced induced structure) is projective. So to establish that $D \cap (\overline{A} - A) \subset D$ is codimension 1, it suffices to establish that for any complete curve $K \subset D$, $K \cap (\overline{A} - A) = neq\emptyset$.

Suppose the contrary, i.e. suppose that there is a complete curve $K \subset D \cap A$. Now after replacing K by a ramified cover of K, we may assume that K actually factors through the stack of D and not just the coarse moduli scheme of D. We may also assume that K is smooth. Let $\pi : C \to K$ be the pullback to K of the universal curve, let $s_1, s_2 : K \to C$ be the two markings and let $h : C \to X$ be the map to X. Since the image of K is contained in A, π is smooth. Now after again replacing K by a ramified cover of K, we may suppose that there is a third section $s_3 : K \to C$ such that $h(s_1(K)), h(s_2(K))$ and $h(s_3(K))$ are disjoint: simply take any hyperplane section H of X which contains neither p nor q and replace K by the normalization of the multisection $h^{-1}(H)$ of π. But then, since any rational curve with three distinct marked points is canonically isomorphic to $(\mathbb{P}^1, 0, 1, \infty)$, we may identify $(\pi : C \to K, s_1, s_2, s_3)$ with $(\pi_1 : K \times \mathbb{P}^1, t_0, t_1, t_\infty)$ where $t_i(x) = (x, i) \in K \times \mathbb{P}^1$ for all $x \in K$. But then the morphism $h : \mathbb{P}^1 \times K \to X$ along with the sections t_0, t_1 satisfies the hypothesis of theorem 61 This contradicts that $K \subset \mathcal{M}(X, \phi)$. So we conclude that $D \cap (\overline{A} - A) \subset D$ is a codimension 1 subvariety. Since this is true for every irreducible component, we conclude that $\overline{A} - A \subset \overline{A}$ is a codimension 1 subvariety.

By the proof of corollary 37, we know that $\overline{A} - A$ is contained in $\overline{\mathcal{M}}_{0,2}(X, e) - \mathcal{M}(X, \phi)$. Thus $\overline{A} - A$ is the union of its (locally closed) intersections with the other components $\mathcal{M}(X, \psi)$. Thus there is some ψ such that $\mathcal{M}(X, \psi) \cap (\overline{A} - A)$ has codimension 1 in $\overline{A}$.

Let us consider the other components of the CD decomposition of $\overline{\mathcal{M}}_{0,2}(X,e)$. One of the combinatorial data we have to consider is the simple combinatorial datum $\phi' : \tau \to \tau$ where the underlying tree of τ is the tree with two vertices v_1, v_2, one edge $\{f_1, f_2\}, \partial f_1 = v_1, \partial f_2 = v_2$ and two tails $\partial f_3 = v_2, \partial f_4 = v_4$ and such that $\beta(v_1) = e, \beta(v_2) = 0$. But clearly $\mathcal{M}(X, \phi')$ will not intersect any of the fibers $E^{-1}(p,q)$ with $p \neq q$.

Second we should consider all components $\mathcal{M}(X, \psi)$ such that ψ is linear. But for general $p, q \in E(\overline{A})$, $\{p,q\}$ lies on no line $L \subset X$. Thus no component $\mathcal{M}(X,\psi)$ can dominate $E(\overline{A})$. So we can discount all linear components.

Next we should consider all components $\mathcal{M}(X,\psi)$ such that ϕ is not moderate. But by theorem 67, $\dim(\mathcal{M}(X,\psi)) \leq 2e-1$, so it cannot have codimension 1 in $\overline{A}$.

All that is left are the components $\mathcal{M}(X,\psi)$ such that ψ is moderate and nonlinear. By induction the 1, 2, and 3 have already been proved for all $e' < e$. And the induction arguments in section 2.3 for $\psi : \sigma_1 \to \sigma_2$ with canonical contraction $\sigma_1 \to \tau_2(e)$ only use 1, 2, and 3 for $e' < e$. Therefore we know that for each such ψ, $\mathcal{M}(X,\psi)$ has pure dimension $\dim(X,\psi)$. And by the analogous (trivial) calculation to lemma 50, we see that $\dim(X,\psi) \leq 2e+1$. The only conclusion is that $\dim(A) \leq 2e+2$. Of course this is equivalent to $\dim(A) = 2e+2$. Also, since $\mathcal{M}(X,\psi) \cap (\overline{A} - A)$ has dimension $2e+1 = \dim(\mathcal{M}(X,\psi)$, we conclude that there is an irreducible component $B \subset \mathcal{M}(X,\psi)$ such that $\overline{B} \subset \overline{A}$. This proves 1 and 3.

All that remains is to prove 2. Let $p \subset X_f$ and consider the preimage $e_{f_1}-1(p) \subset \overline{A}$. Let $D \subset e_{f_1}^{-1}(p)$ be any irreducible component. Since this is a union of subsets $E^{-1}(p,q)$, we know that $D \cap (\overline{A} - A)$ has codimension 1 in D. So it intersects some $\mathcal{M}(X,\psi)$ in a set of codimension 1. Since p is not an Eckardt point we can show that ϕ is moderate – indeed for ψ which are not moderate, the only fibers of $e_{f_1} : \mathcal{M}(X,\psi) \to X$ with dimension $> 2e-2$ are the fibers over Eckardt points. Also, since for a general point $q \in e_{f_2}(D)$, the pair $\{p,q\}$ lies on no lines, we see that ψ must be nonlinear. And then we conclude by theorem 66, that $\mathcal{M}(X,\psi) \cap D \cap (\overline{A} - A)$ is contained in the fiber of $e_{f_1} : \mathcal{M}(X,\psi) \to X$ over p which has dimension $leq(2e+1) - 3 = 2e-2$. Thus $\dim(D) \leq 2e-1$ when $p \in X_f$.

If $p \in X$ is an Eckardt point, then by the same argument each irreducible component D of $e_{f_1}^{-1}(p) \cap \overline{A}$ contains some locus $\mathcal{M}(X, \psi) \cap D$ as a codimension 1 subvariety. As before, we can rule out all linear ψ. If ψ is moderate, then by theorem 66, we conclude that $\dim(D) \leq 2e$. But even for ψ which aren't moderate, by theorem 67 we know $\dim(\mathcal{M}(X, \psi)) \leq 2e - 1$ so again we conclude $\dim(D) \leq 2e$. This proves 2. This finishes the proof of theorem 62.

2.8. Bibliography

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