A NOTE ON HURWITZ SCHEMES OF COVERS OF A POSITIVE GENUS CURVE

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Abstract. Let B be a smooth, connected, projective complex curve of genus h. For $w \geq 2d$ we prove the irreducibility of the Hurwitz stack $\mathcal{H}^{d,w}_{S_d}(B)$ parametrizing degree d covers of B simply-branched over w points, and with monodromy group S_d.

1. Introduction

Suppose that B is a smooth, connected, projective complex curve of genus h. Let $d > 0$ and $w \geq 0$ be integers such that $g := d(h-1) + \frac{w}{2} + 1$ is a nonnegative integer (in particular w is even). We define $\mathcal{H}^{d,w}(B)$ to be the open substack of the Kontsevich moduli stack $\overline{\mathcal{M}}_{g,0}(B,d)$ parametrizing stable maps $f : X \to B$ such that X is smooth and f is finite with only simple branching. Let $\mathrm{br}(f) \subset B$ denote the branch divisor of f. If we choose a basepoint $b_0 \in B - \mathrm{br}(f)$ and an identification $\phi : f^{-1}(b_0) \to \{1, \ldots, d\}$, there is an induced monodromy homomorphism $\tilde{\phi} : \pi_1(B - \mathrm{br}(f), b_0) \to S_d$ which associates to any loop $\gamma : [0,1] \to B$ with $\gamma(0) = \gamma(1) = b_0$, the permutation of $f^{-1}(b_0)$ determined by analytic continuation along γ. In particular, the subgroup $\mathrm{image}(\tilde{\phi}) \subset S_d$ is well-defined up to conjugation independently of ϕ. The corresponding conjugacy class of subgroups determines a locally constant function on $\mathcal{H}^{d,w}(B)$. Given a subgroup $G \subset S_d$, we define $\mathcal{H}^{d,w}_G(B)$ to be the open and closed substack of $\mathcal{H}^{d,w}(B)$ parametrizing stable maps $f : X \to B$ whose corresponding monodromy group is conjugate to G. We are particularly interested in $\mathcal{H}^{d,w}_{S_d}(B)$, the stack parametrizing Hurwitz covers of B with full monodromy group.

Theorem 1.1. *If $w \geq 2d$, then $\mathcal{H}^{d,w}_{S_d}(B)$ is a connected, smooth, finite-type Deligne-Mumford stack over $\mathbb{C}$.*

The fact that $\mathcal{H}^{d,w}_{S_d}(B)$ is a finite-type Deligne-Mumford stack follows from the fact that $\overline{\mathcal{M}}_{g,0}(B,d)$ is a finite-type Deligne-Mumford stack. The fact that $\mathcal{H}^{d,w}_{S_d}(B)$ is smooth follows from a trivial deformation theory computation. So the content of theorem 1.1 is that $\mathcal{H}^{d,w}_{S_d}(B)$ is connected.

This is a classical fact when $h = 0$, i.e. for branched covers of $\mathbb{P}^1$, (c.f. [1], [5], and for a modern account [3, prop. 1.5]). This fact is well-known to experts, but there seems to be no reference. We used theorem 1.1 in our paper [4], and so we present a proof below. We wish to thank Ravi Vakil for useful discussions.

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2. Setup

Our eventual goal is to prove theorem 1.1, but for most of this paper, we shall work with schemes which admit étale maps to $\mathcal{H}^{d,w}_{S_d}(B)$. Suppose $\Sigma \subset B$ is a finite subset, and suppose $b_0 \in \Sigma$ is a point. We define $M^{d,w}(B,\Sigma,b_0)$ to be the fine moduli scheme parametrizing pairs $(f : X \to B, \phi)$ where $f : X \to B$ is a stable map in $\overline{\mathcal{M}}_{g,0}(B,d)$ and where $\phi : f^{-1}(b_0) \to \{1,\dots,d\}$ are such that

(1) f is finite,
(2) f is unramified over Σ, and
(3) ϕ is a bijection.

Using known results on the Kontsevich moduli space $\overline{\mathcal{M}}_{g,0}(B,d)$, it is easy to show that $M^{d,w}(B,\Sigma,b_0)$ is a nonempty, smooth, quasi-projective scheme of dimension w. By [2], there is a branch morphism $\mathrm{br} : M^{d,w}(B,\Sigma,b_0) \to (B-\Sigma)_w$ where $(B-\Sigma)_w$ is the wth symmetric power parametrizing effective degree w divisors on $B-\Sigma$. It is clear that br is quasi-finite, and thus $\mathrm{br} : M^{d,w}(B,\Sigma,b_0) \to (B-\Sigma)_w$ is dominant. We denote by $(B-\Sigma)^o_w \subset (B-\Sigma)_w$ the Zariski open subset parametrizing reduced effective divisors of degree w in $B-\Sigma$. We define $H^{d,w}(B,\Sigma,b_0) \subset M^{d,w}(B,\Sigma,b_0)$ to be the preimage under br of $(B-\Sigma)^o_w$.

For each pair $(f : X \to B, \phi)$ in $H^{d,w}(B,\Sigma,b_0)$ with branch divisor $\mathrm{br}(f)$, there is an induced monodromy homomorphism $\tilde{\phi} : \pi_1(B - \mathrm{br}(f), b_0) \to S_d$ where S_d is the symmetric group of permutations of $\{1,\dots,d\}$. The image of $\tilde{\phi}$ determines a locally constant function on $H^{d,w}(B,\Sigma,b_0)$. Because $M^{d,w}(B,\Sigma,b_0)$ is smooth and $H^{d,w}(B,\Sigma,b_0)$ is dense in $M^{d,w}(B,\Sigma,b_0)$, this locally constant function extends to all of $M^{d,w}(B,\Sigma,b_0)$. Given a subgroup $G \subset S_d$ we define $M^{d,w}_G(B,\Sigma,b_0)$ (resp. $H^{d,w}_G(B,\Sigma,b_0)$) to be the open and closed subscheme of $M^{d,w}(B,\Sigma,b_0)$ (resp. $H^{d,w}(B,\Sigma,b_0)$) on which the image of $\tilde{\phi}$ equals G.

Let $F : \mathcal{X}(w) \to H^{d,w}_{S_d}(B,\Sigma,b_0) \times B$ be the pullback of the universal stable map, i.e. $\mathcal{X}(w)$ parametrizes data $(f : X \to B, \phi, x)$ where $(f : X \to B, \phi) \in H^{d,w}_{S_d}(B,\Sigma,b_0)$ and $x \in X$, and $F(f : X \to B, \phi, x) = (f : X \to B, \phi, f(x))$. We denote by $U \subset \mathcal{X}(w) \times_{F,F} \mathcal{X}(w)$ the open subscheme of the fiber product of $\mathcal{X}(w)$ with itself over $H^{d,w}_{S_d}(B,\Sigma,b_0) \times B$ parametrizing data $(f : X \to B, \phi, x_1, x_2)$ such that $x_1 \neq x_2$, and such that $f(x_1) = f(x_2)$ is neither in S nor equal to any branch point of f. We define $\mathcal{X}_2(w)$ to be the quotient of U by the obvious involution $(f : X \to B, \phi, x_1, x_2) \sim (f : X \to B, \phi, x_2, x_1)$. We denote by $\mathcal{X}^e_2(w)$ the open subscheme of the e-fold fiber product of $\mathcal{X}_2(w)$ with itself over $H^{d,w}_{S_d}(B,\Sigma,b_0)$ parametrizing data $(f : X \to B, \phi, \{x^1_1, x^2_2\}, \dots, \{x^e_1, x^e_2\})$ such that $f(x^1_1), \dots, f(x^e_1)$ are all distinct points in $B - \Sigma$. Notice that the projection $\mathcal{X}^e_2(w) \to H^{d,w}_{S_d}(B,\Sigma,b_0)$ is flat. The condition that the image of $\tilde{\phi}$ be all of S_d, and therefore doubly-transitive, implies that $\mathcal{X}_2(w) \to H^{d,w}_{S_d}(B,\Sigma,b_0)$ has irreducible fibers. Therefore also $\mathcal{X}^e_2(w) \to H^{d,w}_{S_d}(B,\Sigma,b_0)$ has irreducible fibers.

For each $(f : X \to B, \phi, \{x^1_1, x^1_2\}, \dots, \{x^e_1, x^e_2\})$ in $\mathcal{X}^e_2(w)$ we can associate a pair $(f_a : X_a \to B, \phi_a)$ in $H^{d,w+2e}_{S_d}(B,\Sigma,b_0)$ as follows:

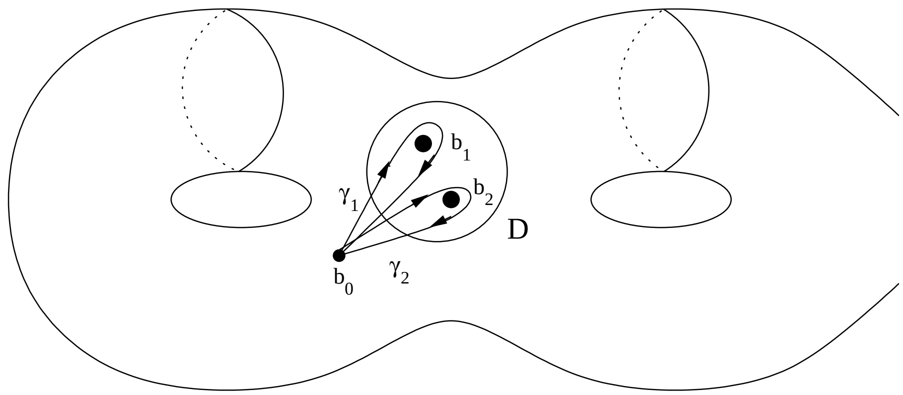

FIGURE 1. Adjacent branch points with equal monodromy

(1) We define X_a to be the e-nodal curve whose normalization is of the form $u : X \to X_a$ such that $u(x_1^i) = u(x_2^i)$ for each $i = 1, \dots, e$,
(2) we define $f_a : X_a \to B$ to be the unique morphism such that $f = f_a \circ u$, and
(3) we define ϕ_a to be the unique map such that $\phi = \phi_a \circ u$.

This association defines a regular morphism $G_{w,e} : \mathcal{X}_2^e(w) \to M_{S_d}^{d,w+2e}(B, \Sigma, b_0)$.

We give a topological application of the morphism $G_{w,e}$. Suppose we have a pair $(f : X \to B, \phi)$ in $H_{S_d}^{d,w+2}(B, \Sigma, b_0)$ and $D \subset B$ is a closed disk which is disjoint from Σ, such that $D \cap \mathrm{br}(f)$ consists of two branch points b_1, b_2 which are contained in the interior of D. Define $U = B - D$ and suppose that $\tilde{\phi} : \pi_1(U - \mathrm{br}(f), b_0) \to S_d$ is surjective. Suppose moreover that f is trivial over the boundary ∂D of D, i.e. $f^{-1}(\partial D)$ consists of d disjoint circles each of which maps homeomorphically to ∂D. Choose simple closed loops γ_1, γ_1 around b_1 and b_2 as displayed in Figure 1. Then $\tilde{\phi}(\gamma_1)$ and $\tilde{\phi}(\gamma_2)$ both equal the same transposition $\tau = (j, k)$.

Lemma 2.1. *With the notations and assumptions in the last paragraph, there exists a pair $(f_a : X_a \to B, \phi_a)$ in $H_{S_d}^{d,w}(B, \Sigma, b_0)$, a datum $(f_a : X_a \to B, \phi_a, \{x_j, x_k\})$ in $\mathcal{X}_2(w)$, and an analytic isomorphism $h : f^{-1}(U) \to f_a^{-1}(U)$ such that*
(1) *$f|_{f^{-1}(U)} = (f_a)|_{f_a^{-1}(U)} \circ h$ and $\phi = \phi_a \circ h$, and*
(2) *the image by $G_{w,1}$ of $(f_a : X_a \to B, \phi_a, \{x_j, x_k\})$ lies in the same connected component of $M_{S_d}^{d,w+2}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$.*

Proof. We may choose an analytic isomorphism of the disk $D \subset B$ with the unit disk $\Delta \subset \mathbb{C}$ such that b_1 and b_2 map to the two roots of $x^2 = t_0$ for some $t_0 \in \Delta - \{0\}$. Let x be the coordinate on Δ. Consider the map $f^{-1}(D) \to D$. For each $i \neq j, k$ the connected component of $f^{-1}(D)$ corresponding to i maps isomorphically to D. The connected component of $f^{-1}(D)$ corresponding to j and k is identified with the covering C_{t_0} of Δ given by $C_{t_0} = \{(x, y) \in \mathbb{C}^2 : x \in \Delta, y^2 - (x^2 - t_0) = 0\}$. For

$t \in \Delta$, consider the family of covers $C_t = \{(x,y) \in \mathbb{C}^2 : x \in \Delta, y^2 - (x^2 - t) = 0\}$. By the Riemann existence theorem, for each $t \in \Delta$ there is a pair $(f_t : X_t \to B, \phi_t)$ in $M^{d,w+2}(B, \Sigma, b_0)$ such that the restriction of f_t to $f_t^{-1}(U)$ is identified with the restriction of f to $f^{-1}(U)$ and such that the restriction of f_t to D consists of $d-2$ copies of D mapping isomorphically to D (one copy for each $i \neq j, k$), and the connected component corresponding to j and k is identified with $C_t \to \Delta$. We will see that $(f_0 : X_0 \to B, \phi_0)$ is in the image of $G_{w,1} : \mathcal{X}_2(w) \to M^{d,w+2}(B, \Sigma, b_0)$.

Define $u : X_a \to X_0$ to be the normalization and define $\{x_j, x_k\}$ to be the preimage of the node $x_0 \in X_0$. We define $f_a : X_a \to B$ to be $f_a = f_0 \circ u$ and $\phi_a = \phi_0 \circ u$. Notice that $f_a : X_a \to B$ is unbranched over D. Define x_j (resp. x_k) to be the preimage of $f_0(x_0)$ on the sheet of $f_a^{-1}(D)$ corresponding to $j \in \{1, \ldots, d\}$ (resp. to $k \in \{1, \ldots, d\}$). We have an identification of $u^{-1}(f_0^{-1}(U))$ with $f_0^{-1}(U)$. Therefore we have an identification $h : f^{-1}(U) \to f_a^{-1}(U)$ commuting with f, f_a and with ϕ, ϕ_a. In particular, we conclude that $\tilde{\phi}_a : \pi_1(U - \mathrm{br}(f_a), b_0) \to S_d$ is identified with $\tilde{\phi} : \pi_1(U - \mathrm{br}(f), b_0) \to S_d$ and so is surjective. So $(f_a : X_a \to B, \phi_a)$ is in $H^{d,w}_{S_d}(B, \Sigma, b_0)$. Clearly $(f_a : X_a \to B, \phi_a, \{x_j, x_k\})$ is in $\mathcal{X}_2(w)$ and, by construction, its image under $G_{w,1}$ is $(f_0 : X_0 \to B, \phi_0)$. Since $(f_0 : X_0 \to B, \phi_0)$ is in the same connected component of $H^{d,w+2}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$, this proves the lemma. □

Lemma 2.2. *With the notations and assumptions in lemma 2.1, suppose given a transposition $(j_b, k_b) \in S_d$. Then there exists a pair $(f_b : X_b \to B, \phi_b)$ in $H^{d,w+2}_{S_d}(B, \Sigma, b_0)$, and an analytic isomorphism $h : f^{-1}(U) \to f_b^{-1}(U)$ such that:*

1. *$br(f_b) = br(f)$,*
2. *$f|_{f^{-1}(U)} = f_b|_{f_b^{-1}(U)} \circ h$ and $\phi = \phi_b \circ h$,*
3. *$\tilde{\phi}_b(\gamma_1) = \tilde{\phi}_b(\gamma_2) = (j_b, k_b)$, and*
4. *$(f_b : X_b \to B, \phi_b)$ is in the same connected component of $H^{d,w+1}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$.*

Proof. Let $(f_a : X_a \to B, \phi_a, \{x_j, x_k\})$, $h_a : f^{-1}(U) \to f_a^{-1}(U)$ be as constructed in the proof of lemma 2.1. Define b_1 to be $f_a(x_j) = f_a(x_k)$. Define $(X_a)_2 \to B - (\Sigma \cup \mathrm{br}(f))$ to be the fiber of $\mathcal{X}_2(w) \to H^{d,w}_{S_d}(B, \Sigma, b_0)$ over $(f_a : X_a \to B, \phi_a)$. Notice that $(X_a)_2 \to B - (\Sigma \cup \mathrm{br}(f))$ is an unbranched covering space. Define x_{j_b} (resp. x_{k_b}) to be the elements of $f_a^{-1}(b_1)$ which lie on the sheets of $f_a^{-1}(D)$ corresponding to $j_b \in \{1, \ldots, d\}$ (resp. $k_b \in \{1, \ldots, d\}$). Because $\tilde{\phi}_a : \pi_1(B - \mathrm{br}(f_a), b_1) \to S_d$ is surjective, in particular it is doubly transitive. Therefore $(X_a)_2$ is irreducible and $(f_a : X_a \to B, \phi_a, \{x_{j_b}, x_{k_b}\})$ is in the same connected component of $\mathcal{X}_2(w)$ as $(f_a : X_a \to B, \phi_a, \{x_j, x_k\})$. So the image $G_{w,1}(f_a : X_a \to B, \phi_a, \{x_{j_b}, x_{k_b}\})$ is in the same connected component of $M^{d,w+2}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$.

Consider the same family of covers $C_t \to \Delta$ as in the proof of lemma 2.1, with the roles of j, k replaced by j_b, k_b. By the Riemann existence theorem there exists a pair $(f_b : X_b \to B, \phi_b)$ and an isomorphism $h_b : f_a^{-1}(U) \to f_b^{-1}(U)$ commuting with f_a, f_b and ϕ_a, ϕ_b such that $f_b^{-1}(D) \to D$ is identified with the covering $C_1 \to \Delta$. We define $h : f^{-1}(U) \to f_b^{-1}(U)$ to be $h_b \circ h_a$. Then

$(f_b : X_b \to B, \phi_b)$ and h satisfy items (1), (2), and (3) of the lemma. Moreover $(f_b : X_b \to B, \phi_b)$ is in the same connected component of $M^{d,w+2}_{S_d}(B, \Sigma, b_0)$ as $G_{w,2}(f_a : X_a \to B, \phi_a, \{x_{j_b}, x_{k_b}\})$. Thus $(f_b : X_b \to B, \phi_b)$ is in the same connected component of $M^{d,w+2}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$. Since both pairs are in $H^{d,w+2}_{S_d}(B, \Sigma, b_0)$ and since $M^{d,w+2}_{S_d}(B, \Sigma, b_0)$ is smooth, we conclude that both pairs are in the same connected component of $H^{d,w+2}_{S_d}(B, \Sigma, b_0)$. □

3. Branching monodromy

Fix a closed disk $D \subset B$ disjoint from Σ. Fix a path from b_0 to the boundary ∂D. Denote by U the open subset $B - D \subset B$. In most of this section we will restrict our attention to the analytic open subset $V \subset M^{d,w}_{S_d}(B, \Sigma, b_0)$ parametrizing $(f : X \to B, \phi)$ such that $\mathrm{br}(f)$ is contained in the interior of D. By assumption, the monodromy group of f, i.e. the image of $\tilde{\phi}$, is all of S_d. For each connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V$, the function which associates to each $(f : X \to B, \phi)$ the image of $\tilde{\phi} : \pi_1(D - \mathrm{br}(f), b_0) \to S_d$ is constant. We call this subgroup the *branching monodromy group* of f (of course it depends on the choice of D and the path from b_0 to ∂D).

Since $H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V$ is the complement of proper analytic subvarieties of the complex manifold V, each connected component of V is the closure of a unique connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V$. For each subgroup $G \subset D$ let us denote by $V_G \subset V$ the open and closed submanifold on which the image of $\tilde{\phi} : \pi_1(D - \mathrm{br}(f), b_0) \to S_d)$ equals G. The goal of this section is to prove that when $w \geq 2d$, every connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$ has nonempty intersection with V_{S_d}, i.e. there is a pair $(f : X \to B, \phi)$ in this connected component and in V which has branching monodromy group equal to S_d.

Suppose that $(f : X \to B, \phi) \in H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V_G$. Because G is generated by transpositions, there is a partition $(A_1, \dots, A_r)$ of $\{1, \dots, d\}$ such that $G = S_{A_1} \times \cdots \times S_{A_r}$ where $S_{A_m} \subset S_d$ consists of those permutations which act as the identity on each subset $A_n \subset \{1, \dots, d\}$ for which $n \neq m$. In other words, G is the subgroup of permutations which stabilize each subset $A_m \subset \{1, \dots, d\}$.

Choose a system of loops $\gamma_1, \dots, \gamma_w$ as in Figure 2. Denote by τ_i the transposition $\tilde{\phi}(\gamma_i)$. Then each τ_i lies in one of the subgroups $S_{A_{m(j)}}$.

Suppose that γ_i and γ_{i+1} are adjacent loops such that τ_i lies in S_{A_m} and τ_{i+1} lies in S_{A_n} with $m \neq n$. Consider the element σ_i of the braid group which interchanges the branch points b_i and b_{i+1} as shown in Figure 3. The result is to replace τ_i by τ_{i+1} and to replace τ_{i+1} by $\tau_{i+1}\tau_i\tau_{i+1}$. Since A_m and A_n are disjoint, we have $\tau_{i+1}\tau_i\tau_{i+1} = \tau_i$. In other words, the result is to interchange τ_i and τ_{i+1}. Note that this operation does not change $G \subset S_d$. By repeating this process, we may

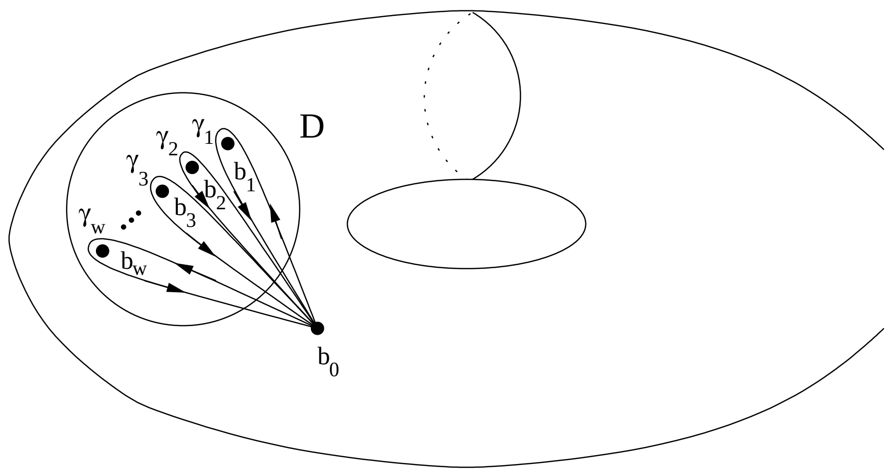

FIGURE 2. Branch points contained in the disk D

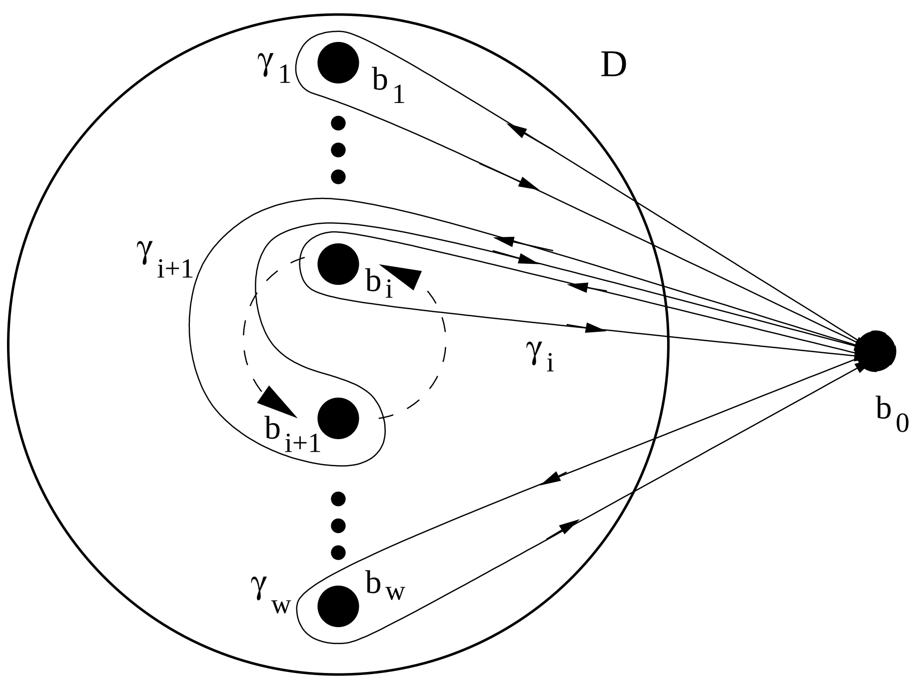

FIGURE 3. Braid move exchanging two branch points

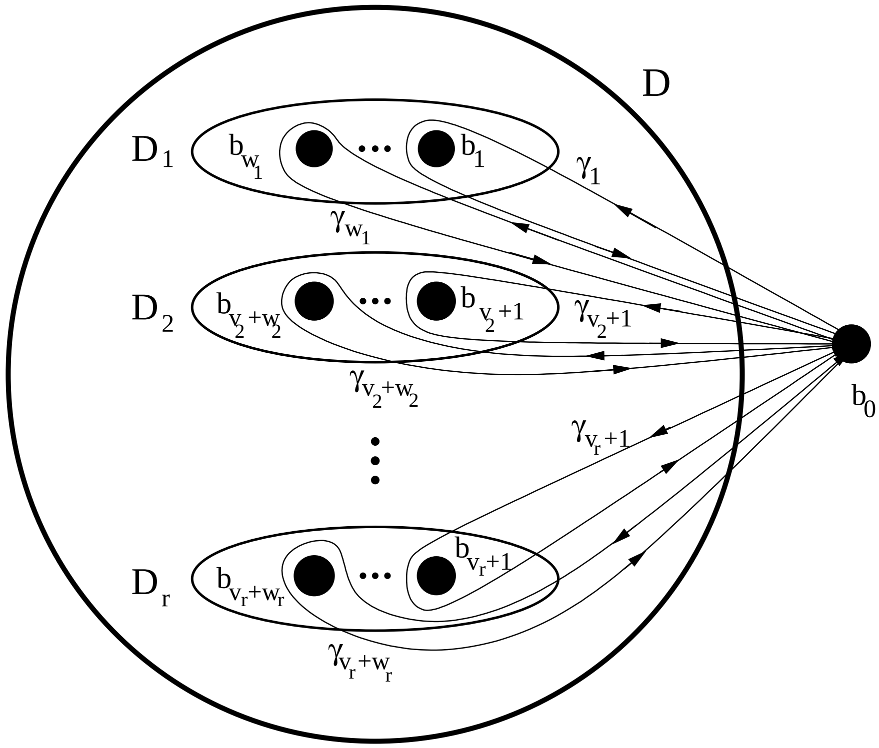

FIGURE 4. Branch points in standard position

arrange that there are integers $w_0 = 0, w_1, \ldots, w_r$ with the following property: for $m = 1, \ldots, r$, denote $v_m = w_0 + \cdots + w_{m-1}$; then for each $m = 1, \ldots, r$, each transpositions τ_i with $v_m + 1 \leq i \leq v_{m+1}$ is in S_{A_m}. Notice that since these transpositions generate S_{A_m}, we have $w_m \geq \#A_m - 1$. Stated more precisely, we have proved that given a pair $(f : X \to B, \phi)$ in $H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V_G$, in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V_G$ there is a pair $(f_a : X_a \to B, \phi_a)$ and an isomorphism $h : f^{-1}(U) \to (f_a)^{-1}(U)$ such that:

(1) $\mathrm{br}(f_a) = \mathrm{br}(f)$,
(2) $f|_{f^{-1}(U)} = (f_a)|_{f_a^{-1}(U)} \circ h$ and $\phi = \phi_a \circ h$, and
(3) the transpositions $\tau_i = \tilde{\psi}(\gamma_i)$ satisfy $\gamma_i \in S_{A_m}$ for $v_m + 1 \leq i \leq v_{m+1}$.

We say that a pair $(f_a : X_a \to B, \phi_a)$ satisfying item (3) is in *standard position.* For each $m = 1, \ldots, r$, choose a subdisk $D_m \subset D$ as in Figure 4 which contains the loops γ_i for $(w_0 + \cdots + w_{m-1}) + 1 \leq i \leq w_0 + \cdots + w_{m-1} + w_m$. Note that any braid move in D_m has no effect on the branch points belonging to D_n with $n \neq m$.

Proposition 3.1. *Suppose that $(f : X \to B, \phi)$ in $H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V_G$ is in standard position. Suppose $w_m \geq 2\#A_m$. Then there are braid moves in D_m transforming $(\tau_{v_m+1}, \ldots, \tau_{v_m+w_m}$ into $(\tau'_1, \ldots, \tau'_{w_m-2}, \tau, \tau)$ such that $\tau'_1, \ldots, \tau'_{w_m-2}$ generate S_{A_m}.*

Proof. Define $g = \tau_{v_m+1} \cdot \dots \cdot \tau_{v_{m+1}}$. By [6, theorem 1], the braid group of D_m acts transitively on the set

$$O_g := \{(\tau_1, \dots, \tau_{w_m}) \in S_{A_m} | \text{ each } \tau_i \text{ a transposition },\\ \langle \tau_1, \dots, \tau_{w_m} \rangle = S_{A_m}, \tau_1 \cdot \dots \cdot \tau_{w_m} = g\}.$$

Thus it suffices to find $(\tau'_1, \dots, \tau'_{w_m-2}, \tau, \tau)$ as above which lies in O_g.

Suppose that g has cycle type $(\lambda_1, \dots, \lambda_s)$ for some partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s)$ of $\#A_m$. Define $\lambda_0 = 0$ and for each $k = 1, \dots, s$, define $\mu_k = \lambda_0 + \dots + \lambda_{k-1}$. Then we may order the elements of A_m so that g is the permutation

$$g = (\mu_1 + 1, \dots, \mu_1 + \lambda_1)\,(\mu_2 + 1, \dots, \mu_2 + \lambda_2) \dots (\mu_s + 1, \dots, \mu_s + \lambda_2)\,. \tag{1}$$

Of course this ordering has nothing to do with the ordering induced by ϕ.

For each $k = 1, \dots, s$, consider the ordered sequence of transpositions, which is defined to be empty if $\lambda_k = 1$, and for $k > 1$ is defined to be

$$I_k = ((\mu_k + 1, \mu_k + 2), (\mu_k + 2, \mu_k + 3), \dots, (\mu_k + \lambda_k - 1, \mu_k + \lambda_k))\,. \tag{2}$$

Thus I_k contains $\lambda_k - 1$ transpositions. Next consider the sequence of transpositions

$$I_{s+1} = ((\mu_1, \mu_2), (\mu_1, \mu_2), (\mu_2, \mu_3), (\mu_2, \mu_3), \dots, (\mu_{s-1}, \mu_s), (\mu_{s-1}, \mu_s))\,. \tag{3}$$

The concatenated sequence $I = I_1 \cup \dots \cup I_s \cup I_{s+1}$ has length $L := \sum_k (\lambda_k - 1) + 2(s-1) = \#A_m + s - 2 \leq 2\#A_m - 2$. The product of these transpositions is g, and these transpositions generate S_{A_m}. Since the sign of g is both $(-1)^{w_m}$ and $(-1)^L$, we have that $w_m - L$ is divisible by 2. And the assumption that $w_m \geq 2\#A_m$, implies that $w_m - L \geq 2$. If we choose any transposition $\tau \in S_{A_m}$, and let J be the constant sequence of length $w_m - L$, $J = (\tau, \tau, \dots, \tau)$, then we have that the concatenated sequence $I \cup J$ is an element of O_g satisfying the hypotheses of the proposition. ∎

Corollary 3.2. *Given $w' \geq w$ with $w' \geq 2d, w \geq 2d - 2$, set $e = \frac{w'-w}{2}$. Suppose given a pair $(f : X \to B, \phi) \in H^{d,w'}_{S_d}(B, \Sigma, b_0) \cap V_G$. Then there is a pair $(f_a : X_a \to B, \phi_a) \in H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V_G$, an isomorphism $h : f^{-1}(U) \to (f_a)^{-1}(U)$ and a datum $(f_a : X_a \to B, \phi_a, \{x^1_1, x^1_2\}, \dots, \{x^e_1, x^e_2\})$ in $\mathcal{X}^e_2(w)$ such that*

1. *$f_a(x^i_j) \in D$ for $i = 1, \dots, e$ and for $j = 1, 2$,*
2. *$f|_{f^{-1}(U)} = (f_a)|_{f_a^{-1}(U)} \circ h$ and $\phi = \psi \circ h$, and*
3. *the image of $(f_a : X_a \to B, \phi_a, \{x^1_1, x^1_2\}, \dots, \{x^e_1, x^e_2\})$ under $G_{w,e}$ is contained in the same connected component of $H^{d,w'}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$.*

Proof. We prove this by induction on $w' - w$. For $w' = w$, there is nothing to prove. Suppose $w' - w > 0$ and suppose the proposition has been proved for all smaller values of $w' - w$. We note by proposition 3.1 that there is a map $(f_c : X_c \to B, \phi_c)$ and $h_c : f^{-1}(U) \to f_c^{-1}(U)$ satisfying the conditions of that proposition. If we define D' to be a small disk containing the branch points of f_c corresponding to the transposition τ, then $(f_c : X_c \to B, \phi_c)$ and D' satisfy the hypothesis of lemma 2.1. By that lemma, there is a datum $(f_b : X_b \to B, \phi_b, \{x_j, x_k\}) \in \mathcal{X}_2(w' - 2)$ and an isomorphism $h_b : f_a^{-1}(B - D') \to f_b^{-1}(B - D')$ satisfying the conditions of that lemma.

If $w' = w + 2$, we are done by taking $(f_a : X_a \to B, \phi_a, \{x_1^1, x_2^1\}) = (f_b : X_b \to B, \phi_b, \{x_j, x_k\})$ and taking $h = h_b \circ h_c$. Therefore suppose that $w' > w + 2$. Now $(f_b : X_b \to B, \phi_b)$ is in $H^{d,w'-2}_{S_d}(B, \Sigma, b_0) \cap V_G$. Since $(w' - 2) - w < w' - w$, by the induction assumption there exists a datum $(f_a : X_a \to B, \phi_a, \{x_1^1, x_2^1\}, \ldots, \{x_1^{e-1}, x_2^{e-1}\})$ in $\mathcal{X}_2^{e-1}(w)$ and $h_a : f_b^{-1}(U) \to f_a^{-1}(U)$ satisfying the conditions of our corollary. Up to deforming this datum slightly, we may suppose that the isomorphism h_a extends to a larger open set which contains $x_j, x_k \in X_b$, and, defining $x_1^e = h_a^{-1}(x_j)$ and $x_2^e = h_a^{-1}(x_k)$, the datum $(f_a : X_a \to B, \phi_a, \{x_1^1, x_2^1\}, \ldots, \{x_1^{e-1}, x_2^{e-1}\}, \{x_1^e, x_2^e\})$ is in $\mathcal{X}_2^e(w)$. We define $h = h_a \circ h_b \circ h_c$. The image of this datum under $G_{w,e}$ is contained in the same connected component as the image of $(f_b : X_b \to B, \phi_b, \{x_j, x_k\})$ under $G_{w'-2,1}$. So the corollary is proved by induction. $\square$

Corollary 3.3. *If $w \geq 2d$, then for any pair $(f : X \to B, \phi) \in H^{d,w}_{S_d}(B, \Sigma, b_0)$ in V, there is a pair $(f_a : X_a \to B, \phi_a) \in H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V_{S_d}$ and an isomorphism $h : f^{-1}(U) \to (f_a)^{-1}(U)$ such that*

1. *$f|_{f^{-1}(U)} = (f_a)|_{f_a^{-1}(U)} \circ h$ and $\phi = \phi_a \circ h$, and*
2. *$(f_a : X_a \to B, \phi_a)$ is in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$.*

Proof. By corollary 3.2, it suffices to consider the case that $w = 2d$. Suppose the branching monodromy group of $(f : X \to B, \phi)$ is $G = S_{A_1} \times \cdots \times S_{A_r}$. We will prove the result by induction on r. If $r = 1$, there is nothing to prove. So assume that $r > 1$ and assume the result is proved for all smaller values of r.

Since $\sum_m (w_m - 2\#A_m)$ equals $w - 2d = 0$, there is some m such that $w_m \geq 2\#A_m$. Without loss of generality, suppose $w_1 \geq 2\#A_1$. By proposition 3.1, we may suppose that the transpositions in D_1 are of the form $(\tau_1, \ldots, \tau_{w_1-2}, \tau, \tau)$ such that $\tau_1, \ldots, \tau_{w_1-2}$ generate S_{A_1}. But then, choosing a small disk D' which contains only the branch points b_{2w_1-1} and b_{2w_1}, we see that $(f : X \to B, \phi)$ and D' satisfy the hypothesis of lemma 2.2. Suppose that $j_b \in A_1$ and $k_b \in A_2$. By lemma 2.2, we can find a pair $(f_b : X_b \to B, \phi_b) \in H^{d,w}(B, \Sigma, b_0)$ and $h_b : f^{-1}(B - D') \to (f_b)^{-1}(B - D')$ such that

1. $f|_{f^{-1}(B-D')} = (f_b)|_{f_b^{-1}(B-D')} \circ h$ and $\phi = \phi_b \circ h$,
2. $\mathrm{br}(f_b) = \mathrm{br}(f)$,
3. the transposition of $(f_b : X_b \to B, \phi_b)$ corresponding to γ_{w_1-1} and γ_{w_1} is (j_b, k_b), and
4. $(f_b : X_b \to B, \phi_b)$ is in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$.

Since the branching monodromy group of $(f_b : X_b \to B, \phi_b) \in H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V$ *outside* of D' already generates $S_{A_1} \times S_{A_2}$, when we add the transposition (j_b, k_b) we conclude the branching monodromy group of $(f_b : X_b \to B, \phi_b)$ is $S_{A_1 \cup A_2} \times S_{A_3} \times \cdots \times S_{A_r}$. By the induction assumption, there is $(f_a : X_a \to B, \phi_a)$ and

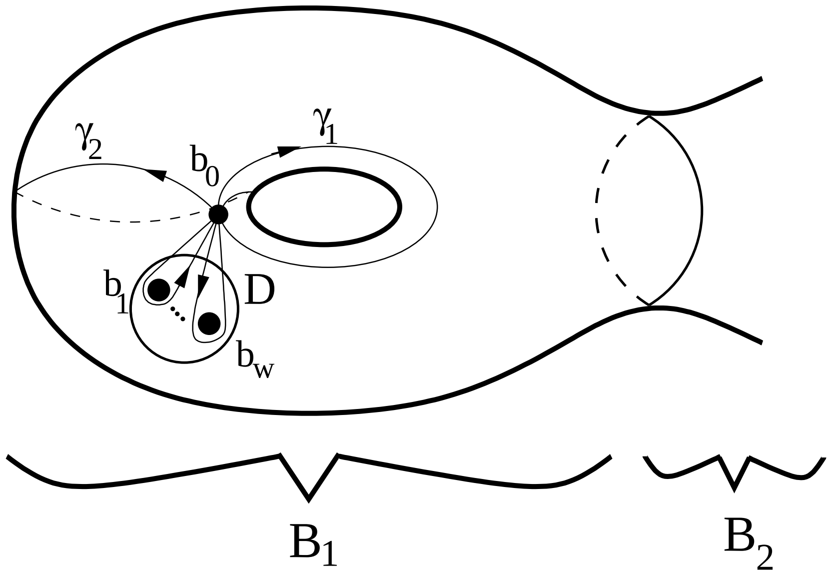

FIGURE 5. The subset B_1

$h_a : f_b^{-1}(U) \to f_a^{-1}(U)$ satisfying the conditions of our corollary where $(f : X \to B, \phi)$ is replaced by $(f_b : X_b \to B, \phi_b)$. Then defining $h = h_a \circ h_b$, we see that $(f_a : X_a \to B, \phi_a)$ and h satisfy the conditions of the corollary for $(f : X \to B, \phi)$, and the corollary is proved by induction. □

4. Induction Argument

In this section we will prove that for $w \geq 2d$, $H^{d,w}_{S_d}(B, \Sigma, b_0)$ is connected. The basic strategy is as follows: If $h = g(B) = 0$, then this is a classical result due to Hurwitz (see the references in the introduction). Suppose given a disk $D \subset B$ and two pairs $(f_1 : X_1 \to B, \phi_1)$ and $(f_2 : X_2 \to B, \phi_2)$ such that all branch points of f_1 and f_2 are contained in D and such that both f_1 and f_2 are trivial over $B - D$, i.e. $f_i^{-1}(B - D) \to B - D$ is just d isomorphic copies of $B - D$ for $i = 1, 2$. Then the genus 0 argument shows that $(f_1 : X_1 \to B, \phi_1)$ and $(f_2 : X_2 \to B, \phi_2)$ are contained in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$. So the argument is reduced to proving that given a general pair $(f : X \to B, \phi)$ with branch points in D, we can perform braid moves such that $f^{-1}(B - D) \to B - D$ is trivial.

Suppose $g \geq 1$ and choose a disk $D \subset B_1 \subset B$ situated as in Figure 5 and which is disjoint from S. Let V be as in section 3 with respect to this disk D. Every connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$ clearly intersects V. So to prove that $H^{d,w}_{S_d}(B, \Sigma, b_0)$ is connected, it suffices to prove that for any two pairs $(f_1 : X_1 \to B, \phi_1)$ and $(f_2 : X_2 \to B, \phi_2)$ in $H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V$ both pairs in the same connected component. We prove this by induction on g through a sequence of intermediate steps (showing each pair is in the same connected component as a pair with some special properties, and finally linking up the resulting pairs).

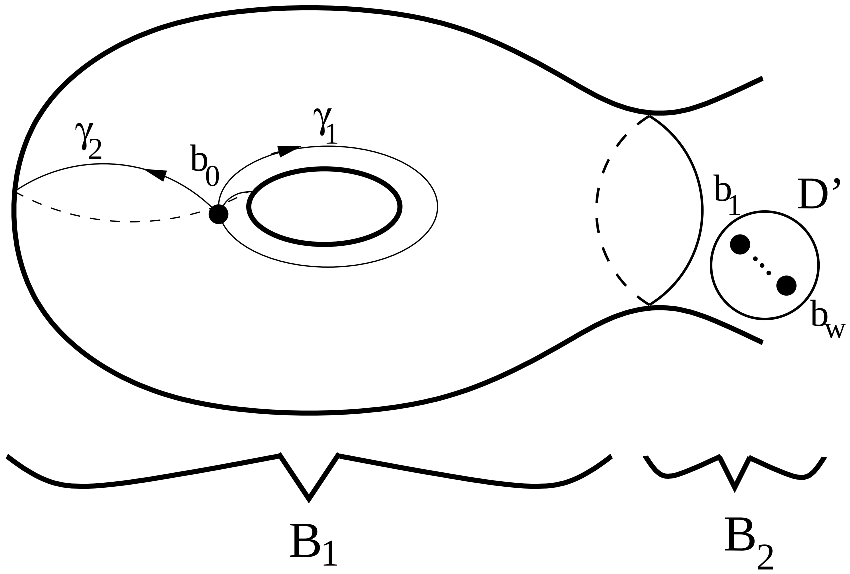

FIGURE 6. The disk D'

Let $D' \subset B_2 \subset B$ be as in Figure 6. Let V' be as in section 3 with respect to D'. We say that $(f : X \to B, \phi)$ in $H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V'$ is B_1-trivial if $f^{-1}(B_1) \to B_1$ is a trivial cover.

Proposition 4.1. *Suppose $w \geq 2d$. Any pair $(f : X \to B, \phi) \in H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V$ is in the same connected component as a pair $(f_a : X_a \to B, \phi_a)$ which is B_1-trivial.*

Proof. We prove this in a number of steps. The idea is to apply braid moves to reduce $\tilde{\phi}(\gamma_1)$ and $\tilde{\phi}(\gamma_2)$ to the identity. Finally we will move all the branch points out of B_1 along a specified path to give a B_1-trivial pair.

Our first braid move is displayed in Figure 7. It consists of choosing the final branch point b_w, moving b_w across the loop γ_1, without crossing γ_2, and continuing along the loop "parallel" to γ_2 to return b_w into D. If the resulting cover is $(f_b : X_b \to B, \phi_b)$, then we clearly have $\tilde{\phi}_b(\gamma_1) = \tilde{\phi}(\gamma_1)\tilde{\phi}(\gamma)$ and $\tilde{\phi}_b(\gamma_2) = \tilde{\phi}(\gamma_2)$. So the result is to multiply the permutation of γ_1 by the permutation of γ while leaving the permutation of γ_2 unchanged.

Our second braid move is exactly like our first braid move with the roles of γ_1 and γ_2 switched. It is illustrated in Figure 8 We choose the first branch point b_1, move b_1 across the loop γ_2, without crossing γ_1, and then continue along the loop "parallel" to γ_1 we return b_1 into D. If the resulting cover if $(f_b : X_b \to B, \phi_b)$, then we clearly have $\tilde{\phi}_b(\gamma_2) = \tilde{\phi}(\gamma_2)\tilde{\phi}(\gamma)$ and $\tilde{\phi}_b(\gamma_1) = \tilde{\phi}(\gamma_1)$. So the result is to multiply the permutation of γ_2 by the permutation of γ while leaving the permutation of γ_1 unchanged. Notice that in both of these moves, we are not concerned about the

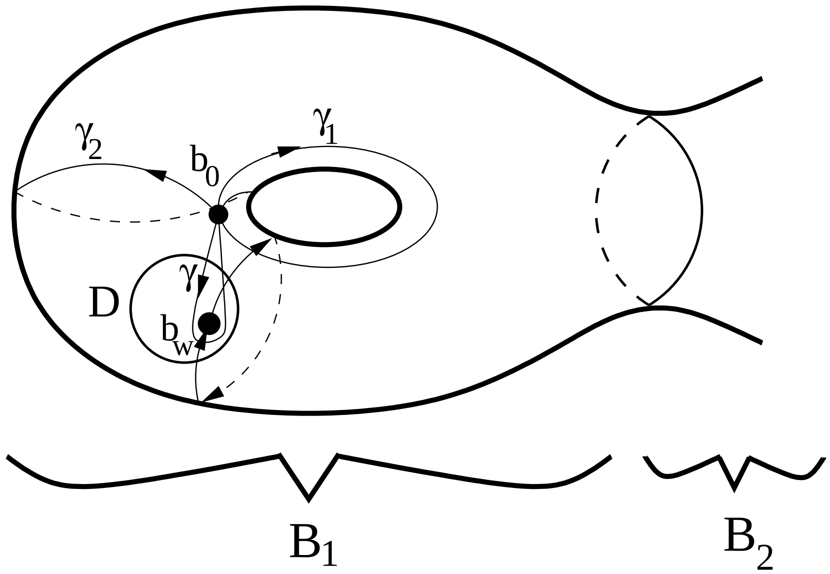

Figure 7. The first braid move

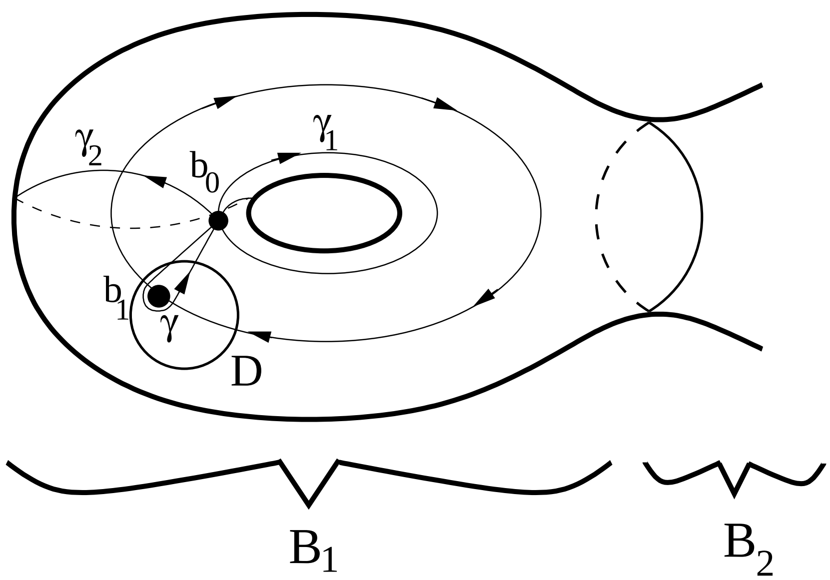

Figure 8. The second braid move

effect of the braid move on the branching monodromy of D (we may always use corollary 3.3 to "repair" the branching monodromy of D).

The main claim is that these braid moves along with corollary 3.3 suffice to trivialize the permutations of γ_1 and γ_2. Suppose given $(f : X \to B, \phi) \in H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V$. Suppose that $\tilde{\phi}(\gamma_1)$ has cycle type $\lambda = (\lambda_1 \geq \cdots \geq \lambda_s)$ and $\tilde{\phi}(\gamma_2)$ has cycle type $\mu = (\mu_1 \geq \cdots \geq \mu_t)$. Define $|\lambda| = \sum_m (\lambda_m - 1) = d - s$

and define $|\mu| = \sum_n (\mu_n - 1) = d - t$. We claim that there is a pair $(f_b : X_b \to B, \phi_b) \in H^{d,w}_{S_d}(B, \Sigma, b_0) \cap V$ such that:

(1) $\tilde{\phi}_b(\gamma_1) = \tilde{\phi}_b(\gamma_2) = 1$, and
(2) $(f_b : X_b \to B, \phi_b)$ is contained in the same connected component of $H^{d,w}_{S_d}$ as $(f : X \to B, \phi)$.

We will prove this by induction on $|\lambda| + |\mu|$. If $|\lambda| + |\mu| = 0$, i.e. $\lambda = \mu = 1^d$, we may simply take $(f_b : X_b \to B, \phi_b) = (f : X \to B, \phi)$. Therefore suppose that $|\lambda| + |\mu| > 0$ and, by way of induction, suppose the result is proved for all smaller values of $|\lambda| + |\mu|$. We make one reduction at the outset: by corollary 3.3, we may replace $(f : X \to B, \phi)$ with a pair which is equivalent over $B - D$, but whose branching monodromy group is all of S_d.

Suppose first that $|\lambda| > 0$. Let $\sigma \in S_d$ be the λ_1-cycle occurring in $\tilde{\phi}(\gamma_1)$ and suppose $\tau \in S_d$ is a transposition such that $\sigma\tau$ is a $(\lambda_1 - 1)$-cycle. By proposition 3.1, we may replace $(f : X \to B, \phi)$ by a pair which is equivalent over $B - D$, and whose sequence of transpositions is of the form $(\tau_1, \ldots, \tau_{w-2}, \tau, \tau)$. If we apply our first braid move, the resulting cover $(f_c : X_c \to B, \phi_c)$ is such that $|\lambda_c| = |\lambda| - 1$ and $|\mu_c| = |\mu|$ so that $|\lambda_c + |\mu_c| < |\lambda| + |\mu|$. By the induction assumption applied to $(f_c : X_c \to B, \phi_c)$, we conclude there exists a pair $(f_b : X_b \to B, \phi_b)$ with $\tilde{\phi}_b(\gamma_1) = \tilde{\phi}_b(\gamma_2) = 1$ and which is in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$ as $(f_c : X_c \to B, \phi_c)$. By construction, $(f_c : X_c \to B, \phi_c)$ is in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$. Thus $(f_b : X_b \to B, \phi_b)$ satisfies conditions (1) and (2) above.

The second possibility is that $|\lambda| = 0$ but $|\mu| > 0$. Let $\sigma \in S_d$ be the μ_1-cycle occurring in $\tilde{\phi}(\gamma_2)$ and suppose $\tau \in S_d$ is a transposition such that $\sigma\tau$ is a $(\mu_1 - 1)$-cycle. By an obvious generalization of proposition 3.1, we may replace $(f : X \to B, \phi)$ by a pair which is equivalent over $B - D$ and whose sequence of transpositions is of the form $(\tau, \tau, \tau_1, \ldots, \tau_{w-2})$. If we apply our second braid move, the resulting cover $(f_c : X_c \to B, \phi_c)$ is such that $|\lambda_c| = |\lambda| = 0$ and $|\mu_c| = |\mu| - 1$ so that $|\lambda_c| + |\mu_c| < |\lambda| + |\mu|$. By the induction assumption applied to $(f_c : X_c \to B, \phi_c)$, we conclude there exists a pair $(f_b : X_b \to B, \phi_b)$ with $\tilde{\phi}_b(\gamma_1) = \tilde{\phi}_b(\gamma_2) = 1$ and which is in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$ as $(f_c : X_c \to B, \phi_c)$. By construction, $(f_c : X_c \to B, \phi_c)$ is in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$ as $(f : X \to B, \phi)$. Thus $(f_b : X_b \to B, \phi_b)$ satisfies conditions (1) and (2) above. So in both the first and second case, we conclude that the claim is true for $(f : X \to B, \phi)$. So the claim is proved by induction.

Now we prove the proposition. By the claim, we may suppose that $(f : X \to B, \phi)$ is such that $\tilde{\phi}(\gamma_1) = \tilde{\phi}(\gamma_2) = 1$. Finally we move the disk D and all its branch points out of B_1 to D' as shown in Figure 9. Let $(f_a : X_a \to B, \phi_a)$ be the resulting pair. Notice that since the path of D never crosses γ_1 or γ_2, we still have $\tilde{\phi}_a(\gamma_1) = \tilde{\phi}_b(\gamma_2) = 1$. As the fundamental group $\pi_1(B_1, b_0)$ is generated by γ_1 and γ_2, we conclude that $(f_a : X_a \to B, \phi_a)$ is trivial over B_1. Thus $(f_a : X_a \to B, \phi_a)$ is B_1-trivial, and the proposition is proved. $\square$

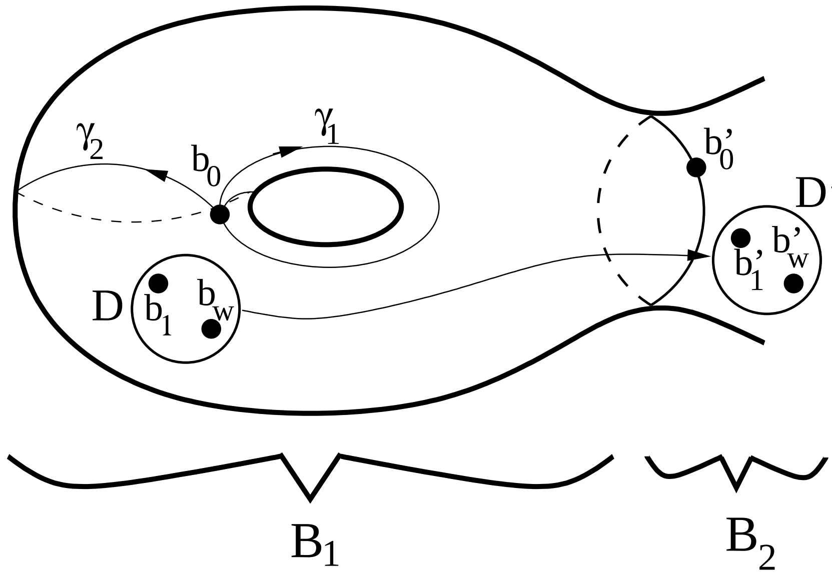

FIGURE 9. Moving D to make the cover B_1-trivial

Now we are ready to prove the theorem.

Theorem 4.2. *If $w \geq 2d$, then $H^{d,w}_{S_d}(B, \Sigma, b_0)$ is connected.*

Proof. The proof is by induction on the genus h of B. If $h = 0$, the theorem is due to Hurwitz (see the references in the introduction). Thus suppose $h > 0$, and by way of induction suppose that the theorem is proved for all genera smaller than h. Suppose that $\Sigma \subset \Sigma' \subset B$. There is a natural map $H^{d,w}_{S_d}(B, \Sigma', b_0) \to H^{d,w}_{S_d}(B, \Sigma, b_0)$ whose image is a dense Zariski open set. So if $H^{d,w}_{S_d}(B, \Sigma', b_0)$ is connected, it follows that $H^{d,w}_{S_d}(B, \Sigma, b_0)$ is also connected. Therefore we may enlarge S, if need be, so that it contains a point b_0' in the boundary circle $B_1 \cap B_2$ (and such that this is the only point of S on the boundary circle).

Now by proposition 4.1, we see that every connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$ contains a B_1-trivial pair. So to finish the proof, it suffices to prove that for two B_1-trivial pairs, say $(f_1 : X_1 \to B, \phi_1)$ and $(f_2 : X_2 \to B, \phi_2)$, there are braid moves which change the first pair to the second. Let $U \subset B_2$ denote the interior of B_2, i.e. the complement of the boundary circle. Choose a path γ in B_1 from b_0 to b_0' and in this way identify $\phi_i : f_i^{-1}(b_0) \to \{1, \ldots, d\}$ with $\phi_i' : f_i^{-1}(b_0') \to \{1, \ldots, d\}$. Now U is homeomorphic to $B' - \{b_0'\}$ for some Riemann surface B' of genus $h - 1$ and for some point $b_0' \in B'$. Let $\Sigma' \subset B'$ denote the union of the image of $\Sigma \cap U$ and $\{b_0'\}$. Then the restricted covers $(f_i : f_i^{-1}(U \to U, \phi_i)$ for $i = 1, 2$ are equivalent to covers $(f_i' : X_i' \to B', \phi_i')$ in $H^{d,w}_{S_d}(B', \Sigma', b_0')$. By the induction assumption, we know that $H^{d,w}_{S_d}(B', \Sigma', b_0')$ is connected. Therefore there is a path $\alpha : [0, 1] \to (B' - \Sigma')^0_w$ such that

(1) $\alpha(0) = \mathrm{br}(f_1')$,
(2) $\alpha(1) = \mathrm{br}(f_2')$, and

(3) if $\tilde{\alpha} : [0,1] \to H^{d,w}_{S_d}(B', \Sigma', b'_0)$ is the lift with $\tilde{\alpha}(0) = (f'_1 : X'_1 \to B', \phi'_1)$, then $\tilde{\alpha}(1) = (f'_2 : X'_1 \to B', \phi'_2)$.

Using our homeomorphism, we may identify α with a path $\beta : [0,1] \to (U - \Sigma \cap U)^0_w$ such that $\beta(0) = \mathrm{br}(f_1)$ and $\beta(1) = \mathrm{br}(f_2)$. It follows that if $\tilde{\beta} : [0,1] \to H^{d,w}_{S_d}(B, \Sigma, b_0)$ is the lift with $\tilde{\beta}(0) = (f_1 : X_1 \to B, \phi_1)$, then $\tilde{\beta}(1) = (f_2 : X_2 \to B, \phi_2)$. This proves that $(f_1 : X_1 \to B, \phi_1)$ and $(f_2 : X_2 \to B, \phi_2)$ lie in the same connected component of $H^{d,w}_{S_d}(B, \Sigma, b_0)$. It follows that $H^{d,w}_{S_d}(B, \Sigma, b_0)$ is connected, and the theorem is proved by induction. $\square$

Now we can prove theorem 1.1. There is a forgetful map $H^{d,w}_{S_d}(B, \Sigma, b_0) \to \mathcal{H}^{d,w}_{S_d}(B)$. This morphism is étale with dense image. Since $H^{d,w}_{S_d}(B, \Sigma, b_0)$ is connected, it follows that $\mathcal{H}^{d,w}_{S_d}(B)$ is also connected, which proves theorem 1.1.

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